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Degree Diameter Problem on Oxide Networks

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Abstract

The degree diameter problem is the problem of finding the largest graph (in terms of number of vertices) subject to the constraints on the degree and the diameter of the graph. Beyond the degree constraint there is no restriction on the number of edges (apart from keeping the graph simple) so the resulting graph may be thought of as being embedded in the complete graph. In a generalization of this problem, the graph is considered to be embedded in some connected host graph. This article considers embedding the graph in the oxide network and provides some exact values and some upper and lower bounds for the optimal graphs.

Keywords: Oxide network; Degree; Diameter; Silicon vertex; Oxygen vertex; Closed ball; Tetrahedron; Triangle

Introduction

All graphs considered in this paper are simple, finite and undirected. The degree of a vertex in the graph G is the number of edges incident with that vertex in G. The maximum and minimum degree of the graph G is denoted by Δ(G) and δ(G), respectively. The distance between two vertices x and y of G is the length of the shortest path between them. The distance between a vertex x and the set A is defined as $d(x,A)=\min_{a\in A} d(x,a)$. The diameter of the graph G is denoted by D and is defined as the maximum pairwise distance between the vertices of the graph G. For a given graph G and positive integers Δ and D, $N_G(\Delta,D)$ denote the order of a largest subgraph of G with given maximum degree Δ and given diameter D. For definitions and notations not defined here we refer to the text [1-4]. The degree/diameter problem (DDP) is a quest to find the largest graphs, in terms of the number of vertices, given constraints on the maximum degree of the graph and its diameter [5-10]. For a thorough survey of the state of the problem [11,12]. Because the maximum degree is the only restriction on the distribution of edges, there is considerable freedom in placing edges so as to avoid violating the diameter constraint. The degree diameter problem of honeycomb and triangular networks have been studied [5,6]. The largest sub graph $N_G(\Delta,D)$ have been determined with multidimensional rectangular Mesh and the multidimensional hexagonal grid as host graphs [7,10].

Silicates are obtained by fusing metal oxides or metal carbonates with sand. Essentially all the silicates contain Sio_4 tetrahedra. In chemistry, the corner vertices of sio_4 tetrahedron represent oxygen ions and the center vertex represents the silicon ion [2]. In graph theory, the tetrahedra unit sio_4 is the complete graph K_4. We call the corner vertices as oxygen nodes and the center vertex as silicon node.

The graph of silicate network can be constructed in different ways [3]. The silicate network SL(1) is a cyclic silicate with six Sio_4 tetrahedra units. The silicate network SL(2) is obtained by adding six units of SL(1) such that each outer SL(1) shares two consecutive tetrahedra of inner SL(1). Inductively, silicate network SL(n) of dimension n is obtained from SL(n-1) by adding a layer of SL(1) around the boundary of SL(n-1). The number of nodes in SL(n) is $15n^2+3n$ and the number of edges of SL(n) is $36n^2$. In Figure 1a, a silicate network of dimension two is shown [2].

The Graph of Oxide Network

When all the silicon nodes are deleted from a silicate network, we obtain a new network which we shall call as an oxide network. An n-dimensional oxide network is denoted by OX(n) having number of vertices $9n^2+3n$ and number of edges $18n^2$ [4]. An oxide network of dimension 2 is shown in Figure 1b [4]. The distance of a vertex x∈V(G) from a triangle T is $\min_{y\in V(T)} d_G(x,y)$ and will be denoted by $d_G(x,T)$, where the vertex y is the closest vertex of T from x. The properties of oxide networks have been studied in various aspect [2-4]. In this paper, we have studied the degree diameter problem of oxide networks.

In the following sections, we consider the cases when Δ=4 and Δ=1,2,3 separately.

Values for Δ=4

Let X denote the infinite oxide network in the Euclidean plane and we consider X as a host graph. Note that Δ(X)=4.

Proposition 1.1

For even D ≥ 4, Let the induced sub graph of X denoted by O_D

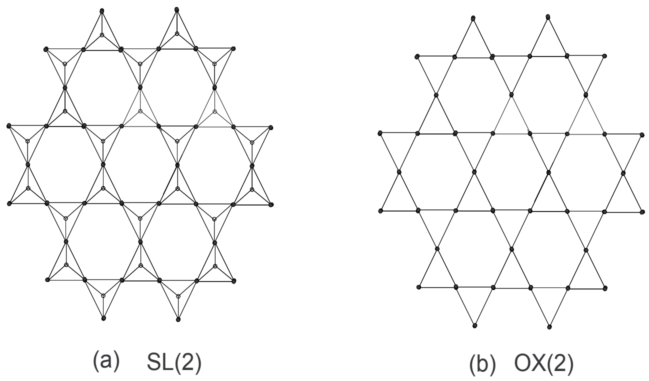

Figure 1: Graph.

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is a closed ball of radius $\frac{D}{2}$ with center as a common vertex m of two triangles with vertex set $\{x \in V(X) : d_x(x,m) \le \frac{D}{2}\}$ then

$$|V(O_D)| = \begin{cases} 9k^2 + 5k, & for\ D = 4k. \\ 9k^2 + 13k + 5, & for\ D = 4k + 2 \end{cases} \quad (1)$$

where k∈N.

Proof

For even D ≥ 4, O_D is the closed ball of radius $\frac{D}{2}$ with center as a common vertex of two triangles and Δ=4. Now we draw horizontal lines on the vertices of O_D and count the vertices on each horizontal line one by one then adding them for top to bottom we have.

For D=4k, k∈ℕ, then

$|V(O_D)|=2k+(k+1)+\ldots+(4k-3)+(2k-1)+(4k-1)+2k+(4k+1)+2k+(4k-1)+(2k-1)+\ldots+(2k+3)+(k+1)+2k$

$= 9k^2+5k$

For D=4k+2, k∈ℕ, then

$|V(O_D)|=(k+1)+(2k+3)+\ldots+(4k-1)+2k+(4k+1)+(2k+1)+(4k+3)+(2k+1)+(4k+1)+2k+(4k-1)+\ldots+(2k+3)+(k+1)$

$= 9k^2+13k+5$

In the Figures 2a and 2b the graphs O_D for D=4 and D=6 are depicted where the central vertex n of O_D is depicted by ⊗.

Proposition 1.2

For odd D>4, Let the induced sub graph of X denoted by

O_D is a closed ball of radius $\frac{D-1}{2}$ with center as a triangle with vertex set $\{x \in V(X) : d_x(x,m) \le \frac{D-1}{2}\}$, where n is the vertex of the central triangle which is closest to x then

$$|V(O_D)| = \begin{cases} 9k^2 + 9k + 3 & for\ D = 4k + 1. \\ 9k^2 + 18k + 9, & for\ D = 4k + 3. \end{cases} \quad (2)$$

where k∈N

Proof

For odd D>4, O_D is the closed ball of radius $\frac{D-1}{2}$ with center as a triangle and Δ=4. Now we draw horizontal lines on the vertices of O_D and count the vertices on each horizontal line one by one then adding them for top to bottom we have.

For D=4k+1, k∈ℕ, then

$|V(O_D)|=(2k+2)+(k+1)+\ldots+(4k-2)+(2k-1)+4k+2k+(4k+2)+(2k+1)+4k+2k+(4k-2)+\ldots+(k+1)+(2k+2)$

$= 9k^2+9k+3$

For D=4k+3, k∈ℕ, then

$|V(O_D)|=(k+2)+(2k+4)+\ldots+4k+(2k+1)+(4k+2)+(2k+2)+(4k+4)+(2k+1)+(4k+2)+2k+4k+\ldots+(k+1)+(2k+2)$

$= 9k^2+18k+9$

In the Figures 3a and 3b the graphs O_D for D=5 and D=7 are depicted. Where the vertices of central triangle of O_D are denoted by a,b,c and is depicted by ⊗.

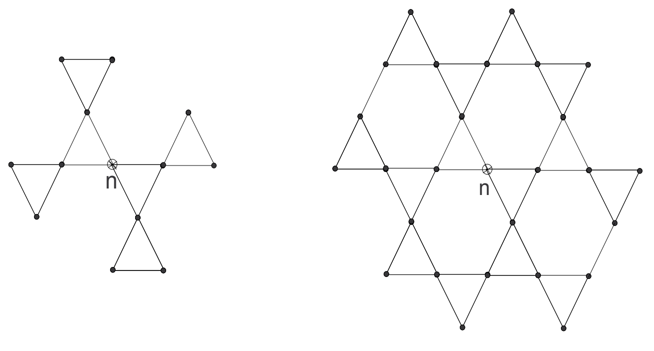
Hence we proved the following statement.

Theorem 1.3

Let X be the infinite oxide network. Then for the largest connected sub graph of X of maximum degree Δ=4 and diameter D in terms of vertices we have the following two cases.

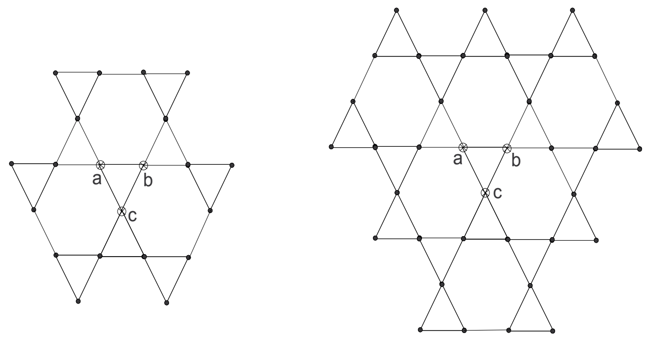
Case (i)

For D=4k,k∈ℕ, the largest connected subgraph of X with Δ=4 denoted by H_D is obtained by connecting the two oxygen vertices of the closed ball O_D with only one vertex of X. Such subgraph H_D for D=4 is shown in Figure 4.



(a) (b)

Figure 2: O_D for D=4 and D=6.



(a) (b)

Figure 3: O_D for D=5 and D=7.

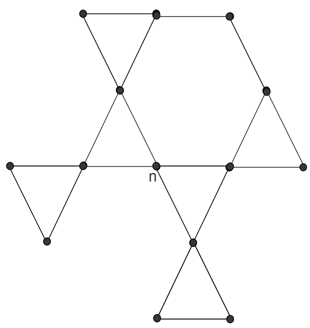


Figure 4: H_D for D=4.

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$N_X(4,D)=|V(H_D)|=|V(O_D)|+1$ for $D=4k, k\in\mathbb{N}$.

Case (ii)

For $D=4k+r, k\in\mathbb{N}, r\in\{1,2,3\}$, the largest subgraph of X with $\Delta=4$ is O_D itself.

$N_X(4,D)=|V(O_D)|$ for $D=4k+r, k\in\mathbb{N}, r\in\{1,2,3\}$.

The following statement is an immediate consequence of the previous theorem.

Corollary 1.4

Let X be the infinite oxide network. Let Δ,D be positive integers, $\Delta \le 4$ and $D=4k+r, k\in\mathbb{N}, r\in\{0,1,2,3\}$.Then we have.

Case (i)

$N_X(\Delta,D) \le |V(H_D)|$ for $D=4k, k\in\mathbb{N}$.

Case (ii)

$N_X(\Delta,D) \le |V(O_D)|$ for $D=4k+r, k\in\mathbb{N}, r\in\{1,2,3\}$.

Values for $\Delta \le 3$

Values for Δ=1,2

Since for $\Delta \ge \Delta(X)$ we have

$N_X(\Delta,D)=|V(H_D)|=|V(O_D)|+1$ for $D=4k, k\in\mathbb{N}$. And

$N_X(\Delta,D)=|V(O_D)|$ for $D=4k+r, k\in\mathbb{N}, r\in\{1,2,3\}$.

Now we consider the cases when maximum degree $\Delta \le 3$.

Since for $\Delta=1$ and diameter $D=4k+r, k\in\mathbb{N}, r\in\{0,1,2,3\}$, the largest sub graph of X is a complete graph K_2.

Hence $N_X(1,D)=2$.

Now we discuss the case when $\Delta=2$.

Theorem 2.1

Let X be the infinite oxide network, then for $D \ge 4$ we have $N_X(2,D)=2D+1$.

Proof

Case (i)

For $D=4k, k\in\mathbb{N}$, clearly the graph H_D contains a cycle. Thus for $D=4k, k\in\mathbb{N}$, a largest subgraph of X having maximum degree 2 is a cycle of length 2D+1.

In the Figure 5a, for D=4 and $\Delta=2$, such a cycle n→a→b→c→d→e→f→g→h→n of length 2D+1 in H_D is shown. which implies that For $D=4k, k\in\mathbb{N}$, such a sub graph of H_D exist.

Case (ii)

For $D=4k+r, k\in\mathbb{N}, r\in\{1,2,3\}$, Clearly the graph O_D contains a cycle. Thus for $D=4k+r, k\in\mathbb{N}, r\in\{1,2,3\}$, a largest subgraph of X having maximum degree 2 is a cycle of length 2D+1.

In the Figure 5b, such a cycle x→i→j→k→l→m→o→p→q→r→s→x for D=5 in O_D is shown. In the Figure 5c, such a cycle n→t→u→v→w→x→y→z→a→b→c→d→e→n for D=6 in O_D is shown. In the Figure 5d, such a cycle y→f→g→h→i→j→k→l→m→n→o→p→q→r→s→y for D=7 in O_D is shown, which implies that For $D=4k+r, k\in\mathbb{N}, r\in\{1,2,3\}$, such a subgraph of O_D exist.

Which completes the proof

Bounds for Δ=3

Now we find the bounds for $\Delta=3$.

For even $D \ge 4$, we have the following theorem.

Theorem 2.2

Let X be the infinite oxide network and $D \ge 4$ be an even positive integer.

$$|N_x(3,D)| \ge \begin{cases} 10, & \text{for } D=4. \\ 38, & \text{for } D=8. \\ |V(H_D)|-(2K+6), & \text{for} D=4k+8, k\in\mathbb{N} \end{cases} \quad (3)$$

and

$$|N_x(3,D)| \ge \begin{cases} 21, & \text{for } D=6. \\ |V(O_D)|-(K+5), & \text{for} D=4k+8, k\in\mathbb{N} \end{cases} \quad (4)$$

Proof

Case (i)

Clearly the largest degree of the graphs W_D (which are sub graphs of H_D, shown in Figures 6a and 6b is at most 3. Where the isolated vertices shown in it do not belongs to the vertex set of W_D. One can also check that the diameter of the graphs W_D is D. Hence the theorem holds for D=4,8.

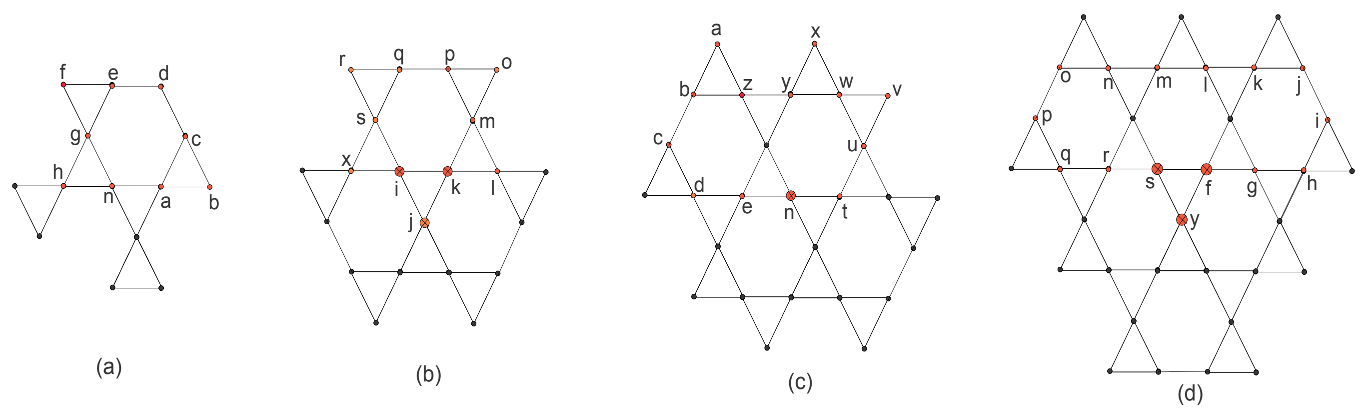

Figure 5: Graphs having cycles on 2D+1 vertices.

Now we consider the graphs W_D (which are sub graphs of H_D for $D=4k+8, k\in\mathbb{N}$). Such graphs W_D for D=12,16,20 depicted in Figures 6c, 7 and 8, clearly having degree at most 3. Where the isolated vertices shown in the graphs do not belongs to the vertex set of W_D. Also one can easily check the diameter of W_D is D.

Since $|V(H_D)|=|V(W_D)|+$ the number of isolated vertices, for $D=4k+8, k\in\mathbb{N}$.

The number of isolated vertices form a sequence (2k+6) in the graphs W_D for $D=4k+8$, $k\in\mathbb{N}$. Hence the theorem holds for $D=4k+8$, $k\in\mathbb{N}$.

Case (ii)

Clearly the maximum degree of the graph W_D (which is a sub graph of O_D, shown in Figure 9a is at most 3. Where the isolated vertices shown in it do not belong to the vertex set of W_D. One can also check that the diameter of the graphs W_D is D. Hence the theorem holds for D=6.

Now we consider the graphs W_D (which are sub graphs of O_D for $D=4k+6, k\in\mathbb{N}$) clearly having degree at most 3. Such graphs W_D for D=10,14,18 are depicted in Figures 9b, 9c and 10. Where the isolated vertices shown in the graphs do not belong to the vertex set of W_D. Note

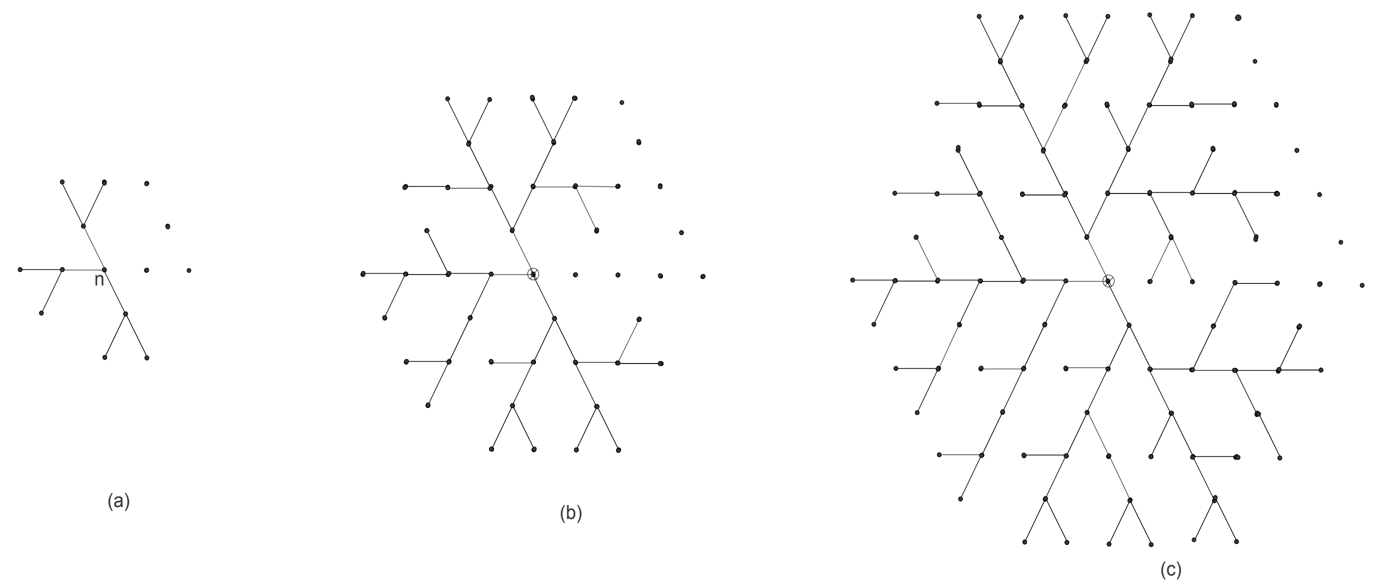

Figure 6: W_D for D=4,8,12.

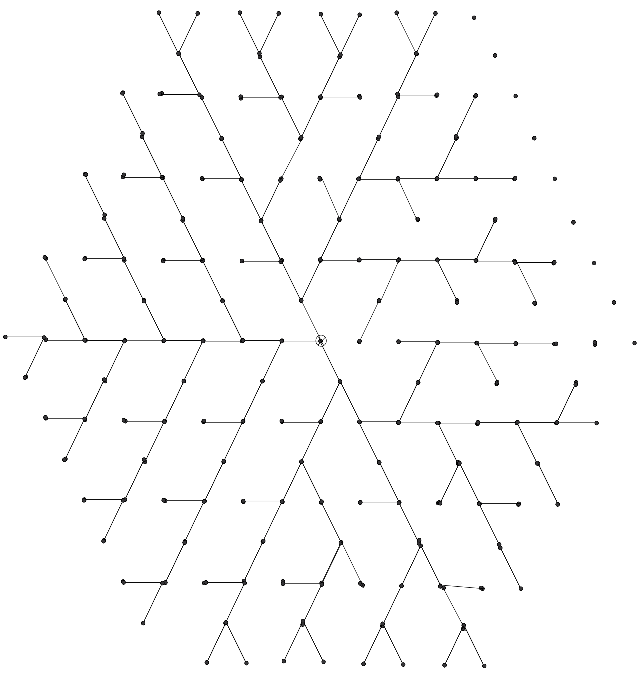

Figure 7: W_D for D=16.

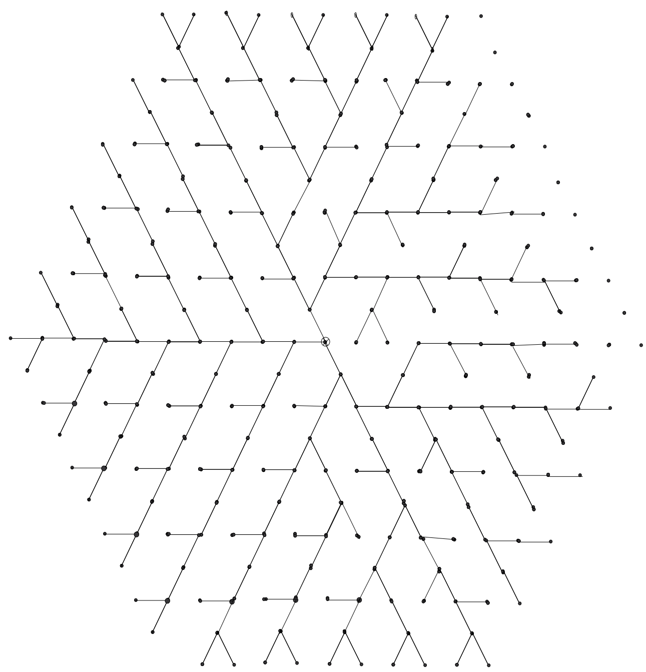

Figure 8: W_D for D=20.

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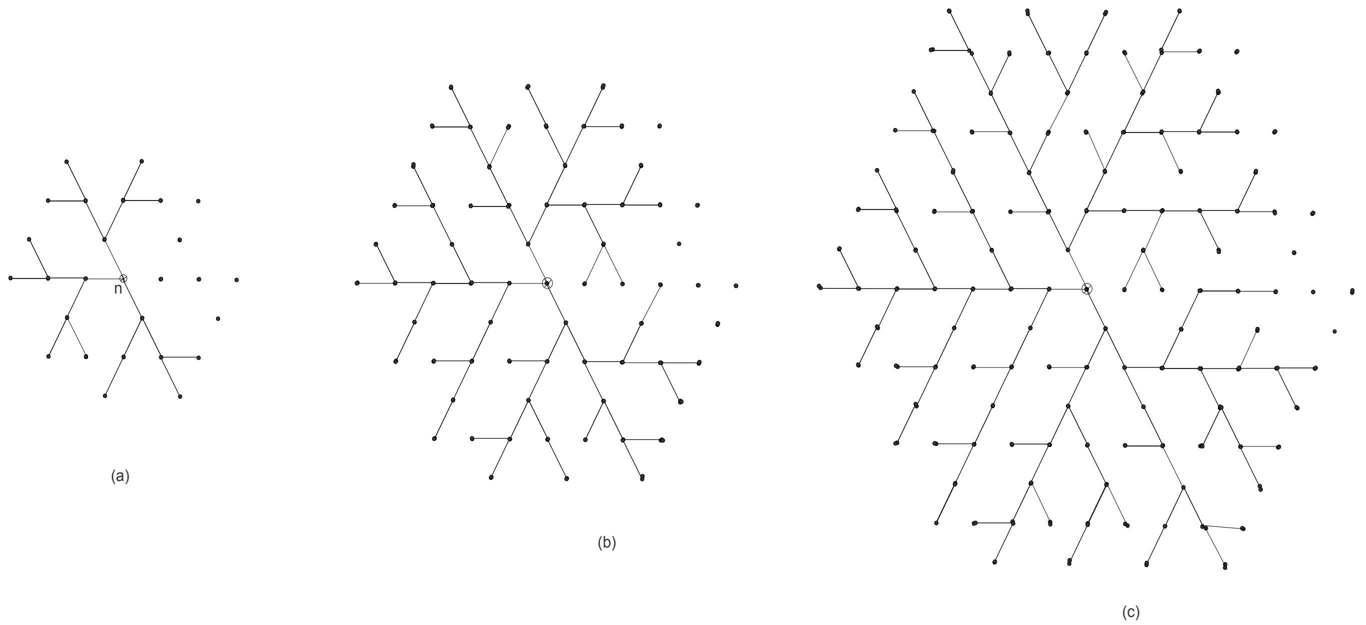

Figure 9: W_D for D=6,10,14.

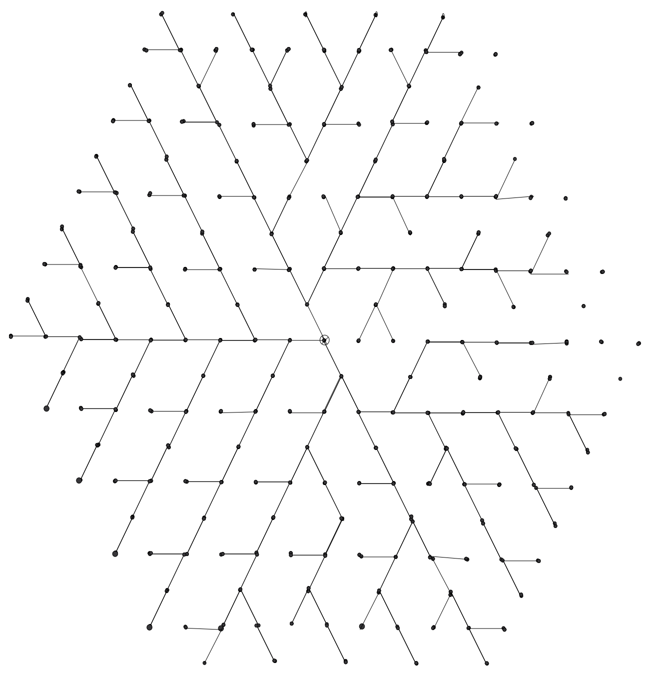

Figure 10: W_D for D=18.

that the central vertex n of W_D corresponds to the central vertex of O_D and is depicted by ⊗. Now we prove that the diameter of W_D is D. For this we have to prove that $dist_{W_D}(n,x) \le \frac{D}{2}$ for every vertex x of W_D. Also one can easily check that all the vertices of W_D are at distance at most $\frac{D}{2}$ from the central vertex n. Hence the diameter of W_D is at most D. Since $|V(O_D)| = |V(W_D)|$+ the number of isolated vertices, for D=4k +6,k∈ℕ.

The number of isolated vertices form a sequence (k+5) in the graphs W_D for D=4k+6,k∈ℕ. Hence the theorem holds for D=4k+6, k∈ℕ.

Which completes the proof.

For odd D>4, we prove the following theorem.

Theorem 2.3

Let X be the infinite oxide network and D>4 be an odd positive integer. Then

$$|N_x(3,D)| \ge \{|V(O_D)| - 9, \quad for D = 4k+8, k \in \mathbb{N} \text{ for D=4k+1,k } \in \mathbb{N}. \tag{5}$$

and

$$|N_x(3,D)| \ge \begin{cases} 24, & for\ D = 6. \\ |V(O_D)| - 9, & for D = 4k+8, k \in \mathbb{N} \end{cases} \tag{6}$$

Proof:

Case (i)

Now we consider the graphs W_D (which are sub graphs of O_D for D=4k+1,k∈ℕ). Such graphs W_D for D=5,9,13,17 are depicted in Figures 11a, 11b, 11c and 12, clearly having degree at most 3. Where the isolated vertices shown in the graphs do not belongs to the vertex set of W_D. Note that the central triangle of W_D corresponds to the central triangle of O_D and its vertices are shown by ⊗. Now we prove that the diameter of W_D is D. One can also check that all the vertices are at distance at most $\frac{D-1}{2}$ from the closest vertex of central triangle in W_D. Hence the diameter of W_D is at most D.

Since $|V(O_D)| = |V(W_D)|$+ the number of isolated vertices, for D=4k+1,k ∈ℕ.

The number of isolated vertices is 9 in the graphs W_D for D=4k+1,k∈ℕ. Hence the theorem holds for D=4k+1, k ∈ℕ.

Case (ii)

Clearly the maximum degree of the graph W_D(which is a sub graph

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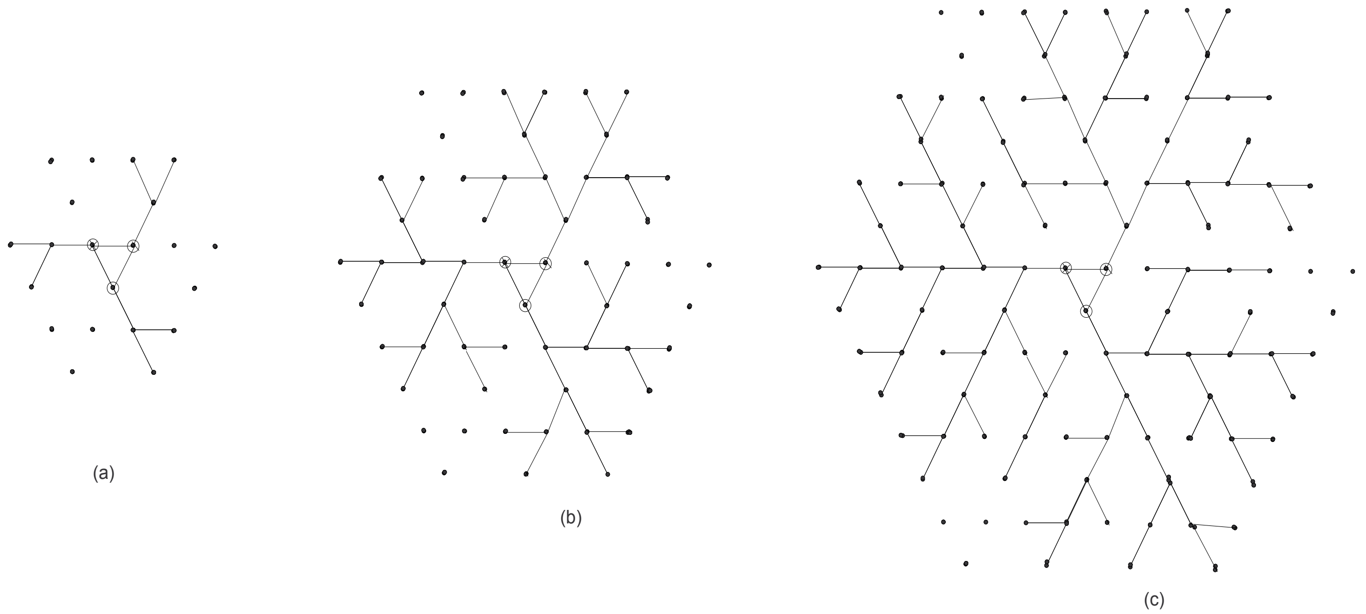

Figure 11: W_D for D=5,9,13.

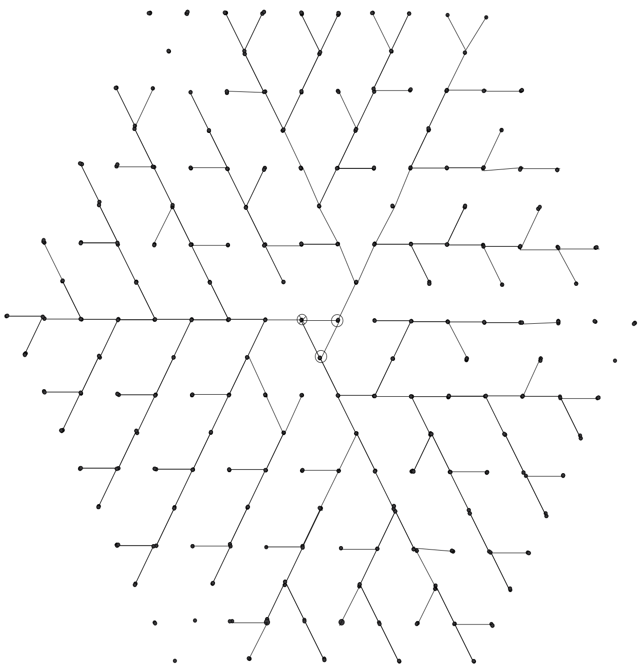

Figure 12: W_D for D=17.

of O_D, shown in Figure 13a is at most 3. Where the isolated vertices shown in it do not belongs to the vertex set of W_D. One can also check that the diameter of the graph W_D is D because all the vertices are at distance at most $\frac{D-1}{2}$ from the closest vertex of central triangle in W_D. The vertices of the central triangle of W_D are emphasized by ⊗. Hence the theorem holds for D=7.

Now we consider the graphs W_D (which are sub graphs of O_D for D=4k +7,k∈ℕ) . Such graphs W_D for D=11,15,19 depicted in Figures 13b, 13c and 14, clearly having degree at most 3. Where the isolated vertices shown in the graphs do not belongs to the vertex set of W_D. Note that the central triangle of W_D corresponds to the central triangle of O_D and its vertices are shown by ⊗. Hence it remains to show that the diameter of W_D is D. As can be seen, all the vertices are at distance at most $\frac{D-1}{2}$ from the closest vertex of central triangle in W_D. Hence the diameter of W_D is at most D.

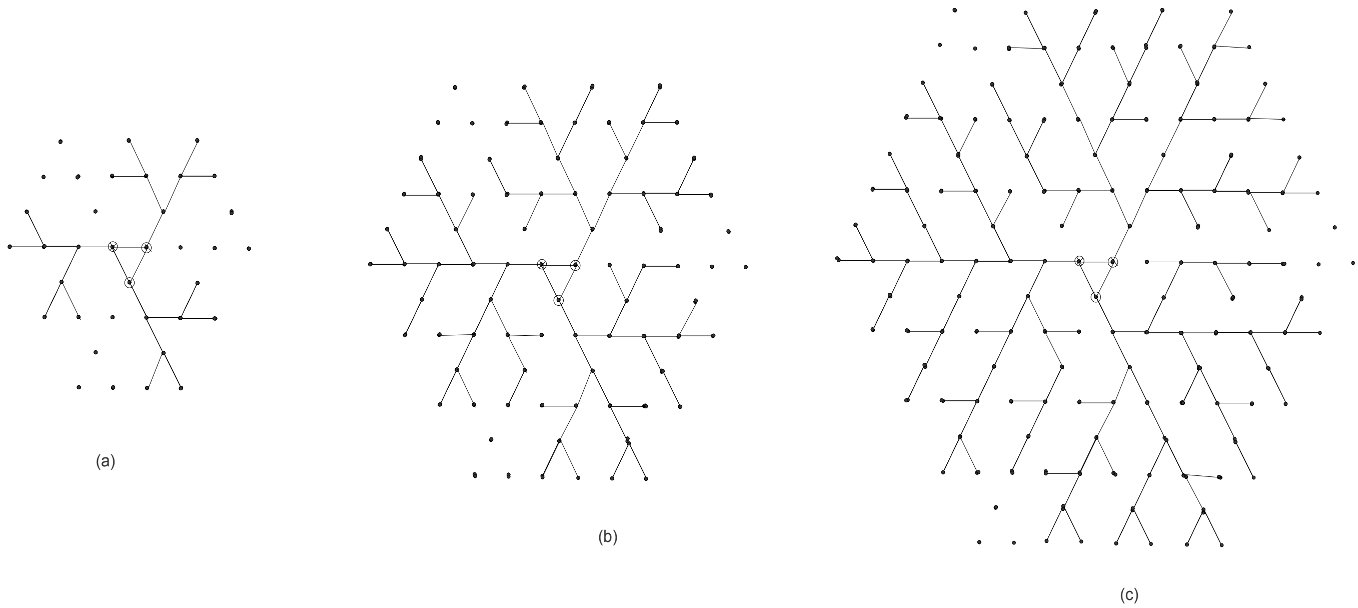

Figure 13: W_D for D=7,11,15.

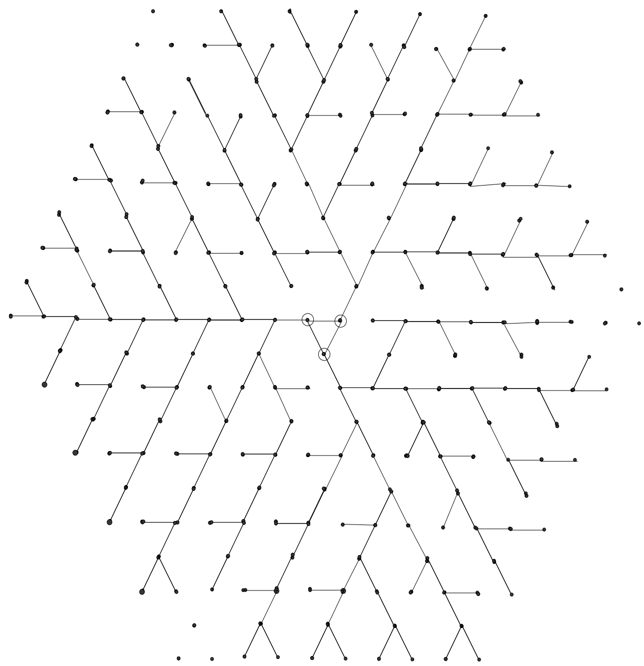

Figure 14: W_D for D=19.

Since $|V(O_D)|=|V(W_D)|$+the number of isolated vertices, for D=4k+7,k∈ℕ.

The number of isolated vertices is 9 in the graphs W_D for D=4k+7,k∈ℕ. Hence the theorem holds for D=4k+7,k∈ℕ.

Which completes the proof.

Using the fact that $N_G(\Delta-1,D) \le N_G(\Delta,D)$ for any graph G, we get the following statement [6].

Corollary 2.4

Let X be the infinite oxide network. Then

$|V(H_D)|-(2k+6) \le N_X(3,D) \le |V(H_D)|$ for D=4k+8,k ∈ℕ.

$|V(O_D)|-(k+5) \le N_X(3,D) \le |V(O_D)|$ for D=4k+6,k∈ℕ.

$|V(O_D)|-9 \le N_X(3,D) \le |V(O_D)|$ for D=4k+1,k∈ℕ.

$|V(O_D)|-9 \le N_X(3,D) \le |V(O_D)|$ for D=4k+7,k∈ℕ.

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