Common Fixed Point Theorem in $T_0$ Quasi Metric Space

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Abstract

In this paper, we prove fixed point theorems for generalized C-contractive and generalized S-contractive mappings in a bi-complete di-metric space. The relationship between q- spherically complete $T_0$ Ultra-quasi-metric space and bi-complete di-metric space is pointed out in proposition 3.2. This work is motivated by Petals and Fvidalis in a $T_0$-ultra-quasi-metric space.

Keywords: Fixed Point; Generalized C-Contraction; Generalized S-Contraction; Spherically Complete; Bi-Complete Di-metric

Introduction

In Agyingi [1] proved that every generalized contractive mapping defined in a q- spherically complete $T_0$-ultra-quasi metric space has a unique fixed point. In Petals and Fvidalis [2] proved that every contractive mapping on a spherically complete non Archimedean normed space has a unique fixed point. Agyingi and Gega proved fixed point theorems in a $T_0$-ultra-quasi-metric space [3-5]. Later many authors published number of papers in this space [6-10].

In this paper we shall prove a fixed point theorem for generalized c- contractive and generalized s-contractive mappings in a bi-complete di-metric space.

If we delete, in the used definition of the pseudo metric $d$ on the set $X$, the symmetry condition, $d(x,y)=d(y,x)$, whenever $x,y \in X$ we are led to the concept of quasi-pseudo metric.

Definition 1.1: Let $(X,m)$ be a metric space. Let $T:X \rightarrow X$ be a map is called a C-contraction if there exist, $0 \leq k < \frac{1}{2}$ such that for all $x,y \in X$ the following inequality holds [10],

$$m(Tx,Ty) \leq km(x,Tx)+m(y,Ty)$$

Definition 2.1: Let $(X,m)$ be a metric space. A map $T:X \rightarrow X$ is called a S-contraction if there exist $0 \leq k < \frac{1}{3}$ such that for all $x,y \in X$ the following inequality holds [10],

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Introduction

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$$m(Tx,Ty) \leq km(x,Tx)+m(y,Ty)$$

Definition 2.2: Let $(X,m)$ be a metric space. A map $T:X \rightarrow X$ is called a S-contraction if there exist $0 \leq k < \frac{1}{3}$ such that for all $x,y \in X$ the following inequality holds [10],

$$m(Tx,Ty) \leq km(x,Tx)+m(y,Ty)+m(x,y)$$

Preliminaries

Now we recall some elementary definitions and terminology from the asymmetric topology which are necessary for a good understanding of the work below.

Definition 2.1: Let $X$ be a non empty set. A function $d:X \times X \rightarrow [0,\infty)$ is called quasi pseudo metric on $X$ if

$$d(x,x)=0, \forall x \in X$$
$$d(x,z) \leq d(x,y)+d(y,z), \forall x,y,z \in X$$

Moreover if $d(x,y)=0=d(y,x)$ then $d$ is said to be a $T_0$ quasi metric or di-metric. The latter condition is referred as the $T_0$ condition.

Let $d$ be quasi-pseudo metric on $X$, then the map $d^{-1}$ defined by $d^{-1}(x,y)=d(y,x)$ whenever $x,y \in X$ is also a quasi-pseudo metric on $X$, called the conjugate of $d$.

It is also denoted by $d'$ or $d^*$. It is easy to verify that the function $d'$ defined by $d'^2=d \vee d^{-1}$

i.e. $d'^2(x,y)=\max\{d(x,y),d(y,x)\}$ defines a metric on $X$ whenever $d$ is a $T_0$ quasi pseudo metric.

In some cases, we need to replace $[0,\infty)$ by $[0,\infty)$ (where for a $d$ attaining the value $\infty$, the triangle inequality is interpreted in the obvious way). In such case we speak of extended quasi-pseudo metric.

Definition 2.2: The di metric space $(X,d)$ is said to be bi complete if the metric space $(R,d')$ is complete.

Example 2.1: Let $X=\mathbb{R}$, we define the real valued map $d$ given by $d(x,y)=|x-y|$, then $(\mathbb{R},d)$ is a di metric space.

Remark 2.1

Let $d$ be quasi-pseudo metric on $X$, then the map $d'^2$ defined by $d'^2(x,y)=d(y,x)$ whenever $x,y \in X$ is also a quasi-pseudo metric on $X$, called the conjugate of $d$.

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Example 2.2: Let $X=[0,\infty)$ define for each $x,y \in X$, $m(x,y)=x$ if $x>y$ and $n(x,y)=x$ if $x<y$. It is not difficult to check that $(X,n)$ is a $T_0$ quasi pseudo metric space. Notice that, for $x,y \in [0,\infty)$, we have $n^2(x,y)=\max\{x,y\}$ if $x+y$ and $n^2(x,y)=0$ if $x=y$, the matrix $n^2$ is complete on $(X,d)$.

Definition 2.3: Let $(X,d)$ be quasi pseudo metric space, for $x,y \in X$ and $x+y \in X$, $B_d(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$

denotes the open ball at $x$. The collection of such balls is a base for a topology $\tau(d)$ induced by $d$ on $X$. Similarly for $x,y \in X$ and $x+y \in X$, $B_d(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$

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\[ C_d(x,e) = \{ y \in X : d(x, y) \leq e \} \]
denotes the closed \( e \)-ball at \( x \).

**Definition 2.4:** Let \((X, d)\) be quasi pseudo metric space. Let \((x_i), i \in I\) be a family of points in \(X\) and let \((y_i), i \in I\) be a family of non negative real numbers.

We say that \( \{C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i)\}, i \in I \) has the mixed binary intersection property provided that \( \{C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i)\}, i \in I \) is the mixed binary intersection property such that

\[ \bigcap_{i=1}^{n} (e_d(x_i, r_i)) \cap (e_{d^{-1}}(x_i, s_i)) = \emptyset \]

**Proposition 2.1:** If \((X, d)\) is an extended Isbell complete quasi pseudo metric space then \((X, d^2)\) is hyper complete. An interesting class of quasi pseudo metric space, for which, we investing a type of completeness are the ultra quasi pseudo metric.

**Definition 2.6:** Let \( X \) be a set \& \( d : X \times X \rightarrow [0, \infty) \) be a function a function mapping into the set \([0, \infty)\) of non negative real’s then \( d \) is ultra quasi pseudo metric on \( X \) if

\[ d(x, x) = 0 \quad \text{for all } x \in X \quad \text{&} \quad d(x, z) \leq \max\{d(x, y), d(y, z)\} \quad \text{whenever } x, y, z \in X \]

The conjugate \( d^{-1} \) of \( d \) where \( d^{-1}(x, y) = d(y, x) \) whenever \( x, y \in X \) is also an ultra quasi pseudo metric on \( X \).

If \( d \) also satisfies the \( T_0 \) – condition, then \( d \) is called a \( T_0 \)- ultra quasi metric on \( X \). Notice that \( d^2 = \sup\{d, d^2\} \) is an ultra-metric on \( X \) whenever \( d \) is a \( T_0 \)- ultra quasi metric.

In a literature, \( T_0 \)- ultra quasi metric spaces are also known as Archimedean \( T_0 \)- quasi metric.

### q-spherically Completeness

In this section we shall recall some results about \( q \)- spherical completeness belonging mainly to [8].

**Definition 3.1:** Let \((X, d)\) be an ultra – quasi pseudo metric space. Let \((x_i), i \in I\) be a family of points in \(X\) and let \((r_i), i \in I\) & \((s_i), i \in I\) be a family of non negative real numbers we say that \((X, d)\) is \( q \)- spherical completeness provided that each \( \{C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i)\}, i \in I \)

Satisfying \( d(x_i, s_i) \leq \max\{r_i, s_i\} \), whenever \( i, j \in I \) is such that

\[ \bigcap_{i=1}^{n} (e_d(x_i, r_i)) \cap (e_{d^{-1}}(x_i, s_i)) = \emptyset \]

**Proposition 3.2:** Each \( q \)-spherically complete \( T_0 \) ultra quasi metric space \((X, d)\) is bi-complete[8].

### Main Results

We recall the following interesting results respectively due to Chatterji [10] and to Shukla [11]

**Theorem 4.1a** A C- contraction on a complete metric space has a unique fixed point.

**Theorem 4.1b** A S- contraction on a complete metric space has a unique fixed point.

Following results generalizes the above theorem to setting of a bi-complete di-metric space.

**Definition 4.1:** Let \((X, d)\) be a quasi pseudo metric space. A map \( T : X \rightarrow X \) is called a \( c \)-pseudo contraction if there exist \( k, 0 \leq k < \frac{1}{2} \) such that for all \( x, y \in X \) the following inequality holds.

\[ d(Tx, Ty) \leq k[d(x, Ty) + d(y, Ty)] \]

**Definition 4.2:** Let \((X, d)\) be a quasi pseudo metric space. A map \( T : X \rightarrow X \) is called a \( S \)-pseudo contraction if there exist \( k, 0 \leq k < \frac{1}{3} \) such that for all \( x, y \in X \) the following inequality holds.

\[ d(Tx, Ty) \leq k[d(x, Ty) + d(x, Ty) + d(x, y)] \]

Now we define following definitions

**Definition 4.3:** Let \((X, d)\) be a quasi pseudo metric space. A map \( T : X \rightarrow X \) is called a generalized \( c \)-pseudo contraction if there exist \( k, 0 \leq k < \frac{1}{4} \) such that for all \( x, y \in X \) the following inequality holds.

\[ d(Tx, Ty) \leq k[d(x, Ty) + d(x, Ty) + d(x, Ty)] \]

**Definition 4.4:** Let \((X, d)\) be a quasi pseudo metric space. A map \( T : X \rightarrow X \) is called a generalized \( S \)-pseudo contraction if there exist \( k, 0 \leq k < \frac{1}{8} \) such that for all \( x, y \in X \) the following inequality holds.

\[ d(Tx, Ty) \leq k[d(x, Ty) + d(x, Ty) + d(x, Ty)] \]

**Theorem 4.1:** Let \((X, d)\) be a bi complete di metric space and let \( T : X \rightarrow X \) be a generalized \( c \)-pseudo contraction then \( T \) has a unique fixed point.

**Proof:** Since \( T : X \rightarrow X \) is a generalized \( c \)-pseudo contraction then there exist \( k, 0 \leq k < \frac{1}{4} \) such that for all \( x, y \in X \) the following inequality holds:

\[ d(Tx, Ty) \leq k[d(x, Ty) + d(y, Ty) + d(x, Ty)] \]

We shall first show that \( T : (X, d') \rightarrow (X, d') \) is a generalized \( c \)-contraction.

Since for any \( x, y \in X \) we have

\[ d^{-1}(Tx, Ty) = d(Ty, Tx) \]

\[ \leq k[d(Ty, y) + d(x, Tx) + d(Ty, x) + d(y, Tx)] \]

\[ \leq k[d^{-1}(y, Ty) + d^{-1}(Tx, x) + d^{-1}(Ty, y) + d^{-1}(Ty, x)] \]

\[ d^{-1}(Tx, Ty) \leq k[d^{-1}(y, Ty) + d^{-1}(x, Ty) + d^{-1}(Ty, x) + d^{-1}(Ty, y)] \]

We see that \( T : (X, d^{-1}) \rightarrow (X, d^{-1}) \) is a generalized \( C \)-pseudo contraction therefore

\[ d(Tx, Ty) \leq k[d(Ty, Ty) + d(x, Ty) + d(Ty, x) + d(x, Ty)] \]

\[ \leq k[d(x, Ty) + d'(x, Ty) + d'(Ty, y) + d'(y, Ty)] \]

and

\[ d^{-1}(Tx, Ty) \leq k[d^{-1}(y, Ty) + d^{-1}(x, Ty) + d^{-1}(Ty, x) + d^{-1}(Ty, y)] \]
\[ \leq k \left[ d^{a}(y, Ty) + d^{a}(x, Tx) + d^{a}(x, Ty) + d^{a}(Tx, y) \right] \] for all \( x, y \in X \)

Hence \( d^{a}(Tx, Ty) \leq k \left[ d^{a}(x, Tx) + d^{a}(y, Ty) + d^{a}(y, Ty) + d^{a}(x, Ty) \right] \) for all \( x, y \in X \)

and so \( T: \{X, d^{a}\} \rightarrow \{X, d^{a}\} \) is a generalized \( C\)-contraction.

By assumption \( (X, d) \) is a bi-complete. Hence \( (X, d^{a}) \) is complete. There fore by theorem (4a) \( T \) has a unique fixed point. This completes the proof.

**Corollary 4.1:** Let \( (X, d) \) be a \( T_{0}\)-Isbell-Complete quasi pseudo metric spaces and \( T: X \rightarrow X \) be a generalized \( C\) – contraction then \( T \) has a unique fixed point.

The proof follows from the proposition 2.1

**Corollary 4.2:** Any generalized \( C\)- pseudo contraction on a q-spherically complete \( T_{0}\) ultra quasi metric space has a unique fixed point.

The proof follows from the proposition 3.1

**Theorem 4.2:**

Let \( (X, d) \) be a bi complete di metric space and let \( T: X \rightarrow X \) be an generalized \( S\) pseudo contraction then \( T \) has a unique fixed point.

**Proof:** As in the previous proof it is enough to prove that \( T: \{X, d^{a}\} \rightarrow \{X, d^{a}\} \) is an generalized \( S\) –contraction.

Since \( T: X \rightarrow X \) be a \( S\) –pseudo contraction then there exist \( k \), \( 0 \leq k < \frac{1}{8} \) such that for all \( x, y \in X \) the following inequality holds:

\[ d^{a}(Tx, Ty) \leq k \left[ d^{a}(x, Tx) + d^{a}(y, Ty) + d^{a}(y, Ty) + d^{a}(x, Ty) + d^{a}(x, Ty) + d^{a}(y, Ty) + d^{a}(x, Ty) + d^{a}(y, Ty) \right] \]

We shall first show that \( T: \{X, d^{a}\} \rightarrow \{X, d^{a}\} \) is a generalized \( C\)-contraction.

Since for any \( x, y \in X \) we have

\[
\begin{align*}
&d^{-1}(Tx, Ty) = d(Ty, Tx) \\
&d(Ty, Tx) \leq k \left[ d^{-1}(Ty, y) + d^{-1}(Ty, x) + d^{-1}(Tx, x) + d^{-1}(Tx, x) + d^{-1}(Ty, y) + d^{-1}(Ty, y) \right] \\
&\leq k \left[ d^{-1}(Ty, y) + d^{-1}(Ty, x) + d^{-1}(Tx, x) + d^{-1}(x, Ty) + d^{-1}(x, Ty) \right] \\
&d^{-1}(Tx, Ty) \leq k \left[ d^{-1}(x, Tx) + d^{-1}(x, Ty) + d^{-1}(Tx, x) + d^{-1}(x, Ty) + d^{-1}(x, Ty) \right]
\end{align*}
\]

We see that \( T: \{X, d^{a}\} \rightarrow \{X, d^{a}\} \) is a pseudo contraction.

Therefore

\[
\begin{align*}
d(Tx, Ty) &\leq k \left[ d(x, Tx) + d(y, Ty) + d(y, Ty) + d(x, Ty) + d(x, Ty) + d(x, Ty) \right] \\
d(Tx, Ty) &\leq k \left[ d^{a}(x, x) + d^{a}(x, y) + d^{a}(x, y) + d^{a}(y, y) + d^{a}(y, y) + d^{a}(y, y) \right]
\end{align*}
\]

and

\[
\begin{align*}
d^{-1}(Tx, Ty) &\leq k \left[ d^{-1}(x, x) + d^{-1}(x, y) + d^{-1}(y, y) + d^{-1}(y, y) + d^{-1}(y, y) \right] \\
&\leq k \left[ d^{a}(x, x) + d^{a}(x, y) + d^{a}(x, y) + d^{a}(y, y) + d^{a}(y, y) \right]
\end{align*}
\]

For all \( x, y \in X \)

Hence

\[
d^{a}(Tx, Ty) \leq k \left[ d^{a}(x, x) + d^{a}(y, y) + d^{a}(x, y) + d^{a}(y, x) \right]
\]

for all \( x, y \in X \) and so \( T: \{X, d^{a}\} \rightarrow \{X, d^{a}\} \) is a generalized \( s\)-contraction.

By assumption \( (X, d) \) is a bi complete. Hence \( (X, d^{a}) \) is complete. There fore by theorem (4a) \( T \) has a unique fixed point. This completes the proof.

**Corollary 4.3:** Let \( (X, d) \) be a \( T_{0}\)-Isbell-Complete quasi pseudo metric spaces and \( T: X \rightarrow X \) be a pseudo contraction then \( T \) has a unique fixed point.

The proof follows from the proposition 2.1

**Corollary 4.4:** Any \( s\)-pseudo contraction on a q-spherically complete \( T_{0}\) ultra quasi metric space has a unique fixed point.

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