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Constructive Homological Algebra

Liangyun Chen*

Department of Mathematics and Statistics Northeast Normal University, P.R China

Description

Constructive algebra can be seen as an abstract version of computer algebra. In computer algebra, on the one hand, one attempts to construct efficient algorithms for solving concrete problems given in an algebraic formulation, where a problem is understood to be concrete if its hypotheses and conclusion have computational content. Constructive algebra, on the other hand, can be understood as a "preprocessing" step for computer algebra that leads to general algorithms, even if they are sometimes not efficient. In constructive algebra, one tries to give general algorithms for solving "virtually any" theorem of abstract algebra.Therefore, a first task in constructive algebra is to define the computational content hidden in hypotheses that are formulated in a very abstract way [1].

For example, what is a good constructive definition of a local ring (i.e., a ring with a unique maximal ideal), a valuation ring (i.e., a ring in which all elements are comparable under division), an arithmetical ring (i.e., a ring which is locally a valuation ring), a ring of Krull dimension ≤ n (i.e., a ring in which every chain p0 ⊂ p1 ⊂···⊂ pk of prime ideals has length k ≤ n), and so on? A good constructive definition must be equivalent to the usual definition within classical mathematics; it must have computational content; and it must be fulfilled by "usual" objects that satisfy the definition. As a typical example, let us consider the classical theorem "any polynomial P in K is a product of irreducible polynomials (K a field)". This leads to an interesting problem: it seems like no general algorithm that produces the irreducible factors. What, then, is the constructive content of this theorem? A possible answer is as follows: when performing computations with P, proceed as if its decomposition into irreducible polynomials were known (at the beginning, proceed as if P were irreducible). When something strange happens (e.g., when the gcd of P and another polynomial Q is a strict divisor of P), use this fact to improve the decomposition of P. This trick was invented in Computer Algebra as the D5-philosophy and later taken up in the form of the dynamical proof method in algebra. It indeed enables one to carry out computations inside the algebraic closure K of K even if it is not possible to effectively construct K, for in general this would require transfinite methods as Zorn's Lemma. The foregoing has been referred to as "dynamical evaluation" of the algebraic closure. From a logical point of view, the "dynamical evaluation" gives a constructive substitute for two highly nonconstructive tools of abstract algebra: the Law of Excluded Middle and Zorn's Lemma [2-4].

For instance, these tools are required in order to "construct" the complete prime factorization of an ideal in a Dedekind domain (i.e., in a Noetherian domain which is locally a valuation domain), while the dynamical method reveals the computational content of this "construction". We refer to for more details on the dynamical proof method in algebra, including a wealth of examples. It is worth mentioning Schuster's new approach with Open Induction. As compared with the dynamical proof method, Schuster's approach is somewhat closer both to everyday mathematical practice, and to the established proof-theoretic methods for the extraction of algorithms, and thus computer programs, from formalized proofs. The approach with Open Induction therefore is a direct competitor of the dynamical proof method, both regarding objectives and techniques. Following this "dynamical" philosophy, the main goal is to find the constructive content hidden in abstract proofs of concrete theorems in Commutative Algebra and especially well-known theorems concerning finitely-generated projective modules (i.e., the images of idempotent matrices) over polynomial rings and syzygy module (i.e., the relations module) of multivariate polynomials with coefficients in a valuation ring, or, more generally, in an arithmetical ring (a ring which is locally a valuation ring). As explained above, the general method consists in replacing some abstract ideal objects whose existence is based on the excluded middle principle and the axiom of choice by incomplete specifications of these objects.

The constructive rewriting of "abstract local-global principles" is very important. In classical proofs using this kind of principle, the argument is "let us see what happens after localization at an arbitrary maximal ideal of R". From a computational point of view, maximal ideals are too abstract objects, particularly if one wishes to deal with a general commutative ring. In the constructive rereading, the argument is "let us see what happens when the ring is a residually discrete local ring", i.e., if ∀x, (x ∈ R× or∀y(1 + xy) ∈ R×). If a constructive proof is obtained in this particular case, the process can be completed by "dynamically evaluating an arbitrary ring R as a residually discrete local ring". For example, in these lecture notes, Dedekind domains will behave dynamically as valuation domains. Dynamical methods were used successfully in order to find constructive substitutes to very elegant abstract theorems such as Quillen Patching, Quillen Induction and Lequain-Simis. The problem of freeness of projective modules over polynomial rings originally raised by Serre in 1955 is approached constructively [5]. Serre remarked that it was not known whether there exist finitely-generated projective modules over A = K[X1,...,Xn], K a field, which are not free. This remark turned into the so-called "Serre's conjecture" or "Serre's problem", stating that indeed there were no such modules.

Conflict of Interest

None.

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***Address for Correspondence:** Liangyun Chen, Department of Mathematics and Statistics Northeast Normal University, P.R China, E-mail: liangyunchen6787@gmail.com*

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