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Convergence Control Parameter Region for an Unsteady Three Dimensional Navier-Stokes Equations of Flow between Two Parallel Disks by using Homotopy Analysis Method

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Abstract

Purpose: The paper aims to find the convergence control parameter region for an unsteady three dimensional Navier-Stokes equations of flow between two parallel disks by using Homotopy Analysis Method.

Findings: The region and value of the convergence control parameter has been found.

Keywords: 3D Navier-Stokes equation; Homotopy analysis method; Convergence; System of nonlinear differential eqiations

Introduction

In mathematics and physics, nonlinear partial differential equations are partial differential equations with nonlinear terms. A few nonlinear differential equations have known exact solutions, but many which are important in applications do not. Sometimes these equations may be linearized by an expansion process in which nonlinear terms are discarded. When nonlinear terms make vital contributions to the solution this cannot be done, but sometimes it is enough to retain a few small ones. Then a perturbation theory may be used to obtain the solution. The differential equations may sometimes be approximated by an equation with small nonlinearities in more than one way, giving rise to different solutions valid over different range of its parameters.

Most scientific and engineering problems are modeled by ordinary differential equations or partial differential equations, Some of them are solved using the analytic methods of perturbation by Nayfeh [1]. In the numerical methods, stability and convergence should be considered so as to avoid divergence or inappropriate results. In the analytic perturbation methods, we should excert the small parameter in the equation. In numerical methods the advantage is that we have to use the small parameter a lot since most problems do not have known analytic solutions, or that if they are known it is too complex to deal with them. The main advantage in analytic method is that it is exact and gives us more context. One of the semi-exact methods which do not need small or large parameters is the Homotopy Analysis Method (HAM), first proposed by Liao in his Ph.D thesis. Liao [2] employed the basic ideas of homotopy in topology to propose a general analytic method for nonlinear problems, namely HAM, which is a powerful analytical method for solving linear and nonlinear differential equations. The HAM also avoids discretization and provides an efficient solution with high accuracy, minimal calculations and avoidance of physically unrealistic assumption. Furthermore, the HAM always provides us with a family of solution expressions with the auxiliary parameter $\hbar$, the convergence region and the rate of each solution might be determined conveniently by the auxiliary parameter $\hbar$. HAM contains the homotopy perturbation method (HPM) discussed by He [3], the Adomian decomposition method (ADM) examined by Allan [4], and the d-expansion method.

The main goal of the present study is to find the value of convergence control parameter for the problem of flow between two disks by the HAM.

Mathematical Formulation

Consider the axis-symmetric flow between two infinite disks with a distance d between them. Both disks are placed in the radial direction with a velocity proportional to the radii. The bottom disk is located in the z = 0 plane. The velocity ratio of the upper disk to the lower one is γ and ε is the amplitude of the disk. For an incompressible fluid without body forces and based on axis symmetric reads from the papers discussed by Dinarvand [5] and Munnavar [6].

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z}=0, \tag{1}$$

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}=\nu(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}), \tag{2}$$

$$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}=\nu\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{\partial^2 u_z}{\partial z^2}\right). \tag{3}$$

Where the velocity vector $\overline{V}=(u_r,u_z),\nu$ is the kinematic viscosity. By using von Karman type similarity transformations, similarity functions can be sought as follows,

$$u_r=rF(\eta),\, and \;\; u_z=dH(\eta),$$

where $\eta=\frac{z}{d}=\frac{z}{\gamma t}$ is the similarity variable. Substituting the similarity functions into the equations (1), (2) and (3). Therefore, the governing equations yields a similarity equation group

$$\begin{cases} F'' &= Re(F^2+ReF+2ReHF+ReH^2F), \\ H' &= -2F. \end{cases} \tag{4}$$

with boundary conditions

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$$F(0) = 1 + \varepsilon cost, \quad H(0) = H(1) = 0 \quad and \quad F(1) = \gamma. \tag{5}$$

Where $Re = \frac{d^2}{\nu}$ is the Reynolds number of the wall and γ is the parameter of the upper disk showing the velocity ratio of the upper disk to the bottom disk. Without loss of generality, we assumed that $0 \leq \gamma \leq 1$.

Analytical solution with HAM

Due to basic idea of HAM, as described in detail by Liao [7,8], according to the boundary conditions (5), we choose

$$F_0(\eta) = 1 - (1 + \varepsilon cost - \gamma)\eta + \varepsilon cost, \tag{6}$$

$$H_0(\eta) = 0, \tag{7}$$

as initial guesses of F(η), and H(η) which satisfy the boundary conditions (5). Besides, we select the auxiliary linear operators $L_1(F)$, and $L_2(H)$ as

$$\mathcal{L}_1(F) = F'', \tag{8}$$

$$\mathcal{L}_2(H) = H', \tag{9}$$

satisfying the follwing properties

$$\mathcal{L}_1(c_1\eta + c_2) = 0, \tag{10}$$

$$\mathcal{L}_2(c_3) = 0. \tag{11}$$

Where c_i, I=1, 2, 3 are arbitrary constants. If q ∈ [0, 1] is an embedding parameter and ħ is an auxiliary nonzero parameter, then the zeroth-order deformation equations are of the following form,

$$(1-q)\mathcal{L}_1\left[\hat{F}(\eta;q) - F_0(\eta)\right] = q\hbar\mathcal{N}_1\left[\hat{H}(\eta;q), \hat{F}(\eta;q)\right], \tag{12}$$

$$(1-q)\mathcal{L}_2\left[\hat{H}(\eta;q) - H_0(\eta)\right] = q\hbar\mathcal{N}_2\left[\hat{H}(\eta;q), \hat{F}(\eta;q)\right]. \tag{13}$$

subject to the boundary conditions

$$\hat{F}(0;q) = 1 + \varepsilon cost, \quad \hat{F}(1;q) = \gamma,$$

$$\hat{H}(0;q) = \hat{H}(1;q) = 0,$$

in which we define the nonlinear operators $\mathcal{N}_1$ and $\mathcal{N}_2$ as

$$\mathcal{N}_1\left[\hat{H}(\eta;q), \hat{F}(\eta;q)\right] = \frac{\partial^2 \hat{F}(\eta;q)}{\partial\eta^2} - Re(2Re(\hat{H}(\eta;q))(\hat{F}(\eta;q)) + Re(\hat{F}(\eta;q))$$

$$+(\hat{F}(\eta;q))^2 + Re(\hat{H}(\eta;q))^2(\hat{F}(\eta;q)),$$

$$\mathcal{N}_2\left[\hat{H}(\eta;q), \hat{F}(\eta;q)\right] = \frac{\partial \hat{H}(\eta;q)}{\partial\eta} + 2\hat{F}(\eta;q).$$

Clearly, when q = 0 the zero-order deformation equations (12) and (13) give rise to:

$$\hat{F}(\eta;0) = F_0(\eta), \ \hat{H}(\eta;0) = H_0(\eta), \ \hat{P}(\eta;0) = P_0(\eta). \tag{14}$$

when *q*=1, they become:

$$\hat{F}(\eta;1) = F(\eta), \ \hat{H}(\eta;1) = H(\eta), \ \hat{P}(\eta;1) = P(\eta). \tag{15}$$

As q increases from 0 to 1, $\hat{F}(\eta;q)$ and $\hat{H}(\eta;q)$ vary from $F_0(\eta)$ and $H_0(\eta)$ to F(η) and H(η).

Expanding $\hat{F}(\eta)$ and $\hat{H}(\eta)$ in Maclaurin series with respect to the embedding parameter q and equations (14) and (15), we obtain

$$\hat{F}(\eta;q) = F_0(\eta) + \sum_{m=1}^{+\infty} F_m(\eta)q^m, \tag{16}$$

$$\hat{H}(\eta;q) = H_0(\eta) + \sum_{m=1}^{+\infty} H_m(\eta)q^m, \tag{17}$$

where

$$F_m(\eta) = \frac{1}{m!}\frac{\partial^m \hat{F}(\eta,q)}{\partial q^m}\Big|_{q=0},$$

$$H_m(\eta) = \frac{1}{m!}\frac{\partial^m \hat{H}(\eta,q)}{\partial q^m}\Big|_{q=0}.$$

As pointed by Liao [9], the convergence of the series (16) – (17) strongly depend upon auxiliary parameter ħ. Assume that ħ is selected such that the series (16) – (17) are convergent at q = 1 then due to equations (14) and (15) we have

$$F(\eta) = F_0(\eta) + \sum_{m=1}^{+\infty} F_m(\eta), \tag{18}$$

$$H(\eta) = H_0(\eta) + \sum_{m=1}^{+\infty} H_m(\eta). \tag{19}$$

Differentiating the zero-order deformation equations (12) and (13) m times with respect to q, then setting q = 0 and finally dividing by m! we have the m^{th} - order deformation equations [10-18].

$$\mathcal{L}_1\left[F_m(\eta) - \chi_m F_{m-1}(\eta)\right] = \hbar R_{1,m}(\eta), \tag{20}$$

$$\mathcal{L}_2\left[H_m(\eta) - \chi_m H_{m-1}(\eta)\right] = \hbar R_{2,m}(\eta). \tag{21}$$

with the following boundary conditions

$$F_m(0) = F_m(1) = 0 \text{ and } H_m(0) = H_m(1) = 0, \tag{22}$$

where

$$R_{1,m}(\eta) = \frac{\partial^2 F_{m-1}(\eta)}{\partial\eta^2} - Re\sum_{i=0}^{m-1}(F_i(\eta)F_{m-1-i}(\eta) + ReF_i(\eta)$$

$$+2ReH_i(\eta)F_{m-1-i}(\eta) + \sum_{l=0}^{i} ReH_i(\eta)H_{i-1}(\eta)F_{m-1-i}(\eta),$$

$$R_{2,m}(\eta) = \frac{\partial H_{m-1}(\eta)}{\partial\eta} + 2F_{m-1}(\eta),$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases}$$

Then the solutions for equations (20) and (21) can be expressed by:

$$F_m(\eta) = \chi_m F_{m-1} + \hbar\mathcal{L}_1^{-1}[R_{1,m}] + c1\eta + c2,$$

$$H_m(\eta) = \chi_m H_{m-1} + \hbar\mathcal{L}_2^{-1}[R_{2,m}] + c3.$$

Where c1, c2, c3 are integral constants can be found by boundary conditions (22). For example, we can obtain the following result for solving the first-order deformation equation by using symbolic software *MATHEMATICA*, and successively obtain [19-25],

$$F_1(\eta) = \frac{-1}{24}\hbar(-1+n)nRe(9 - 9n + 3n^2 + 8Re - 4nRe + 4v + 4nv - 4n^2v$$

$$+4Rev + 4nRev + 2V^2 + 2nv^2 + 2n^2v^2 - 4(-3 - 2Re + n(3 + Re - v)$$

$$+n^2(-1+v) - v)Cos[t] + (3 - 3n + n^2)Cos[2t]),$$

$$H_1(\eta) = \hbar(-3 + 2n + n^2(-1+v) + v - (3 - 2n + n^2)Cos[t].$$

Convergence of HAM solution

The totally analytic series solutions of the functions F(η) and H(η) are given in equations (18) – (1819). The convergence of these series and the rate of approximation for the HAM strongly depends upon the value of the auxiliary parameter ħ, as pointed out by Liao

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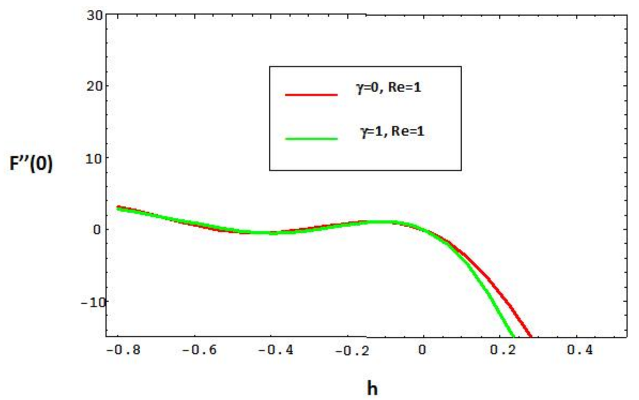

Figure 1: The h – curve of F''(0), obtained by 11th order approximation of the HAM for γ=0, γ=1, ε=1 and t=1 with Re=1.

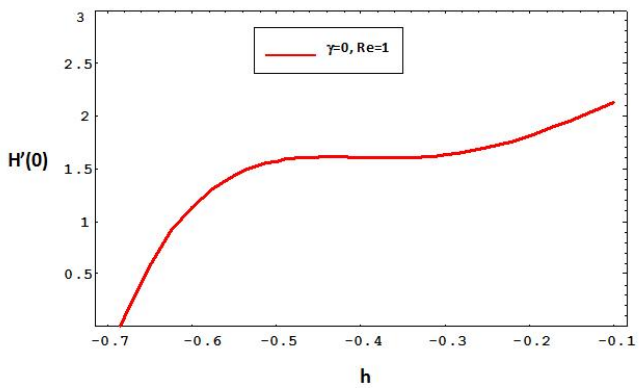

Figure 2: The h – curve of H''(0), obtained by 11th order approximation of the HAM for γ=0, γ=1, ε=1 and t=1 with Re=1.

[8]. In general, by means of the *ħ*-curve, it is straightforward to choose a proper value of ħ to control the convergence of the approximation series. To find the range of the admissible values of ħ, ħ - curves of $F''(0)$ and $H'(0)$ obtained by the 11th order approximation of the HAM for $\gamma = 0$ and $\gamma = 1$ at Re = 1, t = 1 and ε = 1 are plotted in Figures 1 and 2, respectively. From these figures, the valid regions of ħ correspond to the line segments nearly parallel to the horizontal axis. Sometimes this region is not perfectly flat to the slowly convergence rate of the series solution which was discussed by Liao in his book Homotopy Analysis Method. However a value of ħ can be picked up. Therefore in this problem we can choose ħ = -0.4 [26-28].

Conclusion

In this paper, the HAM was used for finding the convergence control parameter of the system of nonlinear ODE derived from von Karman type similarity transform for the unsteady state three dimensional Navier-Stokes equations of flow between two parallel disks. Unlike perturbation methods, the HAM does not depend on any small physical parameters. Thus homotopy analysis method is valid for both weakly and strongly nonlinear problems. Different from all other analytic methods, the homotopy analysis method provides us a simple way to adjust and control the convergence region of the series solution by means of auxiliary parameter ħ. Thus, the auxiliary parameter ħ plays a vital role within the frame of HAM which can be determined by the ħ curves.

References

1. Nayfeh AH (1979) Introduction to perturbation techniques. Wiley, New York.
2. Liao SJ (2005) Comparison between the homotopy analysis method and Homotopy Perturbation Method. Journal of Computer and Applied Mathematics 169: 1186-1194.
3. He JH (2006) Homotopy Perturbation Method for solving boundary value problems. Physics Letters A 350: 87-88.
4. Allan FM (2007) Derivation of the Adomain Decomposition Method using Homotopy Analysis Method. Applied Mathematics and Computation 190: 6-14.
5. Dinarvand S, Rashidi MM, Shahmohamadi H (2009) Analytic approximate solution of three dimentional Navier-Stokes equations of flow between two streatchable disks. Wiley online library, Wiley periodicals.
6. Munawar S, Ali A, Saleem N, Naqeep A (2014) Swirling flow over an oscillatory stretchable disk. Journal of Mechanics 30: 339-347.
7. Liao SJ (2003) Beyond Perturbation: Introduction to homotopy analysis method. Chapman and Hall, CRC Press, Boca Raton.
8. Liao SJ (2003) On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet. Journal of Fluid Mechanics 488: 189-212.
9. Liao SJ (2004) On the homotopy analysis for nonlinear problems. Applied Mathematics and Computation 147: 499-513.
10. Abbasbandy S (2006) The application of homotopy analysis method to nonlinear equations arising in heat transfer. Physics Letters A 360: 109-113.
11. Allan FM, Syam MI (2005) On the analytic solution of non-homogeneous Blasius problem. Journal of Computer and Applied Mathematics 182: 362-371.
12. Allan FM (2009) Constructions of analytic solution to chaotic dynamical systems using the homotopy analysis method. Chaos Solitons Fractals 39: 1744-1752.
13. Bouremel Y (2007) Explicit series solution for the Glauert-jet problems by means of homotopy analysis method. Communications in Nonlinear Science Numerical Simulation 12: 714-724.
14. Ganji DD, Hosseini MJ, Shayegh J (2007) Some non linear heat transfer equations solved by three approximate methods. International Journal of Heat Mass Transfer 34: 1003-1016.
15. Hayat T, Ahmad N, Sajid M, Asghar S (2007) On the MHD flow of a second grade fliud in a porous channel. Journal Of Computer and Applied Mathematics 54: 407-414.
16. Hayat T, Sajid M, Ayub M (2007) A note on series solution for generalized Couette flow. Communications in Nonlinear Science Numerical Simulation 12: 1481-1487.
17. He JH (2000) A coupling method for homotopy technique and perturbation technique for nonlinear problems. International Journal of Mechanics 35: 37-43.
18. Ibrahim MO, Egbetade SA (2013) On the homotopy analysis method for an seir tuberculosis model. American Journal of Applied Mathematics and Statistics 1: 71-75.
19. Nave O, Lehavi Y, Dshtein VG (2012) Application of the HPM and HAM to the problem of the thermal explosion in a radiation gas with polydisperse fuel spray. Journel of Applied and Computational Mathematics.
20. Rafei M, Ganji DD, Daniali H (2007) Solution of the epidemic model by homotopy perturbation method. Applied Mathematics and Computation 187: 1056-1062.
21. Rafei M, Daniali H, Ganji DD (2007) Variational iteration method for solving the epidemic model and the prey and predator problem. Journal of Computer and Applied Mathematics 186: 1701-1709.
22. Ran XJ, Zhu QY, Li Y (2009) An explicit series solution of the squeezing flow between two infinite plates by means of the homotopy analysis method. Communications in Nonlinear Science Numerical Simulation 14: 119-132.
23. Rand RH, Armbruster D (1987) Perturbation methods, bifurcation theory and computer algebraic. Applied Mathematical Sciences, Springer-Verlag, New York.

24. Rashidi MM, Domairry G, Dinarvand S (2009) Approximate solutions for the Burger and regularized long wave equations by means of the homotopy analysis method. Communications in Nonlinear Science Numerical Simulation 14: 708-717.

25. Tsai CC (2012) Homotopy method of fundamental solutions for solving certain nonlinear partial differential equations. Engineering Analysis with Boundary Elements 36: 1226-1234.

26. White FM (1991) Viscous fluid flow (2nd edn). McGraw-Hill, New York.

27. Ziabakhsh Z, Domairry G (2009) Solution of the laminar viscous flow in a semi-porous channel in presence of a uniform magnetic field by using the homotopy analysis method. Communications in Nonlinear Science Numerical Simulation 14: 1284-1294.

28. Zurigat M, Momani S, Alawneh A (2013) The multistage HAM: Application to a biochemical reaction model of fractional order 91: 1030-1040.