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Review Article

Contractions of 3-Dimensional Representations of the Lie Algebra $\mathfrak{sl}(2)$

Jan Smotlacha and Goce Chadzitaskos

Department of Physics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Brehova 7, 110 00 Prague, Czech Republic
Address correspondence to Goce Chadzitaskos, goce.chadzitaskos@fjfi.cvut.cz

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Abstract A theory of grading preserving contractions of representations of Lie algebras has been developed. In this theory, grading of the given Lie algebra is characterized by two sets of parameters satisfying a derived set of equations. Here we introduce a list of resulting 3-dimensional representations for the $\mathbb{Z}_3$-grading of the $\mathfrak{sl}(2)$ Lie algebra.

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1 Introduction

Contraction, the concept named by Inönü and Wigner [3], is suitable for unification of theories, studying relations between symmetries, etc. The first papers about this concept were published in 1950's by Segal [6]. The next papers parallel to Inönü and Wigner came from Saletan [5] and Doebner and Melsheimer [2]. In this paper, contractions of representations are studied.

The problem is the following: starting with a given Lie algebra characterized by its commutation relations, a question arises: which Lie algebras can be obtained using graded contractions? This is a natural way of obtaining Lie algebras, starting with a fixed one. Moreover, for n-dimensional representations of the initial Lie algebra, using another tool of the subject, namely the contraction of representations, we can produce new representations of Lie algebras. Using both concepts simultaneously, we get representations of Lie algebras which arise as the graded contractions.

The contribution of this paper is the investigation of the $\mathbb{Z}_3$-graded contraction of 3-dimensional representation of the complex Lie algebra $\mathfrak{sl}(2)$. In Sections 2 and 3, graded contractions are defined and basic features of $\mathfrak{sl}(2)$ are described. We study which Lie algebras can be obtained by contractions of $\mathfrak{sl}(2)$. In Section 4, we introduce the definition of graded contractions of representations of Lie algebras. It is a tool for construction of new Lie algebras from a given Lie algebra. Section 5 describes all contractions of 3-dimensional representations of $\mathbb{Z}_3$-graded $\mathfrak{sl}(2)$. For this purpose, results published in [1,4] will be applied.

2 Graded contractions of Lie algebras

Graded contractions of Lie algebras are a tool for the production of new Lie algebras from the original Lie algebra. In this section, we introduce the definition of this term.

Definition 1. Let $n \in \mathbb{N}$ and let $\mathfrak{g}$ be a Lie algebra. A grading is the decomposition [1]

$$\mathfrak{g} = \bigoplus_{i=0}^{n-1} \mathfrak{g}_i,$$

where the commutators satisfy

$$\left[\mathfrak{g}_j, \mathfrak{g}_k\right] \subseteq \mathfrak{g}_{j+k},$$

and the addition of indices is taken modulo n. Then the algebra $\mathfrak{g}$ is $\mathbb{Z}_n$-graded.

Definition 2. The graded contraction of the given Lie algebra $\mathfrak{g}$ is the Lie algebra $\mathfrak{g}^\epsilon$ which is isomorphic to the given Lie algebra $\mathfrak{g}$ as a linear space, and with commutation relations preserving the grading $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$:

$$\left[\mathfrak{g}_j, \mathfrak{g}_k\right]_\epsilon = \epsilon_{jk}\left[\mathfrak{g}_j, \mathfrak{g}_k\right] \subseteq \epsilon_{jk}\mathfrak{g}_{j+k}. \tag{2.1}$$

The contraction parameters ϵ_{jk} should satisfy the equations

$$\epsilon_{jk}\epsilon_{j+k,m} = \epsilon_{km}\epsilon_{j,k+m}. \tag{2.2}$$

We define the contraction matrix ϵ as

$$\epsilon = \begin{pmatrix} \epsilon_{00} & \epsilon_{01} & \cdots & \epsilon_{0,n-1} \\ \epsilon_{10} & \epsilon_{11} & \cdots & \epsilon_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \epsilon_{n-1,0} & \epsilon_{n-1,1} & \cdots & \epsilon_{n-1,n-1} \end{pmatrix}.$$

From (2.1), it follows that $\epsilon_{jk} = \epsilon_{kj}$ (i.e., the matrix is symmetric). Taking into account (2.2), we get in the case of $\mathbb{Z}_3$-graded contractions the equations

$$\epsilon_{00}\epsilon_{01} = \epsilon_{01}^2, \quad \epsilon_{00}\epsilon_{02} = \epsilon_{02}^2, \quad \epsilon_{00}\epsilon_{12} = \epsilon_{02}\epsilon_{12}, \quad \epsilon_{01}\epsilon_{11} = \epsilon_{02}\epsilon_{11},$$
$$\epsilon_{01}\epsilon_{12} = \epsilon_{02}\epsilon_{12}, \quad \epsilon_{01}\epsilon_{22} = \epsilon_{02}\epsilon_{22}, \quad \epsilon_{11}\epsilon_{22} = \epsilon_{01}\epsilon_{12}, \quad \epsilon_{11}\epsilon_{22} = \epsilon_{02}\epsilon_{12}.$$

Then we have (up to equivalence of matrices corresponding to the change of bases of the subspaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$) these possible forms of the contraction matrix [4]:

$$\begin{gathered} \epsilon_I = \begin{pmatrix} 1&1&1\\1&0&0\\1&0&0 \end{pmatrix}, \quad \epsilon_{II} = \begin{pmatrix} 1&1&1\\1&1&0\\1&0&0 \end{pmatrix}, \quad \epsilon_{III} = \begin{pmatrix} 0&0&0\\0&1&1\\0&1&0 \end{pmatrix}, \quad \epsilon_{IV} = \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \\ \epsilon_V = \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \quad \epsilon_{VI} = \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \quad \epsilon_{VII} = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \quad \epsilon_{VIII} = \begin{pmatrix} 1&1&0\\1&0&0\\0&0&0 \end{pmatrix}. \end{gathered} \tag{2.3}$$

3 The case of $\mathfrak{sl}(2)$

The Lie algebra $\mathfrak{sl}(2)$ is one of the main kinds of the simple Lie algebras—$\mathfrak{sl}(n+1) \equiv A_n$. The dimension is given by the number of its generators, and it is n^2+2n (i.e. the dimension of $\mathfrak{sl}(2)$ is 3). We denote its generators h, e, f. They satisfy the commutation relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h. \tag{3.1}$$

In the case of the $\mathbb{Z}_3$-grading, we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where

$$\mathfrak{g}_0 = \{h\}_{lin}, \quad \mathfrak{g}_1 = \{e\}_{lin}, \quad \mathfrak{g}_2 = \{f\}_{lin}. \tag{3.2}$$

We apply the matrices from (2.3) on (2.1) and use the notation of (3.2), then we have only 4 distinct possibilities how the commutation relations (3.1) can be changed:

(1) $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = 0$ which corresponds to the Euclidean algebra $\mathfrak{e}(2)$ coinciding with the matrices ϵ_I and ϵ_{II},
(2) $[h,e] = 0$, $[h,f] = 0$, $[e,f] = h$ corresponding to the Heisenberg algebra $\mathfrak{h}$ coinciding with the matrices ϵ_{III} and ϵ_V,
(3) $[h,e] = 0$, $[h,f] = 0$, $[e,f] = 0$ corresponding to the Abelian algebra $\mathfrak{c}$ coinciding with the matrices ϵ_{IV}, ϵ_{VI} and ϵ_{VII},
(4) $[h,e] = 2e$, $[h,f] = 0$, $[e,f] = 0$ corresponding to the algebra which we denote $\mathfrak{l}$, coinciding with the matrix ϵ_{VIII}.

4 Graded contractions of representations of $\mathfrak{sl}(2)$

Starting with the definition of graded contractions of representations, we find different sets of equivalent contractions of Lie algebras and representations for the $\mathbb{Z}_3$-grading, and we ask which Lie algebras correspond to the representations given by the $\mathbb{Z}_3$-graded contractions of the 3-dimensional representations of the Lie algebra $\mathfrak{sl}(2)$.

Definition 3. Let $\mathfrak{g}$ be a $\mathbb{Z}_p$-graded Lie algebra, where $p \in \mathbb{N}$, and let V be a $\mathfrak{g}$-module. Then assume that the action of $\mathbb{Z}_p$-graded $\mathfrak{g}$ on V respects the grading on $V = \bigoplus_{m=0}^{q-1} V_m$, that is

$$0 \neq T(\mathfrak{g}_j)V_m \subseteq V_{j+m}, \tag{4.1}$$

where V_j, $j = 0, \dots, q-1$, defines the corresponding $\mathbb{Z}_q$-grading of V and the addition of indices is modulo q. If one denotes the contracted action $T(\mathfrak{g})$ on V as

$$T(\mathfrak{g}_j)^{\psi} \cdot V_m \subseteq \psi_{jm} T(\mathfrak{g}_j)V_m, \tag{4.2}$$

where ψ_{jm} are the contraction parameters, then

$$\big[T(\mathfrak{g}_j), T(\mathfrak{g}_k)\big]_{\epsilon}^{\psi} \cdot |m\rangle = \psi_{km}\psi_{j,k+m} T(\mathfrak{g}_j)T(\mathfrak{g}_k)|m\rangle - \psi_{jm}\psi_{k,j+m} T(\mathfrak{g}_k)T(\mathfrak{g}_j)|m\rangle \subseteq \epsilon_{jk}\psi_{j+k,m} T(\mathfrak{g}_{j+k})|m\rangle, \tag{4.3}$$

where $|m\rangle$, $m = 0, \dots, q-1$, are the base vectors of the subspace V_m, ϵ and ϵ_{jk} denote the contraction of $\mathfrak{g}$ and the corresponding parameters. The relations (4.3) are definitely satisfied when

$$\epsilon_{jk}\psi_{j+k,m} = \psi_{km}\psi_{j,k+m} = \psi_{jm}\psi_{k,j+m}. \tag{4.4}$$

One uses the notation V^{ψ} for V considered as graded $T(\mathfrak{g}^{\epsilon})$-module, where ψ satisfies (4.4). V^{ψ} is then called the contraction of the representation V with respect to the contraction ϵ.

We write the contraction parameters into the matrix as

$$\psi = \begin{pmatrix} \psi_{00} & \psi_{01} & \cdots & \psi_{0,n-1} \\ \psi_{10} & \psi_{11} & \cdots & \psi_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \psi_{n-1,0} & \psi_{n-1,1} & \cdots & \psi_{n-1,n-1} \end{pmatrix}. \tag{4.5}$$

In the case of contractions of $\mathbb{Z}_3$-graded representations of $\mathbb{Z}_3$-graded Lie algebra, the system of equations (4.4) gives the following relations:

$$\begin{aligned} \epsilon_{00}\psi_{0m} &= \psi_{0m}^2, \\ \epsilon_{01}\psi_{1m} &= \psi_{1m}\psi_{0,m+1} = \psi_{1m}\psi_{0m}, \\ \epsilon_{02}\psi_{2m} &= \psi_{2m}\psi_{0,m+2} = \psi_{2m}\psi_{0m}, \\ \epsilon_{11}\psi_{2m} &= \psi_{1m}\psi_{1,m+1}, \\ \epsilon_{22}\psi_{1m} &= \psi_{2m}\psi_{2,m+2}, \\ \epsilon_{12}\psi_{0m} &= \psi_{2m}\psi_{1,m+2} = \psi_{1m}\psi_{2,m+1}, \end{aligned} \tag{4.6}$$

where $m \in \{0,1,2\}$. From this it follows that (4.6) leads to the system of 18 equations, where for the simplification we suppose that either $\psi_{jk} = 0$ or $\psi_{jk} = 1$ for arbitrary j,k. Now we denote $\mathfrak{S}$ the set of the right-hand sides of (4.6). We divide this set into six parts, each containing a triplet of numbers corresponding to a concrete set of equations in (4.6):

$$\begin{aligned} \mathfrak{S} = \big(&\psi_{00}^2\ \psi_{01}^2\ \psi_{02}^2 \mid \psi_{10}\psi_{00}\ \psi_{11}\psi_{01}\ \psi_{12}\psi_{02} \mid \psi_{20}\psi_{00}\ \psi_{21}\psi_{01}\ \psi_{22}\psi_{02} \\ &\mid \psi_{10}\psi_{11}\ \psi_{11}\psi_{12}\ \psi_{12}\psi_{10} \mid \psi_{20}\psi_{22}\ \psi_{21}\psi_{20}\ \psi_{22}\psi_{21} \mid \psi_{10}\psi_{21}\ \psi_{11}\psi_{22}\ \psi_{12}\psi_{20}\big). \end{aligned}$$

There is a possibility of the cyclic permutation of columns of the contraction matrix ψ and each of these permutations corresponds to the cyclic permutation of numbers in each of the mentioned triplets. Starting with a given contraction matrix ψ, we denote ψ', respectively ψ'', the contractions matrices, where all the columns are shifted one position to right, respectively to left.

The possible forms of $\mathfrak{S}$ depend on a pair of contraction matrices (ϵ, ψ), where ϵ was assigned in (2.3) and both ϵ and ψ can be found in the list of possible contractions of representations published in [4] which is also written in the appendix of this paper. We can distinguish these 12 possibilities:

(1) $\mathfrak{S} = (1\ 1\ 1 \mid 1\ 1\ 0 \mid 1\ 0\ 0 \mid 1\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$ for combinations $(\epsilon_{II}, \psi_{II.1})$,
(2) $\mathfrak{S} = (1\ 1\ 1 \mid 1\ 0\ 0 \mid 1\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$ for $(\epsilon_I, \psi_{I.1})$,

(3) $\mathfrak{S} = (1\ 1\ 1 \mid 1\ 0\ 0 \mid 0\ 0\ 1 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$ for $(\epsilon_I, \psi_{I.2})$,
(4) $\mathfrak{S} = (1\ 1\ 1 \mid 1\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$ for $(\epsilon_I, \psi_{I.3})$, $(\epsilon_{II}, \psi_{II.2})$, and $(\epsilon_{VIII}, \psi_{VIII.1})$—in this case all the matrices ψ have the same form,
(5) $\mathfrak{S} = (1\ 1\ 1 \mid 0\ 0\ 0 \mid 1\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$ for $(\epsilon_I, \psi_{I.4})$,
(6) $\mathfrak{S} = (1\ 1\ 1 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$ for $(\epsilon_I, \psi_{I.5})$, $(\epsilon_{II}, \psi_{II.3})$, $(\epsilon_{VI}, \psi_{VI.1})$, $(\epsilon_{VII}, \psi_{VII.1})$, and $(\epsilon_{VIII}, \psi_{VIII.2})$ with the same form of the matrices ψ,
(7) $\mathfrak{S} = (1\ 1\ 0 \mid 1\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$ for combinations with the same form of the matrices ψ $(\epsilon_I, \psi_{I.6})$, $(\epsilon_{II}, \psi_{II.4})$, and $(\epsilon_{VIII}, \psi_{VIII.3})$,
(8) $\mathfrak{S} = (1\ 1\ 0 \mid 0\ 0\ 0 \mid 0\ 1\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$ for $(\epsilon_I, \psi_{I.7})$,
(9) $\mathfrak{S} = (1\ 1\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$ for combinations with the same form of the matrices ψ $(\epsilon_I, \psi_{I.8})$, $(\epsilon_{II}, \psi_{II.5})$, $(\epsilon_{VI}, \psi_{VI.2})$, $(\epsilon_{VII}, \psi_{VII.3})$, and $(\epsilon_{VIII}, \psi_{VIII.4})$,
(10) $\mathfrak{S} = (1\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$; here we have 3 distinct sets of pairs with the same matrices ψ:
   - $(\epsilon_I, \psi_{I.9})$, $(\epsilon_{II}, \psi_{II.6})$, $(\epsilon_{VI}, \psi_{VI.5})$, $(\epsilon_{VII}, \psi_{VII.4})$, and $(\epsilon_{VIII}, \psi_{VIII.5})$,
   - $(\epsilon_{VI}, \psi_{VI.3})$ and $(\epsilon_{VII}, \psi_{VII.2})$,
   - $(\epsilon_{VI}, \psi_{VI.4})$ and $(\epsilon_{VIII}, \psi_{VIII.6})$,
(11) $\mathfrak{S} = (0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 1\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$ for combinations with the same form of the matrices ψ: $(\epsilon_{III}, \psi_{III.1})$, $(\epsilon_{IV}, \psi_{IV.1})$, and $(\epsilon_{VII}, \psi_{VII.5})$,
(12) $\mathfrak{S} = (0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0)$; we have 6 distinct sets of pairs of matrices, each of them containing just 1 form of the matrix ψ:
   - $(\epsilon_{III}, \psi_{III.2})$ and $(\epsilon_{IV}, \psi_{IV.2})$,
   - $(\epsilon_V, \psi_{V.1})$,
   - $(\epsilon_V, \psi_{V.2})$ and $(\epsilon_{VI}, \psi_{VI.6})$,
   - $(\epsilon_V, \psi_{V.3})$, $(\epsilon_{VI}, \psi_{VI.8})$, and $(\epsilon_{VII}, \psi_{VII.6})$,
   - $(\epsilon_V, \psi_{V.4})$, $(\epsilon_{VI}, \psi_{VI.9})$, and $(\epsilon_{VIII}, \psi_{VIII.7})$,
   - $(\epsilon_{VI}, \psi_{VI.7})$.

Now we investigate how the contractions in the previous list transform the 3-dimensional representations of $\mathfrak{sl}(2)$. First, we study the cases which coincide with the representations of the Abelian algebra $\mathfrak{c}$. These cases immediately provide the solution, because the Abelian algebra cannot be decomposed into anything simpler. In Section 3, we see that the Abelian algebra $\mathfrak{c}$ coincides with the contraction matrices ϵ_{IV}, ϵ_{VI}, and ϵ_{VII}. This means that we know the solution for the cases (6), (9), (10), (11), and (12) in the previous list.

In order to study the other cases, we start with the 3-dimensional representation of $\mathfrak{sl}(2)$. All irreducible 3-dimensional representations of $\mathfrak{sl}(2)$ are isomorphic with the representation T given by matrices

$$T(e) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad T(f) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{4.7}$$

Now we consider the remaining cases using this representation T, the corresponding contraction matrices ψ from [4] introduced in the list of all possible contractions of 3-dimensional representations and the contraction matrices ψ', ψ'' arising from ψ by the cyclic permutation of columns. According to the properties of commutation relations of the matrices corresponding to the resulting representations, we distinguish representations corresponding to Lie algebras $\mathfrak{e}(2)$, $\mathfrak{c}$, $\mathfrak{l}$, and $\mathfrak{l}'$, where $\mathfrak{l}'$ and $\mathfrak{l}$ differ in exchanging the commutation relations for e and f:

$$[h,e]_{\mathfrak{l}'} = 0, \quad [h,f]_{\mathfrak{l}'} = -2f, \quad [e,f]_{\mathfrak{l}'} = 0.$$

We explain the procedure on the first case, corresponding to

$$\mathfrak{S} = \big(1\ 1\ 1 \mid 1\ 1\ 0 \mid 1\ 0\ 0 \mid 1\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0\ 0\big),$$
$$\epsilon_{II} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

First, we have to find the grading of the $\mathfrak{g}$-module corresponding to (4.1). It appears that the only suitable grading has the form

$$V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.8}$$

Now we take the representations T from (4.7) and we multiply the columns of the particular matrices (from right to left in agreement with the grading (4.8)) by the elements of the rows of the matrix ψ: the first row in ψ corresponds to $T(h)$, the second row corresponds to $T(e)$, the third row corresponds to $T(f)$. By this way, we get a new representation

$$T^{\psi}(e) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^{\psi}(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^{\psi}(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

its commutation relations correspond to the Lie algebra $\mathfrak{l}$. Next, we compose the matrices ψ' and ψ'' which arise by the permutation of the columns of ψ and get

$$\psi' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \psi'' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now we repeat the same procedure as we did for ψ and get the matrices corresponding to representations of $\mathfrak{e}(2)$ for both ψ' and ψ''.

The results for all the cases are listed in Table 1, where for each case in the previous list, the resulting representation is written.

Table 1: Contractions of 3-dimensional $\mathfrak{sl}(2)$ representations.

Order number of the contraction in the previous list	Applied contraction matrix		
	ψ	ψ'	ψ''
(1)	$\mathfrak{l}$	$\mathfrak{e}(2)$	$\mathfrak{e}(2)$
(2)	$\mathfrak{l}$	$\mathfrak{e}(2)$	$\mathfrak{l}'$
(3)	$\mathfrak{e}(2)$	$\mathfrak{l}$	$\mathfrak{l}'$
(4)	$\mathfrak{l}$	$\mathfrak{l}$	$\mathfrak{c}$
(5)	$\mathfrak{c}$	$\mathfrak{l}'$	$\mathfrak{l}'$
(6)	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
(7)	$\mathfrak{l}$	$\mathfrak{l}$	$\mathfrak{c}$
(8)	$\mathfrak{l}'$	$\mathfrak{l}'$	$\mathfrak{c}$
(9)–(12)	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$

5 Conclusion

The presented results of the contractions of 3-dimensional representations of $\mathfrak{sl}(2)$ are applicable in some mathematical areas.

One of them is the area of universal enveloping algebras. The problem, depicted in the following diagram, is the following: we investigate if an adjoint representation ρ represented in a given universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ associated with a Lie algebra $\mathfrak{g}$ can be simplified by a contraction ψ to the corresponding adjoint representation ρ^{ϵ} represented in another universal enveloping algebra $\mathcal{U}(\mathfrak{g}^{\epsilon})$ associated with a Lie algebra $\mathfrak{g}$ which is a contraction of $\mathfrak{g}$. It can be shown that in the case $\mathfrak{g} \equiv \mathfrak{sl}(2)$, this task is solvable for the 3-dimensional representations, but for the higher dimensions it is unfeasible. The main problem consists in the need of the same base for the matrices of both representations. The form of the matrices corresponding to the representation of the contracted $\mathfrak{g}$ in the given base is too difficult to be obtained with the help of the contraction.

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[d, "\epsilon"] & \mathcal{U}(\mathfrak{g}) \arrow[d] \\
\mathfrak{g}^{\epsilon} \arrow[d] & \rho(\mathfrak{g}) \arrow[d, "\psi"] \\
\mathcal{U}(\mathfrak{g}^{\epsilon}) \arrow[r] & \rho^{\epsilon}(\mathfrak{g}).
\end{tikzcd}$$

Appendix

Table A1: List of possible $\mathbb{Z}_3$-graded contractions of Lie algebras [4].

ϵ	Number	ψ	ϵ	Number	ψ	ϵ	Number	ψ
$\begin{pmatrix}1&1&1\\1&0&0\\1&0&0\end{pmatrix}$	I.1	$\begin{pmatrix}1&1&1\\1&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}$	III.1	$\begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}$		VI.8	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$
	I.2	$\begin{pmatrix}1&1&1\\1&0&0\\0&0&1\end{pmatrix}$		III.2	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$		VI.9	$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$
	I.3	$\begin{pmatrix}1&1&1\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	IV.1	$\begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$	VII.1	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$
	I.4	$\begin{pmatrix}1&1&1\\0&0&0\\1&0&0\end{pmatrix}$		IV.2	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$		VII.2	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$
	I.5	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$	V.1	$\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$		VII.3	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$
	I.6	$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix}$		V.2	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix}$		VII.4	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$
	I.7	$\begin{pmatrix}1&1&0\\0&0&0\\0&1&0\end{pmatrix}$		V.3	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$		VII.5	$\begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}$
	I.8	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$		V.4	$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$		VII.6	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$
	I.9	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	VI.1	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix}$	VIII.1	$\begin{pmatrix}1&1&1\\1&0&0\\0&0&0\end{pmatrix}$
$\begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix}$	II.1	$\begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix}$		VI.2	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$		VIII.2	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$
	II.2	$\begin{pmatrix}1&1&1\\1&0&0\\0&0&0\end{pmatrix}$		VI.3	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$		VIII.3	$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix}$
	II.3	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$		VI.4	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$		VIII.4	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$
	II.4	$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix}$		VI.5	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$		VIII.5	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$
	II.5	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$		VI.6	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix}$		VIII.6	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$
	II.6	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$		VI.7	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}$		VIII.7	$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$

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References

[1] M. de Montigny and J. Patera, *Discrete and continuous graded contractions of Lie algebras and superalgebras*, J. Phys. A, 24 (1991), 525–547.
[2] H. D. Doebner and O. Melsheimer, *On a class of generalized group contractions*, Nuovo Cimento A, 49 (1967), 306–311.
[3] E. Inönü and E. P. Wigner, *On the contraction of groups and their representations*, Proc. Nat. Acad. Sci. U. S. A., 39 (1953), 510–524.
[4] R. V. Moody and J. Patera, *Discrete and continuous graded contractions of representations of Lie algebras*, J. Phys. A, 24 (1991), 2227–2257.
[5] E. J. Saletan, *Contraction of Lie groups*, J. Mathematical Phys., 2 (1961), 1–21.
[6] I. E. Segal, *A class of operator algebras which are determined by groups*, Duke Math. J., 18 (1951), 221–265.