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Common Fixed Points of Generalized α - ψ -contractive Type Self-mappings and Multivalued Mappings

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Abstract

In this paper, mutivated by the recent work of samet et al. (Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal. we give some new results on existence of common fixed points for a pair of generalized α-ψ-contractive self-mappings and multivalued mappings. This results extend and improve many existing results in the literature. Some examples are given to illustrate the results.

Keywords: Common fixed point; Generalized α-ψ-contractive; Self-mappings; Multivalued mappings

Introduction

Recently, some results on existing of common fixed point for a pair of mappings or multifunctions have been given. In 2007, Zhang [1] defined a new generalized contractive type condition for a pair of mappings in metric spaces. Let $A\in(0,+\infty]$ and $F:\mathbb{R}_A^+\to\mathbb{R}$. Denote by I[0,A) the collection of all functions $F:\mathbb{R}_A^+\to\mathbb{R}$ satisfing the following conditions:

(i) F(0)=0 and F(t)>0 for each $t\in(0,A)$,

(ii) F is nondecreasing on $\mathbb{R}_A^+$,

(iii) F is continuous.

From Zhang [1] we know that for any $F\in I[0,A)$, $\lim_{n\to\infty}F(\varepsilon_n)=0$, $(\varepsilon_n\in\mathbb{R}_A^+)$ implies $\lim_{n\to\infty}\varepsilon_n=0$. Denote by $\Psi[0,A)$ the family of all functions $\psi:\mathbb{R}_A^+\to\mathbb{R}^+$ which is nondecreasing and right upper semi-continuous such that $\lim_{n\to\infty}\Psi^n(t)=0$ for each $t\in(0,A)$. It is easy to see that for any $\psi\in\Psi[0,A)$ we have $\psi(0)=0$ and $\psi(t)<t$ for each $t\in(0,A)$.

Regarding the above notations, Zhang [1] proved the following theorem on existing of common fixed point for a pair of mappings.

Theorem 1: Let (X,d) be a complete metric space and let $D=\sup\{d(x,y)\,|\,x,y\in X\}$. Set A>D if D<∞ and A=D if D=∞. Suppose that T,S:X→X are two mappings, $F\in I[0,A)$ and $\psi\in\Psi[0,F(A^-))$ satisfing

$$F(d(Tx,Sy))\le\psi(F(M(x,y)))\ \textit{for each}\ x,y\in X$$

where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Sy),\frac{1}{2}(d(Tx,y)+d(x,Sy))\}.$$

Then T and S have a unique common fixed point in X. Moreover, for each $x_0\in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1}=Tx_{2n}$ and $x_{2n+2}=Sx_{2n+1}$ converges to the common fixed point of T and S.

Denote by Ψ the family of all nondecreasing functions ψ:[0,+∞)→[0,+∞) such that $\sum_{n=1}^{+\infty}\psi^n(t)<+\infty$ for all t>0. Let (X,d) be a metric space and α: X×X→[0,∞) be a mapping. A mapping T:X→X is called α-admissible whenever α(x,y)≥1 implies $\alpha(T_x,T_y)\ge1$. In 2011, Samet and Vetro [2] introduced a new type of contraction to a mapping T:X→X, called α-ψ-contractive mappings, that is, $\alpha(x,y)d(T_x,T_y)\le\psi(d(x,y))$ for all x,y∈X and proved the following theorem.

Theorem 2: Let (X,d) be a complete metric space, α:X×X→[0,+∞) a function, ψ∈Ψ and T:X→X be an α-ψ-contractive mapping such that the following assertions hold [2]:

(i) T is α-admissible,

(ii) There exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge1$,

(iii) T is continuous or for any sequence $\{x_n\}$ in X that $\alpha(x_n,x_{n+1})\ge1$ for all n and $x_n\to x$, we have $\alpha(x_n,x)\ge1$ for all n.

Then, T has a fixed point in X.

This result generalized and improved many existing fixed point results for a mapping defined on a complete metric space (X,d), in particular the famous Banach contraction principle. Also, the authors proved that adding the following condition to the conditions of above theorem, the fixed point is unique.

(H):For any x,y∈X there exists z∈X such that α(x,z) ≥ 1 and α(y,z) ≥ 1.

Let (X,d) be a metric space and let CB(X) denote the set of all nonempty closed bounded subsets of X. Let H be the Hausdorff metric on CB(X) with respect to the metric d, that is, $H(A,B)=max\{\sup_{x\in A}d(x,B),\sup_{y\in B}d(y,A)\}$ for all A,B∈CB(X) where $d(y,A)=inf_{x\in A}d(y,x)$. Let $T,S:X\to2^X$ be a pair of multi-valued mappings. It is called that x is a common fixed point of T and S if x∈Tx and x∈Sx. In 2010, Rouhani and Moradi [3] proved the following common fixed point result for multi-valued mappings.

Theorem 3: Let (X,d) be a complete metric space and let T,S:X→CB(X) be two multivalued mappings such that

$$H(Tx,Sy)\le\alpha M(x,y)$$

for all x,y∈X where 0 ≤ α<1 and

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Sy),\frac{1}{2}(d(Tx,y)+d(x,Sy))\}.$$

Then, T and S have a common fixed point in X.

On the other hand in 2013, Beg et al. introduced common fixed

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point results for multivalued mappings defined on a metric space endowed with a graph [4]. In this paper combining ideas of the above mentioned literature we give some new results on common fixed points of α-ψ-contractive self-mappings and multivalued mappings [5-8].

Results

Now we give the main results of this study. Firstly we give some definitions.

Definition 1: Let (X,d) be a metric space and $\alpha:X\times X\to[0,\infty)$ be a mapping. We say that α is symmetric if $\alpha(x,y) \ge 1$ implies $\alpha(y,x) \ge 1$. Also, it is called that α is transitive if $\alpha(x,y) \ge 1$ and $\alpha(y,z) \ge 1$ implies $\alpha(x,z) \ge 1$.

Definition 2: Let (X,d) be a metric space and $T,S:X\to X$ be two mappings. We say that the ordered pair (T,S) is α-admissible whenever $\alpha(x,y) \ge 1$ implies $\alpha(Tx,Sy) \ge 1$.

Theorem 4: Let (X,d) be a complete metric space and let $D=\sup\{d(x,y) \mid x,y\in X\}$. Set A>D if D<∞ and A=D if D=∞. Suppose $\alpha:X\times X\to[0,+\infty)$ be a symmetric and transitive function, $F\in I[0,A)$, $\psi\in\Psi[0,F(A^-))$ and $T,S:X\to X$ are two mappings satisfing

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Sy),\frac{1}{2}(d(Tx,y)+d(x,Sy))\}.$$

where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Sy),\frac{1}{2}(d(Tx,y)+d(x,Sy))\}.$$

Moreover, let the following assertions hold:

(i) (T,S) is α-admissible,

(ii) There exists $x_0\in X$ such that $\alpha(x_0,Tx_0) \ge 1$,

(iii) For any sequence $\{x_n\}$ in X that $\alpha(x_n,x_{n+1}) \ge 1$ for all n and $x_n\to x$, then $\alpha(x_n,x) \ge 1$ for all n.

Then, T and S have a common fixed point in X.

Proof: Define an iterated sequence $\{x_n\}$ in X with $x_{2n+1}=Tx_{2n}$ and $x_{2n+2}=Sx_{2n+1}$. Now $\alpha(x_0,x_1)=\alpha(x_0,Tx_0) \ge 1$. Since (T,S) is α-admissible, then $(x_1,x_2)=\alpha(Tx_0,Sx_1) \ge 1$. Since α is symmetric, then $(x_2,x_1)\ge 1$. So, $\alpha(x_3,x_2)=\alpha(Tx_2,Sx_1) \ge 1$. Continuing this process, we see that $\alpha(x_n,x_{n+1}) \ge 1$ for all n. Now

$$F(d(x_{2n+1},x_{2n+2}))\le\alpha(x_{2n},x_{2n+1})F(d(Tx_{2n},Sx_{2n+1})) \quad (1)$$

(F(M(x_2n,x_2n+1))).

But we have

$$M(x_{2n},x_{2n+1})=\max\{d(x_{2n},x_{2n+1}),d(x_{2n},Tx_{2n}),d(x_{2n+1},Sx_{2n+1}),\frac{1}{2}(d(Tx_{2n},x_{2n+1})+d(x_{2n},Sx_{2n+1})\}$$

$$=\max\{d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),\frac{1}{2}(d(x_{2n+1},x_{2n+1})+d(x_{2n},x_{2n+2}))\}$$

$$d(x_{2n+1},x_{2n+2})>d(x_{2n},x_{2n+1})$$

If we have $d(x_{2n+1},x_{2n+2})>d(x_{2n},x_{2n+1})$, then from (1) we will have

$$F(d(x_{2n+1},x_{2n+2}))\le\psi(F(d(x_{2n+1},x_{2n+2})))<F(d(x_{2n+1},x_{2n+2}))$$

which is a contradiction. So, we can only have the case

$$F(d(x_{2n+1},x_{2n+2}))\le\psi(F(d(x_{2n},x_{2n+1}))). \quad (2)$$

Also

$$F(d(x_{2n+1},x_{2n}))\le\alpha(x_{2n},x_{2n-1})F(d(Tx_{2n},Sx_{2n-1})) \quad (3)$$

(F(M(x_2n,x_2n-1))).

But we have

$$=\max\{d(x_{2n},x_{2n-1}),d(x_{2n},x_{2n+1}),d(x_{2n-1},x_{2n}),\frac{1}{2}(d(x_{2n+1},x_{2n-1})+d(x_{2n},x_{2n}))\}$$

$$=\max\{d(x_{2n},x_{2n-1}),d(x_{2n},x_{2n+1}),d(x_{2n-1},x_{2n}),\frac{1}{2}(d(x_{2n+1},x_{2n-1})+d(x_{2n},x_{2n}))\}$$

$$\le\max\{d(x_{2n},x_{2n-1}),d(x_{2n+1},x_{2n})\}.$$

If we have $d(x_{2n+1},x_{2n})>d(x_{2n},x_{2n-1})$, then from (3) we will have

$$F(d(x_{2n+1},x_{2n}))\le\psi(F(d(x_{2n+1},x_{2n})))<F(d(x_{2n+1},x_{2n}))$$

which is a contradiction. So, we can only have the case

$$F(d(x_{2n+1},x_{2n}))\le\psi(F(d(x_{2n},x_{2n-1}))). \quad (4)$$

From (2) and (4) we see that

$$F(d(x_n,x_{n+1}))\le\psi(F(d(x_{n-1},x_n))), \quad \text{for all } n\in\mathbb{N}. \quad (5)$$

From (5) we get

$$F(d(x_n,x_{n+1}))\le\psi^n(F(d(x_0,x_1))), \quad \text{for all } n\in\mathbb{N}.$$

Tending n to ∞, we obtain $F(d(x_n,x_{n+1}))\to 0$. Hence, $d(x_n,x_{n+1})\to 0$ as $n\to\infty$. Since α is symmetric and transitive, hence we have $\alpha(x_n,x_m)\ge 1$ and $\alpha(x_m,x_n)\ge 1$ for all n, $m\in\mathbb{N}$ which n<m. Now showing that $\{x_n\}$ is a cauchy sequence is comletely similar to theorem 1 in Zhang's [1]. From completeness of (X,d), there exists $x\in X$ such that $x_n\to x$.

Next, we shall show that Tx=x and Sx=x. At first, we have

$$F(d(x_{2n+1},Sx))=F(d(Tx_{2n},Sx))\le\alpha(x_{2n},x)F(d(Tx_{2n},Sx))\le\psi(F(M(x_{2n},x))). \quad (6)$$

But

$$M(x_{2n},x)=\max\{d(x_{2n},x),d(x_{2n},x_{2n+1}),d(x,Sx),\frac{1}{2}(d(x_{2n+1},x)+d(x_{2n},Sx))\}.$$

Tending n to ∞, we obtain $M(x_{2n},x)\searrow d(x,Sx)$. Now tending n to ∞ in (6) and using continuity of F and right upper semi-continuoty of ψ, we get $F(d(x,Sx))\le\psi(F(d(x,Sx)))$ which implies $F(d(x,Sx))=0$ and so $d(x,Sx)=0$. Hence, Sx=x. On the other hand,

$$F(d(Tx,x_{2n+2}))\le\alpha(x,x_{2n+1})F(d(Tx,Sx_{2n+1}))\le\psi(F(M(x,x_{2n+1}))). \quad (7)$$

But

$$M(x,x_{2n+1})=\max\{d(x,x_{2n+1}),d(Tx,x),d(x_{2n+1},x_{2n+2}),\frac{1}{2}(d(Tx,x_{2n+1})+d(x,x_{2n+2}))\}.$$

Tending n to ∞, we obtain $M(x,x_{2n+1})\searrow d(Tx,x)$. Tending n to ∞ in (7), we obtain $F(d(Tx,x))\le\psi(F(d(Tx,x)))$ which implies $F(d(Tx,x))=0$ and so $d(Tx,x)=0$. Hence, Tx=x.

Example 1: Let $X=\left\{\frac{1}{n}\mid n\in\mathbb{N}\right\}\cup\{0\}$ with the usual metric $d(x,y)=|x-y|$. Obviously (X,d) is complete and D=1. Let $F(t)=t^{\frac{1}{t}}$ on [0,e) and F(0)=0, then $F\in\Im[0,A)$ where A=e>D. Let $\psi(t)=\frac{t}{2}$. Then, $\psi\in\Psi[0,e^{\frac{1}{e}})$. Suppose $T,S:X\to X$ be defined by

$$Tx=\begin{cases}0 & x=0,\\ 1 & x=\frac{1}{n},1\le n\le 100,\\ \frac{1}{n+1} & x=\frac{1}{n},n>100\end{cases}$$

and

$$Sy=\begin{cases}0 & y=0,\\ 0 & y=\frac{1}{n},1\le n\le 200,\\ \frac{1}{n+1} & y=\frac{1}{n},n>200.\end{cases}$$

Also, define

$$\alpha(x,y)=\begin{cases}1 & x,y\in\{\frac{1}{n}\mid n>200\}\cup\{0\},\\ 0 & otherwise.\end{cases}$$

Now let $x,y\in\left\{\frac{1}{n}\mid n>200\right\}\cup\{0\}$. If x=0 and $y=\frac{1}{n}$ where n>200, then $F(d(Tx,Sy))=\left(\frac{1}{n+1}\right)^{n+1}$ and $M(x,y)=\max\left\{\frac{1}{n},0,|\frac{1}{n}-\frac{1}{n+1}|,\frac{1}{2}(\frac{1}{n}+\frac{1}{n+1})\right\}=\frac{1}{n}$ Hence

$$F(d(Tx,Sy))=(\frac{1}{n+1})^{n+1}=\frac{1}{n+1}(\frac{1}{n+1})^{n}\le\frac{1}{2}(\frac{1}{n})^{n}=\psi(F(M(x,y)).$$

Similarly, if y=0 and $x=\frac{1}{n}$ where n>200, then $F(d(Tx,Sy))=\left(\frac{1}{n+1}\right)^{n+1}$ and $M(x,y)=\frac{1}{n}$ and hence $F(d(Tx,Sy))\le\frac{1}{2}(\frac{1}{n})^{n}=\psi(F(M(x,y)).$ If $x,y\in\{\frac{1}{n}\mid n>200\}$, then put $y=\frac{1}{m}$ and $y=\frac{1}{m}$ where n,m>200, then

$$F(d(Tx,Sy))=F\left(\left|\frac{1}{n+1}-\frac{1}{m+1}\right|\right)=\left(\frac{|m-n|}{(n+1)(m+1)}\right)^{\frac{(n+1)(m+1)}{|m-n|}}$$

$$=\left(\frac{|m-n|}{(n+1)(m+1)}\right)^{\frac{n+m+1}{|m-n|}}\left(\frac{|m-n|}{nm}\right)^{\frac{nm}{|m-n|}}\left(\frac{mn}{(n+1)(m+1)}\right)^{\frac{nm}{|m-n|}}$$

$$\le\frac{1}{2}(1)\left(\frac{|m-n|}{nm}\right)^{\frac{nm}{|m-n|}}=\psi(F(d(x,y))\le\psi(F(M(x,y)).$$

By definition of α, we see that

$$\alpha(x,y)F(d(Tx,Sy))\le\psi(F(M(x,y)))\ for\ each\ x,y\in X.$$

Also, if we put $x_0=0$, then $\alpha(x_0,Tx_0)=\alpha(0,0)=1$. Obviously, α is symmetric and transitive, (T,S) is α-admissible and the condition (iii) of theorem 4 holds. Hence, by the theorem we can say that T and S have a common fixed point in X. Here $T_0=S_0=0$.

Note that in the above example we can not apply theorem 1. To see this, put $y=\frac{1}{101}$ and $y=\frac{1}{101}$. Then, $F(d(Tx,Sy))=(1)^1=1$ and

$$M(x,y)=\max\left\{|\frac{1}{100}-\frac{1}{101}|,|\frac{1}{100}-1|,|\frac{1}{101}-0|,\frac{1}{2}\left(|1-\frac{1}{101}|+|\frac{1}{100}-0|\right)\right\}$$

$$=\max\left\{\frac{1}{10100},\frac{99}{100},\frac{1}{101},\frac{1}{2}\left(\frac{100}{101}+\frac{1}{100}\right)\right\}=\frac{99}{100}.$$

We see that $F(d(Tx,Sy))=1>(\frac{99}{100})^{\frac{100}{99}}=F(M(x,y)>\psi(F(M(x,y)).$ Hence we can not use theorem 1. Note that theorem 4 is a generalization of theorem 1, because having the contraction condition of theorem 1 it is sufficient to put α(x,y)=1 for each x,y∈X. Then, all of the conditions of theorem 4 are holded. So by the theorem, T and S have a common fixed point in X. Also, the above example shows that this generalization is real [9-11].

Also, we can not apply the contraction $\alpha(x,y)d(Tx,Sy)\le cM(x,y)$ for some c∈[0,1). To see this, put $x=\frac{1}{n}$ and $y=\frac{1}{n+1}$ where n>200. Then,

$$\sup_{x,y\in X}\frac{\alpha(x,y)d(Tx,Sy)}{M(x,y)}\ge\sup_{n>200}\frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}}=1.$$

Let φ:[0,A)→[0,∞) be a Lebesgue integrable function which is summable on each compact subset of [0,A) such that $\int_0^{\varepsilon}\phi(t)dt>0$ for ach ε∈(0,A). Then, we have also the following result to integral type version of α-ψ-contraction for a pair of mappings T and S.

Corollary 1: Let (X,d) be a complete metric space and let $D=\sup\{d(x,y)\mid x,y\in X\}$. Set A>D if D<∞ and A=D if D=∞. Suppose α:X×X→[0,+∞) be a symmetric and transitive function, $\psi\in\Psi[0,\int_0^{A}\phi(t)dt)$ and T,S:X→X are two mappings satisfing

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Sy),\frac{1}{2}(d(Tx,y)+d(x,Sy))\}.$$

where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Sy),\frac{1}{2}(d(Tx,y)+d(x,Sy))\}.$$

Moreover, let the following assertions hold:

(i) (T,S) is α-admissible,

(ii) There exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge1$,

(iii) For any sequence $\{x_n\}$ in X that $\alpha(x_n,x_{n+1})\ge1$ for all n and $x_n\to x$, then $\alpha(x_n,x)\ge1$ for all n.

Then, T and S have a common fixed point in X.

In what follows, we introduce the common fixed point results for a pair of multivalued mappings.

Definition 3: Let (X,d) be a metric space, α:X×X→[0,+∞) be a function and T,S:XCB(X) are two multivalued mappings. We say that the ordered pair (T,S) is α-admissible whenever α(x,y)≥1 implies α(u,v)≥1 for each u∈Tx,v∈Sy.

Theorem 5: Let (X,d) be a complete metric space and ψ∈Ψ be a strictly increasing and right upper semi-continuous function. Suppose α:X×X→[0,+∞) be a symmetric function and T,S:X→CB(X) are two multivalued mappings satisfing

$$\alpha(x,y)H(Tx,Sy)\le\psi(M(x,y)),\ for\ each\ x,y\in X$$

where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Sy),\frac{1}{2}(d(Tx,y)+d(x,Sy))\}.$$

Moreover, let the following assertions hold:

(i) (T,S) is α-admissible,

(ii) There exists $x_0\in x$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\ge1$,

(iii) For any sequence $\{x_n\}$ in X that $\alpha(x_n,x_{n+1})\ge1$ for all n and $x_n\to x$, then $\alpha(x_n,x)\ge1$ for all n.

Then, T and S have a common fixed point in X.

Proof. Obviously, M(x,y)=0 if and only if x=y is a common fixed point of T and S. Hence, we may assume that $M(x_0,x_1)>0$. Now $\alpha(x_0,x_1)\ge1$ implies

$$d(x_1,Sx_1)\le\alpha(x_0,x_1)H(Tx_0,Sx_1)\le\psi(M(x_0,x_1))<M(x_0,x_1).\qquad(8)$$

But

$$M(x_0,x_1)=\max\{d(x_0,x_1),d(x_0,Tx_0),d(x_1,Sx_1),\frac{1}{2}(d(Tx_0,x_1)+d(x_0,Sx_1))\}$$

$$\le\max\{d(x_0,x_1),d(x_1,Sx_1)\}.$$

If $d(x_1,Sx_1)>d(x_0,x_1)$, then from (8), we get $d(x_1,Sx_1)<d(x_1,Sx_1)$ which is a contradiction. Hence, we have only $d(x_0,x_1)\ge d(x_1,Sx_1)$ and so from (8), we have $d(x_1,Sx_1)<d(x_0,x_1)$. Hence, there exists x_2Sx_1 such that $d(x_1,x_2)<d(x_0,x_1)$. Since (T,S) is α-admissible, hence $(x_1,x_2)\ge1$ and since α is symmetric, then $\alpha(x_2,x_1)\ge1$. Now

$$d(Tx_2,x_2)\le\alpha(x_2,x_1)H(Tx_2,Sx_1)\le\psi(M(x_2,x_1)).\qquad(9)$$

But

$$M(x_2,x_1)=\max\{d(x_2,x_1),d(Tx_2,x_2),d(x_1,Sx_1),\frac{1}{2}(d(Tx_2,x_1)+d(x_2,Sx_1))\}$$

$\leq \max\{d(x_2,x_1), d(Tx_2,x_2)\}$.

If $d(Tx_2,x_2)>d(x_2,x_1)$, then from (9), we obtain $d(Tx_2,x_2)\leq\psi(d(Tx_2,x_2))$ which is a contradiction (note that $d(Tx_2,x_2)>d(x_2,x_1)\geq 0$). Hence, we have only $d(x_2,x_1)\geq d(Tx_2,x_2)$ and so from (9), $d(Tx_2,x_2)\leq\psi(d(x_2,x_1))<\psi(d(x_0,x_1))$. Hence, there exists $x_3\in Tx_2$ such that $d(x_3,x_2)<\psi(d(x_0,x_1))$. Since (T,S) is α-admissible, so $(x_3,x_2)\geq 1$ and since α is symmetric, hence $\alpha(x_2,x_3)\geq 1$. Continuing this process, we obtain a sequence $\{x_n\}$ in X such that

$$x_{2n+1}\in Tx_{2n}, x_{2n+2}\in Sx_{2n+1}, \alpha(x_n,x_{n+1})\geq 1, d(x_n,x_{n+1})<\psi^{n-1}(d(x_0,x_1))$$

for all $n\in\mathbb{N}$. From triangle inequality, we conclude that $d(x_n,x_m)\leq\Sigma_{i=n}^{i=m-1}\psi^{i-1}(d(x_0,x_1))\to 0$ as $m>n\to\infty$. Hence, $\{x_n\}$ is a cauchy sequence. From completeness of (X,d), there exists $x\in X$ such that $x_n\to x$.

Now we shall show that $x\in Tx$ and $x\in Sx$. At first, we have

$$M(x_{2n},x)\leq\max\{d(x_{2n},x), d(x_{2n},x_{2n+1}), d(x,Sx), \frac{1}{2}(d(x_{2n+1},x)+d(x_{2n},Sx))\}. \quad (10)$$

But

$$M(x_{2n},x)\leq\max\{d(x_{2n},x), d(x_{2n},x_{2n+1}), d(x,Sx), \frac{1}{2}(d(x_{2n+1},x)+d(x_{2n},Sx))\}.$$

Tending n to ∞, we obtain $M(x_{2n},x)\searrow d(x,Sx)$. Now tending n to ∞ in (10) and using right upper semi-continuoty of ψ, we get $d(x,Sx)\leq\psi(d(x,Sx))$ which implies $d(x,Sx)=0$. Hence, $x\in Sx$. On the other hand,

$$d(Tx,x_{2n+2})\leq\alpha(x,x_{2n+1})H(Tx,Sx_{2n+1})\leq\psi(M(x,x_{2n+1})). \quad (11)$$

But

$$M(x,x_{2n+1})\leq\max\{d(x,x_{2n+1}), d(Tx,x), d(x_{2n+1},x_{2n+2}), \frac{1}{2}(d(Tx,x_{2n+1})+d(x,x_{2n+2}))\}.$$

Tending n to ∞, we obtain $d(x,y)=|x-y|$. Also tending n to ∞ in (11), we obtain $d(Tx,x)\leq\psi(d(Tx,x))$ which implies $d(Tx,x)=0$ and so $d(Tx,x)=0$. Hence, $x\in Tx$.

Example 2: Let $X=[0,\infty)$ with the usual metric $d(x,y)=|x-y|$. Obviously, (X,d) is complete. Let $\psi(t)=\frac{t}{2}$ and suppose $T,S:X\to CB(X)$ be defined by

$$Tx=\begin{cases}[0,\frac{x}{4}] & x\in[0,1],\\ \{2\} & x>1\end{cases}$$

and

$$Sy=\begin{cases}\{\frac{y}{4}\} & y\in[0,1],\\ \{3\} & y>1.\end{cases}$$

Also define

$$\alpha(x,y)=\begin{cases}1 & x,y\in[0,1],\\ 0 & otherwise.\end{cases}$$

Obviously, α is symmetric. If $x,y\in[0,1]$, then

$$H(Tx,Sy)=H([0,\frac{x}{4}],\{\frac{y}{4}\})=\max\{\frac{|x-y|}{4},\frac{y}{4}\}\leq\frac{1}{2}\max\{|x-y|,|y-\frac{y}{4}|\}\leq\psi(M(x,y)).$$

Hence by definition of α, we have $(x,y)H(Tx,Sy)\leq\psi(M(x,y))$ for all $x,y\in X$. It is easy to check that all of other conditions of theorem 5 hold. Hence by the theorem, T and S have a common fixed point in X. In fact, $0\in T_0$ and $0S_0$. Note that in the above example we can not apply theorem in Rouhani's [3]. To see this, put x=2 and y=1. Then,

$H(Tx,Sy))=H(\{2\},\{\frac{1}{4}\})=|2-\frac{1}{4}|=\frac{7}{4}$ and

$$M(x,y)=\max\{|2-1|,0,|1-\frac{1}{4}|,\frac{1}{2}(|2-1|+|2-\frac{1}{4}|)\}=\frac{11}{8}.$$

We see that $H(Tx,Sy))=\frac{7}{4}>\frac{11}{8}=M(x,y)>\quad(M(x,y)$.

Note that theorem 5 is a generalization in Rouhani's [3] because having the contraction condition in Rouhani's [8] it is sufficient to put $\alpha(x,y)=1$ for each $x,y\in X$. Then, we will have all of the conditions of theorem 5 holded. So, by the theorem the multivalued mappings T and S have a common fixed point in X. Also, example 2 shows that this generalization is real.

Also, note that we can not use theorem 4 in Beg and Butt [4] for example 2. Since if we get $E(G)=\{(x,y)\,|\,x,y\in[0,1]\}$ and choose $x=\frac{3}{4}$ and y=1, then

$$H(Tx,Sy))=H([0,\frac{3}{16}],\{\frac{1}{4}\})=\max\{\frac{1}{4},|\frac{1}{4}-\frac{3}{16}|\}=\frac{1}{4}$$

and

$$d(x,y)=|1-\frac{3}{4}|=\frac{1}{4}.$$

We see that $H(Tx,Sy))=\frac{1}{4}=d(x,y)>cd(x,y)$ for any $0\leq c<1$. On the other hand, theorem 2.3 is a generalization of theorem 4 in Beg and Butt [1] because having the contraction condition of theorem 4 in Beg and Butt [1] it is sufficient to define $\alpha(x,y)=1$ if $(x,y)\in E(G)$ and 0 otherwise. Example 2 shows that this generalization is real.

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