

# Deformations of ternary algebras

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## Abstract

The aim of this paper is to extend to ternary algebras the classical theory of formal deformations of algebras introduced by Gerstenhaber. The associativity of ternary algebras is available in two forms, totally associative case or partially associative case. To any partially associative algebra corresponds by anti-commutation a ternary Lie algebra. In this work, we summarize the principal definitions and properties as well as classification in dimension 2 of these algebras. Then we focus ourselves on the partially associative ternary algebras, we construct the first groups of a cohomology adapted to formal deformations and then we work out a theory of formal deformation in a way similar to the binary algebras.

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## Introduction

We are concerned in this work by certain ternary algebraic structures which appear more or less naturally in various domains of theoretical and mathematical physics. Indeed, theoretical physics progress of quantum mechanics and the discovery of the mechanics of Nambu, as well as works of S. Okubo gave impulse to a significant development involving ternary structures. The quark model proposed by Y. Nambu in 1973 [26, 35] represents a particular case of ternary system of algebras and was known since under the name of “Nambu Mechanics”. The cubic matrices and a generalization of the determinant, called the “hyperdeterminant”, also illustrates the ternary algebras. It was first introduced by Cayley in 1840, then found again and generalized by Kapranov, Gelfand and Zelevinskii in 1990 [26]. Other ternary and cubic algebras have been studied by Lawrence, Dabrowski, Nesti and Siniscalco, Plyushchay, Rausch de Traubenberg, and other authors. The ternary operation gives rise to partially associative, totally associative or Lie ternary algebras, one direction of our work is devoted to classification up to isomorphism of ternary algebras in small dimensions. However, in the second part we are interested in the deformation and degeneration of ternary algebras. These two fundamental concepts are useful in order to have more information about the ternary algebra or to construct new algebras starting from a given algebra. The study of deformations of the ternary algebras leads to the cohomology study of these algebras.

The main purpose of our work is to build and study a cohomology for the partially associative ternary algebras, totally associative ternary algebras and the ternary Lie algebras. In this paper we focus on partially associative ternary algebras and express the first cohomology groups adapted to formal deformation.

## 1 Definitions

Let  $V$  be a vector space over  $\mathbb{k}$ , an algebraically closed field of characteristic zero. A ternary operation on  $V$  is a linear map  $m : V \otimes V \otimes V \longrightarrow V$ . Assume that  $V$  is  $n$ -dimensional with  $n$  finite and let  $B = \{e_1, \dots, e_n\}$  be a basis of  $V$ , the ternary operation  $m$  is completely determined by its structure constants  $C_{ijk}^s$  defined by

$$m(e_i \otimes e_j \otimes e_k) = \sum_{s=1}^n C_{ijk}^s e_s \quad (1.1)$$

### 1.1 Totally and partially associative ternary algebras.

**Definition 1.1.** A totally associative ternary algebra is given by a  $\mathbb{k}$ -vector space  $V$  and a ternary operation  $m$  satisfying

$$m(m(x \otimes y \otimes z) \otimes u \otimes v) = m(x \otimes m(y \otimes z \otimes u) \otimes v) = m(x \otimes y \otimes m(z \otimes u \otimes v)) \quad (1.2)$$

for all  $x, y, z, u, v \in V$ .

**Definition 1.2.** A partially associative ternary algebra is given by a  $\mathbb{k}$ -vector space  $V$  and a ternary operation  $m$  satisfying

$$m(m(x \otimes y \otimes z) \otimes u \otimes v) + m(x \otimes m(y \otimes z \otimes u) \otimes v) + m(x \otimes y \otimes m(z \otimes u \otimes v)) = 0 \quad (1.3)$$

for all  $x, y, z, u, v \in V$ .

If there is no ambiguity on the ternary operation and in order to simplify the writing, one will denote  $m(x \otimes y \otimes z) = (xyz)$ .

**Remark 1.1.** Let  $(V, \cdot)$  be a bilinear associative algebra. Then, the ternary operation, defined by

$$m(x, y, z) = (x \cdot y \cdot z) \quad (1.4)$$

determines on the vector space  $V$  a structure of totally associative ternary algebra which is not partially associative.

### 1.2 Free associative ternary algebras.

In this paragraph, we give the construction of the free totally associative ternary algebra and the free partially associative ternary algebra on a finite-dimensional vector space  $V$ . This construction is a particular case of the  $k$ -ary algebras studied by Gnedbaye [18]. The free totally associative ternary algebra is given by the following

**Proposition 1.1.** *Let  $V$  be a finite-dimensional vector space. We set*

$$T^{<2>}(V) := \bigoplus_{n \geq 0} V^{\otimes 2n+1} \quad (1.5)$$

The space  $T^{<2>}(V)$  provided with the ternary operation induced by triple concatenation given by

$$(w_0 w_1 w_2) = x_0^0 \otimes \dots \otimes x_{2n_0}^0 \otimes x_0^1 \otimes \dots \otimes x_{2n_1}^1 \otimes x_0^2 \otimes \dots \otimes x_{2n_2}^2 \quad (1.6)$$

where  $w_i = x_0^i \otimes \dots \otimes x_{2n_i}^i \in V^{\otimes 2n_i+1}$  ( $i = 0, 1, 2$ ) defines the free totally associative ternary algebra on the vector space  $V$ . This ternary algebra is denoted by  $tAss^{<2>}(V)$ .

The free partially associative ternary algebra results from the previous proposition. One denotes the space of the symmetric group  $S_{2n+1}$  on  $V^{\otimes 2n+1}$  by  $(V^{\otimes 2n+1})_{S_{2n+1}}$ . According to the previous proposition, one may consider the symmetric and free totally associative ternary algebra which we denote by  $stAss^{<2>}(V)$  on a vector space  $V$ , described by the space

$$T^{<2>}(V)_{sym} := \bigoplus_{n \geq 0} (V^{\otimes 2n+1})_{S_{2n+1}} \quad (1.7)$$

provided with a ternary operation induced by triple concatenation.

**Proposition 1.2.** *The construction of the free partially associative ternary algebra is represented by a sequence of vector spaces defined by the relation*

$$pAss_0^{\langle 2 \rangle} = V, \quad pAss_n^{\langle 2 \rangle} = \bigoplus_{\substack{0 \leq n_1, n_2 \leq n-1 \\ n_1+n_2 = n-1}} V \otimes pAss_{n_1}^{\langle 2 \rangle} \otimes pAss_{n_2}^{\langle 2 \rangle}, \quad \text{for } n \geq 1 \quad (1.8)$$

We denote the partially associative free ternary algebra on  $V$  by  $pAss^{\langle 2 \rangle}(V)$

### 1.3 Symmetric ternary algebras and ternary Lie algebras

In what follows the ternary operation is denoted by  $[x, y, z]$  for all  $x, y, z$  in  $V$ .

**Definition 1.3.** A ternary algebra on a vector space  $V$  is said symmetric if

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = [x_1, x_2, x_3], \quad \forall \sigma \in S_3 \quad \text{and} \quad \forall x_1, x_2, x_3 \in V \quad (1.9)$$

**Definition 1.4.** A ternary operation is said commutative if

$$\sum_{\sigma \in S_3} Sgn(\sigma)[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = 0, \quad \forall \sigma \in S_3 \quad \text{and} \quad \forall x_1, x_2, x_3 \in V \quad (1.10)$$

**Remark 1.2.** A symmetric ternary operation is commutative.

**Definition 1.5.** An antisymmetric ternary algebra is characterized by the relation

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = Sgn(\sigma)[x_1, x_2, x_3], \quad \forall \sigma \in S_3 \quad \text{and} \quad \forall x_1, x_2, x_3 \in V \quad (1.11)$$

**Definition 1.6.** A ternary Lie algebra is an antisymmetric ternary operation satisfying the generalized Jacobi condition:

$$J(x_1, x_2, x_3, x_4, x_5) = \sum_{\sigma \in S_5} Sgn(\sigma)[[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}], x_{\sigma(4)}, x_{\sigma(5)}] = 0 \quad (1.12)$$

The concept of Lie algebra was generalized to Lie  $n$ -ary algebras by V. Fillipov in 1985 [13] (in Russian) and in addition by Ph. Hanlon and M. Wachs in 1990 [22], see also [33].

**Remark 1.3.** The generalized Jacobi condition is still written

$$\begin{aligned} \sum_{\sigma \in S_5} Sgn(\sigma)[[x_{\sigma(x_1)}, x_{\sigma(x_2)}, x_{\sigma(x_3)}], x_{\sigma(x_4)}, x_{\sigma(x_5)}] = \\ = [[x_{i_1}, x_{i_2}, x_{i_3}], x_{i_4}, x_{i_5}] - [[x_{i_1}, x_{i_2}, x_{i_4}], x_{i_3}, x_{i_5}] \\ + [[x_{i_1}, x_{i_2}, x_{i_5}], x_{i_3}, x_{i_4}] + [[x_{i_1}, x_{i_3}, x_{i_4}], x_{i_2}, x_{i_5}] \\ - [[x_{i_1}, x_{i_3}, x_{i_5}], x_{i_2}, x_{i_4}] + [[x_{i_1}, x_{i_4}, x_{i_5}], x_{i_2}, x_{i_3}] \\ - [[x_{i_2}, x_{i_3}, x_{i_4}], x_{i_1}, x_{i_5}] + [[x_{i_2}, x_{i_3}, x_{i_5}], x_{i_1}, x_{i_4}] \\ - [[x_{i_2}, x_{i_4}, x_{i_5}], x_{i_1}, x_{i_3}] + [[x_{i_3}, x_{i_4}, x_{i_5}], x_{i_1}, x_{i_2}] \end{aligned} \quad (1.13)$$

**Remark 1.4.** One can easily check that for a ternary Lie algebra

$$[x, x, y] = [x, y, y] = [x, y, x] = 0$$

As in the binary case, there is a functor which makes correspond to any partially associative ternary algebra a ternary Lie algebra.

**Proposition 1.3.** *To any partially associative ternary algebra on a vector space  $V$  with ternary operation  $m$ , one associates a ternary Lie algebra on  $V$  defined by the bracket*

$$[x_1, x_2, x_3] = \sum_{\sigma \in S_3} Sgn(\sigma)m(x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}) \quad (1.14)$$

**Proof.** It is clear that the bracket is antisymmetric and direct calculation shows that the generalized Jacobi condition is satisfied.  $\square$

## 2 Algebraic varieties of ternary algebras

Let  $V$  be  $n$ -dimensional  $\mathbb{k}$ -vector space with  $n$  finite and  $\{e_1, \dots, e_n\}$  be a basis of  $V$ . Let  $m$  be a ternary operation on  $V$ . The multilinearity of  $m$  implies that for all  $e_i, e_j, e_k$  one has

$$m(e_i \otimes e_j \otimes e_k) = \sum_{s=1}^n C_{ijk}^s e_s \quad (2.1)$$

Consequently, the ternary operation is completely determined by the set of structure constants:

$$\{X_{ij}\}_{i=1, \dots, n^4} = \{C_{ijk}^s, i, j, k, s = 1, \dots, n\} \subset \mathbb{k}^{n^4} \quad (2.2)$$

### 2.1 Algebraic varieties of totally associative ternary algebras

The set of  $n$ -dimensional totally associative ternary algebras is determined by the following system of algebraic polynomial equations

$$\sum_{i_7=1}^n C_{i_1 i_2 i_3}^{i_7} C_{i_7 i_4 i_5}^{i_6} = \sum_{i_7=1}^n C_{i_2 i_3 i_4}^{i_7} C_{i_1 i_7 i_5}^{i_6} = \sum_{i_7=1}^n C_{i_3 i_4 i_5}^{i_7} C_{i_1 i_2 i_7}^{i_6} \quad (2.3)$$

where  $i_1, \dots, i_6 = 1, \dots, n$ . This set forms a quadratic algebraic variety embedded in  $\mathbb{k}^{n^4}$  and is denoted by  $tAss_n$ .

### 2.2 Algebraic varieties of partially associative ternary algebras

The set of  $n$ -dimensional partially associative ternary algebras is denoted by  $pAss_n$ , it forms an algebraic variety embedded in  $\mathbb{k}^{n^4}$  and is determined by the following polynomial system:

$$\sum_{i_7=1}^n C_{i_1 i_2 i_3}^{i_7} C_{i_7 i_4 i_5}^{i_6} + \sum_{i_7=1}^n C_{i_2 i_3 i_4}^{i_7} C_{i_1 i_7 i_5}^{i_6} + \sum_{i_7=1}^n C_{i_3 i_4 i_5}^{i_7} C_{i_1 i_2 i_7}^{i_6} = 0 \quad (2.4)$$

where  $i_1, \dots, i_6 = 1, \dots, n$ .

### 2.3 Algebraic varieties of ternary Lie algebras

The set of  $n$ -dimensional Lie ternary algebras denoted by  $\mathcal{L}_n^3$  is an algebraic variety embedded in  $\mathbb{k}^{n^2(n-1)(n-2)}$ . The variety of ternary Lie algebras is determined by the following polynomial system :

$$\begin{aligned} & \sum_{i_7=1}^n C_{i_1 i_2 i_3}^{i_7} C_{i_7 i_4 i_5}^{i_6} + \sum_{i_7=1}^n C_{i_1 i_2 i_4}^{i_7} C_{i_7 i_3 i_5}^{i_6} + \sum_{i_7=1}^n C_{i_1 i_2 i_5}^{i_7} C_{i_7 i_3 i_4}^{i_6} + \sum_{i_7=1}^n C_{i_1 i_3 i_4}^{i_7} C_{i_7 i_2 i_5}^{i_6} \\ & - \sum_{i_7=1}^n C_{i_1 i_3 i_5}^{i_7} C_{i_7 i_2 i_4}^{i_6} + \sum_{i_7=1}^n C_{i_1 i_4 i_5}^{i_7} C_{i_7 i_2 i_3}^{i_6} - \sum_{i_7=1}^n C_{i_2 i_3 i_4}^{i_7} C_{i_7 i_1 i_5}^{i_6} \\ & + \sum_{i_7=1}^n C_{i_2 i_3 i_5}^{i_7} C_{i_7 i_1 i_4}^{i_6} - \sum_{i_7=1}^n C_{i_2 i_4 i_5}^{i_7} C_{i_7 i_1 i_3}^{i_6} + \sum_{i_7=1}^n C_{i_3 i_4 i_5}^{i_7} C_{i_7 i_1 i_2}^{i_6} = 0 \end{aligned}$$

where  $i_1, \dots, i_6$  take values 1 to  $n$  and

$$C_{\sigma(i_1)\sigma(i_2)\sigma(i_3)}^{i_4} = Sgn(\sigma)C_{i_1 i_2 i_3}^{i_4}, \quad \forall \sigma \in S_3, \quad i_1, \dots, i_4 = 1, \dots, n$$

## 2.4 Action of $GL_n(\mathbb{k})$ on varieties of ternary algebras

The action of the group  $GL_n(\mathbb{k})$  on an algebraic variety of ternary algebras  $\Upsilon$ , where  $\Upsilon$  indicates  $tAss_n$ ,  $pAss_n$  or  $\mathcal{L}_n^3$ , is defined as follows:

$$GL_n(\mathbb{k}) \times \Upsilon \longrightarrow \Upsilon, \quad (f, m) \mapsto m' = f.m$$

Thus

$$m'(x_1 \otimes x_2 \otimes x_3) = f^{-1}m(f(x_1) \otimes f(x_2) \otimes f(x_3)) \quad (2.5)$$

Let  $m \in \Upsilon$ , the orbit of  $m$  is denoted by  $\theta(m)$  and defined by

$$\theta(m) = \{f.m / f \in GL_n(\mathbb{k})\} \quad (2.6)$$

In other words,

$$m' \in \theta(m) \iff \exists f \in GL_n(\mathbb{k}) : m' = f.m, \quad \theta(m) \subset \Upsilon \quad (2.7)$$

The orbits are in correspondence with the isomorphism classes of  $n$ -dimensional ternary algebras. The stabilizer subgroup of  $m$ ,

$$Stab(m) = \{f \in GL_n(\mathbb{k}) / m = f.m\}$$

is  $Aut(m)$ , the automorphisms group of  $m$ .

The orbit  $\theta(m)$  is identified with the homogeneous space  $\frac{GL_n(\mathbb{k})}{Aut(m)}$ . Thus,

$$\dim \theta(m) = n^2 - \dim Aut(m)$$

The orbit  $\theta(m)$  is provided, when  $\mathbb{k} = \mathbb{C}$  (a complex field), with the structure of a differentiable manifold. In fact,  $\theta(m)$  is image through the action of the Lie group  $GL_n(\mathbb{k})$  of the point  $m$ , considered as a point of  $Hom(V \otimes V \otimes V, V) \times Hom(V, V \otimes V \otimes V)$ .

The Zariski tangent space to  $\Upsilon$  at the point  $m$  corresponds to  $Z^2(m, m)$  and the tangent space to the orbit corresponds to  $B^2(m, m)$ , the cohomology group  $Z^2(m, m)$  and  $B^2(m, m)$  are described in section (4.2).

An algebras whose orbit is open for the topology of Zariski is called rigid and constitute an interesting class for the geometrical study of algebraic varieties [21]. Indeed, the Zariski closure of an open orbit constitutes an irreducible component of the algebraic variety.

The last section of this paper is devoted to classifications up to isomorphism of 2-dimensional partially associative ternary algebras, totally associative ternary algebras and ternary Lie algebras.

## 3 Operads of ternary algebras

An operad is an algebraic tool which provides a modeling of operations with  $n$  variables on a certain type of algebras, such that (Lie algebras, commutative algebras, associative algebras, partially associative ternary algebras and totally associative ternary algebras etc).

The operads of the binary algebras were studied by many authors [14, 28] and those of the ternary algebras by Gnedbaye [19].

An operad (resp. unital operad) on a sequence of  $\mathbb{k}$ -vector spaces denoted  $Vect_{\mathbb{k}}$  is an associative algebra (resp. unital associative algebra) in the monoidal category  $(S - mod, \otimes)$ . More explicitly, a  $\mathbb{k}$ -linear operad is a collection of  $n$ -dimensional vector spaces  $\mathcal{P}(n)$  provided with

actions of the symmetric groups  $S_n$  and a distinguished element  $I$  for  $\mathcal{P}(1)$  (the unit) and compositions

$$\circ_i : \mathcal{P}(k) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(k+n-1), \quad \mu \otimes \nu \longmapsto \mu \circ_i \nu$$

such that for  $\mu \in \mathcal{P}(k)$ ,  $\nu \in \mathcal{P}(n)$ , and for all  $k, n \in \mathbb{N}$  and  $i = 1, \dots, k$  one has

$$\mu \circ_i \nu(x_1, \dots, x_{k+n+1}) = \mu(x_1, \dots, x_{i-1}, \nu(x_i, \dots, x_{i+n-1}), x_{i+n}, \dots, x_{k+n-1})$$

satisfying the following axioms.

- Equivariance: compatibility of the symmetric group action with compositions. By setting

$$\mu(\sigma.(x_1, \dots, x_n)) = \mu^\sigma(x_1, \dots, x_n)$$

the permutations  $\pi \in S_k$  and  $\rho \in S_n$  define  $\sigma = \pi \circ_i \rho \in S_{k+n-1}$ . The equivariance is can be expressed by

$$\mu^\pi \circ_{\pi(i)} \nu^\rho = (\mu \circ_i \nu)^\sigma$$

- Associativity of the compositions: for all  $\lambda \in \mathcal{P}(l)$ ,  $\mu \in \mathcal{P}(k)$ ,  $\nu \in \mathcal{P}(n)$ , we have

$$(\lambda \circ_i \mu) \circ_{j+k-1} \nu = (\lambda \circ_j \nu) \circ_i \mu \quad \text{for } 1 \leq i < j \leq l \quad (3.1)$$

$$(\lambda \circ_i \mu) \circ_{i-1+j} \nu = \lambda \circ_i (\mu \circ_j \nu) \quad \text{for } 1 \leq i \leq l, 1 \leq j \leq k \quad (3.2)$$

- The unit  $I$  is defined by

$$I \circ_1 \mu = \mu = \mu \circ_i I \quad (3.3)$$

for all  $\mu \in \mathcal{P}(k)$  and  $i = 1, \dots, k$ . In this case the operad is unital. For the examples see [9], [28].

The ternary operations lead to the following operads:

- $pAss^{(3)}$ : operad of partially associative ternary algebras,
- $tAss^{(3)}$ : operad of totally associative ternary algebras,
- $Lie^{(3)}$ : operad of ternary Lie algebras,
- $tAssSym^{(3)}$ : operad of totally associative symmetric ternary algebras.

The concept of Koszul duality for the associative algebras is an algebraic theory developed in the seventies by S. Priddy. Later in 1994, V. Ginzburg et M. M. Kapranov in their article [17] generalized this concept for the algebraic operads.

Let  $\mathcal{P}$  be an operad, one denotes by  $\mathcal{P}^!$  the dual operad within the meaning of Koszul duality. For the ternary algebras there are the following results [19].

**Theorem 3.1.** *One has*

$$pAss^{(3)!} = tAss^{(3)}$$

$$tAss^{(3)!} = pAss^{(3)}$$

$$(Lie^{(3)})^! = tAssSym^{(3)}$$

## 4 Formal deformations of ternary algebras

In this section we extend to ternary algebras the formal deformation theory introduced in 1964 by Gerstenhaber [15] for associative algebras, and in 1967 by Nijenhuis and Richardson for Lie algebras [36]. The formal deformations of mathematical objects is one of the oldest technics used by mathematicians. The deformations give more information about the structure of the object, for example one can try to see which properties are stable under deformation.

In this theory the scalar's field is extended to the power series ring. A more general approach was developed by Fialowski and her collaborators, following Schlessinger where a commutative algebra is taken instead of power series ring in one variable. The fundamental results of Gerstenhaber's theory connect deformation theory with the suitable cohomology groups. There is no general cohomology theory. Every structure has a cohomology, a Hochschild cohomology for associative algebras, Chevalley-Eilenberg cohomology for Lie algebras etc. In the following we define the concept of deformation for any ternary operation and define the suitable 1 and 2 cohomology groups for partially associative ternary algebras adapted to formal deformation.

Let  $V$  be a vector space over a field  $\mathbb{k}$  and  $m_0$  be a ternary operation on  $V$ . Let  $\mathbb{k}[[t]]$  be the power series ring in one variable  $t$  and coefficients in  $\mathbb{k}$  and  $V[[t]]$  be the extension of  $V$  by extending the coefficients domain from  $\mathbb{k}$  to  $\mathbb{k}[[t]]$ . Then  $V[[t]]$  is a  $\mathbb{k}[[t]]$ -module and when  $V$  is finite-dimensional and we have

$$V[[t]] = V \otimes_{\mathbb{k}} \mathbb{k}[[t]] \quad (4.1)$$

One notes that  $V$  is a submodule of  $V[[t]]$ . The extension of ternary operations to  $V[[t]]$ ,

$$V[[t]] \otimes V[[t]] \otimes V[[t]] \longrightarrow V[[t]]$$

may be considered by using the  $\mathbb{k}[[t]]$ -linearity as a map

$$V \otimes V \otimes V \longrightarrow V[[t]]$$

**Definition 4.1.** A formal deformation of ternary operation  $m_0$  on  $V$  is given by a ternary operation  $m_t$  defined by

$$m_t : V \otimes V \otimes V \longrightarrow V[[t]] \quad (4.2)$$

$$x \otimes y \otimes z \longmapsto m_t(x \otimes y \otimes z) = \sum_{i \geq 0} m_i(x \otimes y \otimes z) t^i \quad (4.3)$$

where  $m_i \in \text{Hom}(V \otimes V \otimes V, V)$ .

### 4.1 Deformations of partially associative ternary algebras

The deformation of a partially associative ternary algebra is determined by the deformation of the ternary operation.

**Definition 4.2.** Let  $V$  be a  $\mathbb{k}$ -vector space and  $\tau_0 = (V, m_0)$  be a partially associative ternary algebra. A deformation of  $\tau_0$  on  $V$  is given by a linear map

$$m_t : V \otimes V \otimes V \longrightarrow V[[t]]$$

defined by

$$m_t(x \otimes y \otimes z) = \sum_{i \geq 0} m_i(x \otimes y \otimes z) t^i, \quad \text{where } m_i \in \text{Hom}(V^{\otimes 3}, V)$$

satisfying the following condition

$$m_t(m_t(x \otimes y \otimes z) \otimes v \otimes w) + m_t(x \otimes m_t(y \otimes z \otimes v) \otimes w) + m_t(x \otimes y \otimes m_t(z \otimes v \otimes w)) = 0 \quad (4.4)$$

We call the condition (4.4) the deformation equation of partially associative ternary algebra  $\tau_0$ .

### 4.1.1 Deformation equation

In the following, we study the equation (4.4) and thus characterize the deformations of partially associative ternary algebras. The equation may be written

$$\begin{aligned} \sum_{i \geq 0} t^i m_i \left( \sum_{j \geq 0} m_j (x \otimes y \otimes z) t^j \otimes v \otimes w \right) + \sum_{i \geq 0} t^i m_i (x \otimes \sum_{j \geq 0} m_j (y \otimes z \otimes v) t^j \otimes w) \\ + \sum_{i \geq 0} t^i m_i (x \otimes y \otimes \sum_{j \geq 0} m_j (z \otimes v \otimes w) t^j) = 0 \end{aligned} \quad (4.5)$$

or

$$\sum_{i \geq 0} \sum_{j \geq 0} (m_i (m_j (x \otimes y \otimes z) \otimes v \otimes w) + m_i (x \otimes m_j (y \otimes z \otimes v) \otimes w) + m_i (x \otimes y \otimes m_j (z \otimes v \otimes w))) t^{i+j} = 0 \quad (4.6)$$

**Definition 4.3.** We call ternary partial associator the map

$$Hom(V^{\otimes 3}, V) \times Hom(V^{\otimes 3}, V) \longrightarrow Hom(V^{\otimes 5}, V), \quad (m_i, m_j) \longmapsto m_i \circ m_j \quad (4.7)$$

defined for all  $x, y, z \in V$  by

$$\begin{aligned} m_i \circ m_j (x \otimes y \otimes z \otimes v \otimes w) = m_i (m_j (x \otimes y \otimes z) \otimes v \otimes w) + m_i (x \otimes m_j (y \otimes z \otimes v) \otimes w) \\ + m_i (x \otimes y \otimes m_j (z \otimes v \otimes w)) \end{aligned}$$

**Remark 4.1.** This associator can be generalized in the following way. Let

$$f \in Hom(V^{\otimes(2k+1)}, V) \quad \text{and} \quad g \in Hom(V^{\otimes(2s+1)}, V)$$

Then  $f \bullet g \in Hom(V^{\otimes(2(k+s)+1)}, V)$  is defined by

$$f \bullet g(x_1, \dots, x_{2(k+s)+1}) = \sum_{i=1}^{2k+1} f(x_1, \dots, g(x_i, \dots, x_{i+2s}), x_{i+2s+1}, \dots, x_{2(k+s)+1})$$

By using the ternary partial associator, the deformation equation may be written as follows

$$\sum_{i \geq 0} \sum_{j \geq 0} (m_i \circ m_j) t^{i+j} = 0 \quad \text{or} \quad \sum_{k \geq 0} t^k \sum_{i=0}^k m_i \circ m_{k-i} = 0 \quad (4.8)$$

This equation is equivalent to the following infinite system:

$$\sum_{i=0}^k m_i \circ m_{k-i} = 0, \quad k = 0, 1, \dots$$

In particular,

- for  $k = 0$ ,  $m_0 \circ m_0 = 0$ , this corresponds to the partial associativity of  $m_0$ ,
- for  $k = 1$ ,  $m_0 \circ m_1 + m_1 \circ m_0 = 0$ ,
- for  $k = 2$ ,  $m_2 \circ m_0 + m_1 \circ m_1 + m_0 \circ m_2 = 0$ .



### 4.1.2 Equivalent and trivial deformations

In this paragraph, we characterize the equivalent and trivial deformations of a partially associative ternary algebras.

**Definition 4.4.** Given two deformations of a partially associative ternary algebra  $m_t = \sum_{i \geq 0} m_i t^i$  and  $m'_t = \sum_{i \geq 0} m'_i t^i$  of  $m_0$  ( $m_0 = m'_0$ ), we say that they are equivalent if there is a formal isomorphism  $\Phi_t : V \rightarrow V[[t]]$  which is a  $\mathbb{k}[[t]]$ -linear map that may be written in the form

$$\Phi_t = \sum_{i \geq 0} \Phi_i t^i = Id + \Phi_1 t + \Phi_2 t^2 + \dots \quad \text{where } \Phi_i \in \text{End}_{\mathbb{k}}(V) \quad \text{and} \quad \Phi_0 = Id$$

such that

$$\Phi_t \circ m_t = m'_t \circ \Phi_t \tag{4.9}$$

A deformation  $m_t$  of  $m_0$  is said to be trivial if and only if  $m_t$  is equivalent to  $m_0$ .

The condition (4.9) may be written

$$\Phi_t(m_t(x \otimes y \otimes z)) = m'_t(\Phi_t(x) \otimes \Phi_t(y) \otimes \Phi_t(z)), \quad \forall x, y, z \in V \tag{4.10}$$

which is equivalent to

$$\sum_{i \geq 0} \Phi_i \left( \sum_{j \geq 0} m_j(x \otimes y \otimes z) t^j \right) t^i = \sum_{i \geq 0} m'_i \left( \sum_{j \geq 0} \Phi_j(x) t^j \otimes \sum_{k \geq 0} \Phi_k(y) t^k \otimes \sum_{l \geq 0} \Phi_l(z) t^l \right) t^i \tag{4.11}$$

or

$$\sum_{i, j \geq 0} \Phi_i(m_j(x \otimes y \otimes z)) t^{i+j} = \sum_{i, j, k, l \geq 0} m'_i(\Phi_j(x) \otimes \Phi_k(y) \otimes \Phi_l(z)) t^{i+j+k+l}$$

By identification of coefficients, one obtains that the constant coefficients are identical

$$m_0 = m'_0 \quad \text{because} \quad \Phi_0 = Id$$

and for coefficients of  $t$  one has

$$\begin{aligned} \Phi_0(m_1(x \otimes y \otimes z)) + \Phi_1(m_0(x \otimes y \otimes z)) &= m'_1(\Phi_0(x) \otimes \Phi_0(y) \otimes \Phi_0(z)) \\ &\quad + m'_0(\Phi_1(x) \otimes \Phi_0(y) \otimes \Phi_0(z)) \\ &\quad + m'_0(\Phi_0(x) \otimes \Phi_1(y) \otimes \Phi_0(z)) \\ &\quad + m'_0(\Phi_0(x) \otimes \Phi_0(y) \otimes \Phi_1(z)) \end{aligned}$$

from which it follows

$$\begin{aligned} m_1(x \otimes y \otimes z) + \Phi_1(m_0(x \otimes y \otimes z)) &= m'_1(x \otimes y \otimes z) + m_0(\Phi_1(x) \otimes y \otimes z) \\ &\quad + m_0(x \otimes \Phi_1(y) \otimes z) + m_0(x \otimes y \otimes \Phi_1(z)) \end{aligned}$$

Consequently,

$$\begin{aligned} m'_1(x \otimes y \otimes z) &= m_1(x \otimes y \otimes z) + \Phi_1(m_0(x \otimes y \otimes z)) - m_0(\Phi_1(x) \otimes y \otimes z) \\ &\quad - m_0(x \otimes \Phi_1(y) \otimes z) - m_0(x \otimes y \otimes \Phi_1(z)) \end{aligned}$$

## 4.2 Cohomological approach of a partially associative ternary algebras

The study of the deformation equation leads us to certain elements of the cohomology of partially associative ternary algebras. The existence of this cohomology is ensured by the operadic structure of the ternary algebras.

Let  $\tau = (V, m_0)$  be a partially associative ternary algebra on a  $\mathbb{k}$ -vector space  $V$ .

**Definition 4.5.** We call ternary  $p$ -cochain a linear map  $\varphi : V^{\otimes 2p+1} \longrightarrow V$ . The set of  $p$ -cochains on  $V$  is

$$\mathbb{C}^p(\tau, \tau) = \left\{ \varphi : V^{\otimes 2p+1} = \underbrace{V \otimes V \otimes \dots \otimes V}_{2p+1 \text{ times}} \longrightarrow V \right\}$$

The 1-coboundary and 2-coboundary operators for partially associative ternary algebras are defined as follows

**Definition 4.6.** We call ternary 1-coboundary of partially associative ternary algebra  $\tau$  the map

$$\delta^1 : \mathbb{C}^0(\tau, \tau) \longrightarrow \mathbb{C}^1(\tau, \tau), \quad f \longmapsto \delta^1 f$$

defined by

$$\delta^1 f(x \otimes y \otimes z) = f(m_0(x \otimes y \otimes z)) - m_0(f(x) \otimes y \otimes z) - m_0(x \otimes f(y) \otimes z) - m_0(x \otimes y \otimes f(z))$$

**Definition 4.7.** We call ternary 2-coboundary operator of partially associative ternary algebra  $\tau$  the map

$$\delta^2 : \mathbb{C}^1(\tau, \tau) \longrightarrow \mathbb{C}^2(\tau, \tau), \quad \varphi \longmapsto \delta^2 \varphi$$

defined by

$$\begin{aligned} \delta^2 \varphi(x \otimes y \otimes z \otimes v \otimes w) &= m_0[\varphi(x \otimes y \otimes z) \otimes v \otimes w] + m_0[x \otimes \varphi(y \otimes z \otimes v) \otimes w] \\ &\quad + m_0[x \otimes y \otimes \varphi(z \otimes v \otimes w)] + \varphi[m_0(x \otimes y \otimes z) \otimes v \otimes w] \\ &\quad + \varphi[x \otimes m_0(y \otimes z \otimes v) \otimes w] + \varphi[x \otimes y \otimes m_0(z \otimes v \otimes w)] \end{aligned}$$

**Remark 4.2.** The operator  $\delta^2$  can also be defined by

$$\delta^2 \varphi = \varphi \bullet m_0 + m_0 \bullet \varphi$$

where  $\bullet$  is the operation defined in the paragraph (4.1.1). Note that  $\delta^2 \circ \delta^1 = 0$ .

The cohomology spaces relative to these coboundary operators are

**Definition 4.8.** The space of 1-cocycles of  $\tau$  is

$$Z^1(\tau, \tau) = \{f : V \longrightarrow V \mid \delta^1 f = 0\}$$

The space of 2-coboundaries of  $\tau$  is

$$B^2(\tau, \tau) = \{\varphi : V^{\otimes 3} \longrightarrow V \mid \varphi = \delta^1 f, f \in \mathbb{C}^0(\tau, \tau)\}$$

The space of 2-cocycles of  $\tau$  is

$$Z^2(\tau, \tau) = \{f : V^{\otimes 3} \longrightarrow V \mid \delta^2 f = 0\}$$

**Remark 4.3.** One has  $B^2(\tau, \tau) \subset Z^2(\tau, \tau)$ , because  $\delta^2 \circ \delta^1 = 0$ . Note also that  $Z^1(\tau, \tau)$  gives the space of derivations of a ternary algebra  $\tau$ , denoted  $Der(\tau)$ .

**Definition 4.9.** We call the  $p^{th}$  cohomology group of the partially associative ternary algebra  $\tau$  the quotient

$$H^p(\tau, \tau) = \frac{Z^p(\tau, \tau)}{B^p(\tau, \tau)}, \quad p = 1, 2$$

We characterize now deformations in terms of cohomology. Let  $m_t$  be a deformation of a partially associative ternary algebra  $\tau_0 = (V, m_0)$ .

By using the definition of 2-coboundaries and by gathering the first and the last term, the deformation equation (4.4) may be written

$$\delta^2 m_k = \sum_{i=1}^{k-1} m_i \circ m_{k-i} = 0, \quad k = 1, 2, \dots$$

**Lemma 4.1.** *The first term  $m_1$  in the deformation  $m_t$  is a 2-cocycle of the cohomology of the partially associative ternary algebra  $\tau_0$ .*

**Proof.** Take  $k = 1$  in the deformation equation (4.4). □

**Definition 4.10.** Let  $\tau_0 = (V, m_0)$  be a partially associative ternary algebra and  $m_1$  be an element of  $Z^2(\tau_0, \tau_0)$ . The cocycle  $m_1$  is said *integrable* if there exists a deformation  $m_t = \sum_{i \geq 0} m_i t^i$  of  $\tau_0$ .

**Proposition 4.1.** *Let  $m_t$  be a deformation of a partially associative ternary algebra. The integrability of  $m_1$  depends only on its cohomology class.*

**Proof.** We saw in the previous section that if two deformations  $m$  and  $m'$  are equivalent then  $m'_1 = m_1 + \delta\Phi_1$ . Recall that two elements are cohomologous if their difference is a coboundary. Thus,

$$\begin{aligned} \delta m_1 = 0 &\implies \delta m'_1 = \delta(m_1 + \delta\Phi_1) = \delta m_1 + \delta(\delta\Phi_1) = 0 \\ m_1 = \delta g &\implies m'_1 = \delta g - \delta\Phi_1 = \delta(g - \Phi_1) \end{aligned}$$

which end the proof. □

**Proposition 4.2.** *There is, over  $\mathbb{k}[[t]]/t^2$ , a one to one correspondence between the elements of  $H^2(\tau_0, \tau_0)$  and the infinitesimal deformation*

$$m_t(x \otimes y \otimes z) = m_0(x \otimes y \otimes z) + m_1(x \otimes y \otimes z)t, \quad \forall x, y, z \in V \quad (4.12)$$

**Proof.** The proof follows from a direct calculation. □

Assume now that

$$m_t(x \otimes y \otimes z) = m_0(x \otimes y \otimes z) + m_1(x \otimes y \otimes z)t + m_2(x \otimes y \otimes z)t^2 + \dots, \quad \forall x, y, z \in V$$

such that  $m_1 = \dots = m_{n-1} = 0$ .

The deformation equation implies  $\delta m_n = 0$  which means  $m_n \in Z^2(\tau_0, \tau_0)$ . If further  $m_n \in B^2(\tau_0, \tau_0)$ , i.e.  $m_n = \delta g$  with  $g \in Hom(V, V)$ , then using a formal morphism  $\Phi_t = Id + tg$  we obtain that the deformation  $m_t$  is equivalent to the deformation given for all  $x, y, z \in V$  by

$$m'_t(x \otimes y \otimes z) = \Phi_t^{-1} \circ m_t \circ (\Phi_t(x) \otimes \Phi_t(y) \otimes \Phi_t(z)) = m_0(x \otimes y \otimes z) + m'_{n+1}(x \otimes y \otimes z)t^{m+1} + \dots$$

and again  $m_{n+1} \in Z^2(\tau_0, \tau_0)$ .

Thus, we have the following theorem.

**Theorem 4.1.** *Let  $\tau = (V, m_0)$  be a partially associative ternary algebra and  $m_t$  be a one parameter family of deformations of  $m_0$ . Then  $m_t$  is equivalent to*

$$m_t(x \otimes y \otimes z) = m_0(x \otimes y \otimes z) + m'_p(x \otimes y \otimes z)t^p + m'_{p+1}(x \otimes y \otimes z)t^{p+1} + \dots \quad (4.13)$$

where  $m'_p \in Z^2(\tau_0, \tau_0)$  and  $m'_p \notin B^2(\tau_0, \tau_0)$ .

The previous theorem leads to the following fundamental corollary.

**Corollary 4.1.** *If  $H^2(\tau_0, \tau_0) = 0$ , then all deformations of  $\tau_0$  are equivalent to a trivial deformation.*

**Remark 4.4.** A partially associative ternary algebra such that any deformation is equivalent to trivial deformation is called rigid. The previous result gives a sufficient condition for the rigidity. Recall that rigid ternary algebras has a great interest in the study of algebraic varieties of ternary algebras. The Zariski closure of the orbit of a rigid ternary algebra gives an irreducible component.

## 5 Degeneration of ternary algebras

The concept of degeneration (also called contraction) of algebras appeared first in the physics literature (Segal 1951, Inönü and Wigner 1953) [23] to describe the classical mechanics as a degeneration of quantum mechanics. Later in 1961, Saletan [37], generalized this concept and gave a necessary and sufficient condition for the existence of Lie algebra contractions. The concepts of degenerations and deformations give more information about the structure of the object and helps, in general, to construct new algebras. They are also used to study algebraic varieties of algebras (associative, Lie, ...) see (Gabriel, Gerstenhaber, Mazzola, Richardson, Makhlouf, Nijenhuis, Goze, Inönü, Saletan, ...) and in the theory of quantum groups by Celegheni, Giachetti, Sorace et Tarlini [7, 8].

The aim of this section is to introduce the concept of degeneration of ternary algebras.

Let  $\tau = (V, m)$  be a ternary algebra and  $Isom(V)$  be the set of invertible maps of  $End(V)$ . Recall that the action of  $Isom(V)$  on the ternary algebras denoted  $f.\tau$  is defined by

$$(f.m)(x \otimes y \otimes z) = f^{-1}(m(f(x) \otimes f(y) \otimes f(z)))$$

**Definition 5.1.** Let  $\tau_1$  and  $\tau_0$  be two ternary algebras on a  $\mathbb{k}$ -vector space  $V$  and  $f_t$  be a one parameter continuous family of endomorphism of  $V$  such that

$$f_t = f_0 + tf_1 + t^2f_2 + \dots$$

One supposes that  $f_t$  is invertible for  $t \neq 0$ . If the limit  $f_t.\tau_1$  exists when  $t \rightarrow 0$  and is equal to  $\tau_0$ , where  $\tau_0$  belongs to the Zariski closure  $\overline{\theta(\tau_1)}$  of the orbit of  $\tau_1$ , we say that  $\tau_0$  is a degeneration of  $\tau_1$ .

The following proposition gives the connection between degeneration and deformation.

**Proposition 5.1.** *Let  $\tau_0$  be a formal degeneration of  $\tau_1$ , then  $\tau_1$  is a formal deformation of  $\tau_0$ .*

**Proof.** In fact, if  $\tau_0 = \lim_{t \rightarrow 0} f_t.\tau_1$  is the degeneration of  $\tau_1$ , then  $\tau_t = f_t.\tau_1$  is a deformation of  $\tau_0$ .  $\square$

**Remark 5.1.** The converse is in general false.

## 6 Classifications

In this section, we establish the classification up to isomorphism of 2-dimensional partially associative ternary algebras and totally associative ternary algebras. Ternary Lie algebras with dimension lower than 5 correspond to antisymmetric ternary operations.

**Proposition 6.1.** *Any 2-dimensional partially associative ternary algebra is trivial or isomorphic to the ternary algebras defined by the following non trivial product*

$$m(e_1 \otimes e_1 \otimes e_1) = e_2 \quad (6.1)$$

**Proposition 6.2.** *Any nontrivial 2-dimensional totally associative ternary algebra is isomorphic to one of the following totally associative ternary algebras:*

$m_1^1$	$m_1^1(e_1 \otimes e_1 \otimes e_1) = e_1$ $m_1^1(e_1 \otimes e_2 \otimes e_2) = e_1 + e_2$ $m_1^1(e_2 \otimes e_2 \otimes e_1) = e_1 + e_2$	$m_1^1(e_1 \otimes e_1 \otimes e_2) = e_2$ $m_1^1(e_2 \otimes e_1 \otimes e_1) = e_2$ $m_1^1(e_2 \otimes e_2 \otimes e_2) = e_1 + 2e_2$	$m_1^1(e_1 \otimes e_2 \otimes e_1) = e_2$ $m_1^1(e_2 \otimes e_1 \otimes e_2) = e_1 + e_2$
$m_2^1$	$m_2^1(e_1 \otimes e_1 \otimes e_1) = e_1$ $m_2^1(e_1 \otimes e_2 \otimes e_2) = 0$ $m_2^1(e_2 \otimes e_2 \otimes e_1) = 0$	$m_2^1(e_1 \otimes e_1 \otimes e_2) = e_2$ $m_2^1(e_2 \otimes e_1 \otimes e_1) = e_2$ $m_2^1(e_2 \otimes e_2 \otimes e_2) = 0$	$m_2^1(e_1 \otimes e_2 \otimes e_1) = e_2$ $m_2^1(e_2 \otimes e_1 \otimes e_2) = 0$
$m_3^1$	$m_3^1(e_1 \otimes e_1 \otimes e_1) = 0$ $m_3^1(e_1 \otimes e_2 \otimes e_2) = 0$ $m_3^1(e_2 \otimes e_2 \otimes e_1) = 0$	$m_3^1(e_1 \otimes e_1 \otimes e_2) = 0$ $m_3^1(e_2 \otimes e_1 \otimes e_1) = 0$ $m_3^1(e_2 \otimes e_2 \otimes e_2) = e_1$	$m_3^1(e_1 \otimes e_2 \otimes e_1) = 0$ $m_3^1(e_2 \otimes e_1 \otimes e_2) = 0$
$m_4^1$	$m_4^1(e_1 \otimes e_1 \otimes e_1) = 2e_1 + e_2$ $m_4^1(e_1 \otimes e_2 \otimes e_2) = e_1$ $m_4^1(e_2 \otimes e_2 \otimes e_1) = e_1$	$m_4^1(e_1 \otimes e_1 \otimes e_2) = e_1 + e_2$ $m_4^1(e_2 \otimes e_1 \otimes e_1) = e_1 + e_2$ $m_4^1(e_2 \otimes e_2 \otimes e_2) = e_2$	$m_4^1(e_1 \otimes e_2 \otimes e_1) = e_1 + e_2$ $m_4^1(e_2 \otimes e_1 \otimes e_2) = e_1$
$m_5^1$	$m_5^1(e_1 \otimes e_1 \otimes e_1) = e_1$ $m_5^1(e_1 \otimes e_2 \otimes e_2) = 0$ $m_5^1(e_2 \otimes e_2 \otimes e_1) = 0$	$m_5^1(e_1 \otimes e_1 \otimes e_2) = 0$ $m_5^1(e_2 \otimes e_1 \otimes e_1) = 0$ $m_5^1(e_2 \otimes e_2 \otimes e_2) = e_2$	$m_5^1(e_1 \otimes e_2 \otimes e_1) = 0$ $m_5^1(e_2 \otimes e_1 \otimes e_2) = 0$
$m_6^1$	$m_6^1(e_1 \otimes e_1 \otimes e_1) = 0$ $m_6^1(e_1 \otimes e_2 \otimes e_2) = 0$ $m_6^1(e_2 \otimes e_2 \otimes e_1) = 0$	$m_6^1(e_1 \otimes e_1 \otimes e_2) = 0$ $m_6^1(e_2 \otimes e_1 \otimes e_1) = 0$ $m_6^1(e_2 \otimes e_2 \otimes e_2) = e_2$	$m_6^1(e_1 \otimes e_2 \otimes e_1) = 0$ $m_6^1(e_2 \otimes e_1 \otimes e_2) = 0$
$m_7^1$	$m_7^1(e_1 \otimes e_1 \otimes e_1) = e_1$ $m_7^1(e_1 \otimes e_2 \otimes e_2) = e_1$ $m_7^1(e_2 \otimes e_2 \otimes e_1) = e_1$	$m_7^1(e_1 \otimes e_1 \otimes e_2) = e_2$ $m_7^1(e_2 \otimes e_1 \otimes e_1) = e_2$ $m_7^1(e_2 \otimes e_2 \otimes e_2) = e_2$	$m_7^1(e_1 \otimes e_2 \otimes e_1) = e_1 - e_2$ $m_7^1(e_2 \otimes e_1 \otimes e_2) = -e_1 + e_2$
$m_8^1$	$m_8^1(e_1 \otimes e_1 \otimes e_1) = 0$ $m_8^1(e_1 \otimes e_2 \otimes e_2) = 0$ $m_8^1(e_2 \otimes e_2 \otimes e_1) = 0$	$m_8^1(e_1 \otimes e_1 \otimes e_2) = 0$ $m_8^1(e_2 \otimes e_1 \otimes e_1) = 0$ $m_8^1(e_2 \otimes e_2 \otimes e_2) = 0$	$m_8^1(e_1 \otimes e_2 \otimes e_1) = e_1$ $m_8^1(e_2 \otimes e_1 \otimes e_2) = e_2$
$m_9^1$	$m_9^1(e_1 \otimes e_1 \otimes e_1) = 0$ $m_9^1(e_1 \otimes e_2 \otimes e_2) = e_1$ $m_9^1(e_2 \otimes e_2 \otimes e_1) = e_1$	$m_9^1(e_1 \otimes e_1 \otimes e_2) = 0$ $m_9^1(e_2 \otimes e_1 \otimes e_1) = 0$ $m_9^1(e_2 \otimes e_2 \otimes e_2) = e_2$	$m_9^1(e_1 \otimes e_2 \otimes e_1) = 0$ $m_9^1(e_2 \otimes e_1 \otimes e_2) = -e_1$
$m_{10}^1$	$m_{10}^1(e_1 \otimes e_1 \otimes e_1) = e_1$ $m_{10}^1(e_1 \otimes e_2 \otimes e_2) = 0$ $m_{10}^1(e_2 \otimes e_2 \otimes e_1) = 0$	$m_{10}^1(e_1 \otimes e_1 \otimes e_2) = e_2$ $m_{10}^1(e_2 \otimes e_1 \otimes e_1) = 0$ $m_{10}^1(e_2 \otimes e_2 \otimes e_2) = 0$	$m_{10}^1(e_1 \otimes e_2 \otimes e_1) = 0$ $m_{10}^1(e_2 \otimes e_1 \otimes e_2) = 0$

$m_{11}^1$	$m_{11}^1(e_1 \otimes e_1 \otimes e_1) = e_1$ $m_{11}^1(e_1 \otimes e_2 \otimes e_2) = e_1$ $m_{11}^1(e_2 \otimes e_2 \otimes e_1) = e_1$	$m_{11}^1(e_1 \otimes e_1 \otimes e_2) = e_2$ $m_{11}^1(e_2 \otimes e_1 \otimes e_1) = e_2$ $m_{11}^1(e_2 \otimes e_2 \otimes e_2) = e_2$	$m_{11}^1(e_1 \otimes e_2 \otimes e_1) = -e_2$ $m_{11}^1(e_2 \otimes e_1 \otimes e_2) = -e_1$
$m_{12}^1$	$m_{12}^1(e_1 \otimes e_1 \otimes e_1) = e_1$ $m_{12}^1(e_1 \otimes e_2 \otimes e_2) = 0$ $m_{12}^1(e_2 \otimes e_2 \otimes e_1) = 0$	$m_{12}^1(e_1 \otimes e_1 \otimes e_2) = 0$ $m_{12}^1(e_2 \otimes e_1 \otimes e_1) = e_2$ $m_{12}^1(e_2 \otimes e_2 \otimes e_2) = 0$	$m_{12}^1(e_1 \otimes e_2 \otimes e_1) = 0$ $m_{12}^1(e_2 \otimes e_1 \otimes e_2) = 0$
$m_{13}^1$	$m_{13}^1(e_1 \otimes e_1 \otimes e_1) = e_2$ $m_{13}^1(e_1 \otimes e_2 \otimes e_2) = e_2$ $m_{13}^1(e_2 \otimes e_2 \otimes e_1) = e_2$	$m_{13}^1(e_1 \otimes e_1 \otimes e_2) = e_1$ $m_{13}^1(e_2 \otimes e_1 \otimes e_1) = e_1$ $m_{13}^1(e_2 \otimes e_2 \otimes e_2) = e_1$	$m_{13}^1(e_1 \otimes e_2 \otimes e_1) = e_1$ $m_{13}^1(e_2 \otimes e_1 \otimes e_2) = e_2$
$m_{14}^1$	$m_{14}^1(e_1 \otimes e_1 \otimes e_1) = -e_1 + e_2$ $m_{14}^1(e_1 \otimes e_2 \otimes e_2) = e_2$ $m_{14}^1(e_2 \otimes e_2 \otimes e_1) = e_2$	$m_{14}^1(e_1 \otimes e_1 \otimes e_2) = e_1$ $m_{14}^1(e_2 \otimes e_1 \otimes e_1) = e_1$ $m_{14}^1(e_2 \otimes e_2 \otimes e_2) = e_1 + e_2$	$m_{14}^1(e_1 \otimes e_2 \otimes e_1) = e_1$ $m_{14}^1(e_2 \otimes e_1 \otimes e_2) = e_2$

In the case of ternary Lie algebras, we have

**Proposition 6.3.** *Any  $n$ -dimensional antisymmetric ternary operation with  $n \leq 4$  is a ternary Lie algebra.*

These results of classification are obtained either by a direct reasoning or using a formal computation software [29].

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