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Common Fixed Point Theorem in T_0 Quasi Metric Space

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Abstract

In this paper, we prove fixed point theorems for generalized C-contractive and generalized S-contractive mappings in a bi-complete di-metric space. The relationship between q- spherically complete T_0 Ultra-quasi-metric space and bi-complete diametric space is pointed out in proposition 3.2. This work is motivated by Petals and Fvidalis in a T_0-ultra-quasi-metric space

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Introduction

In Agyingi [1] proved that every generalized contractive mapping defined in a q- spherically complete T_0-ultra-quasi metric space has a unique fixed point. In Petals and Fvidalis [2] proved that every contractive mapping on a spherically complete non Archimedean normed space has a unique fixed point. Agyingi and Gega proved fixed point theorems in a T_0-ultra-quasi-metric space [3-5]. Later many authors published number of papers in this space [6-10].

In this paper we shall prove a fixed point theorem for generalized c- contractive and generalized s-contractive mappings in a bi-complete di-metric space.

If we delete, in the used definition of the pseudo metric d on the set X. the symmetry condition, $d(x, y)=d(y, x)$, whenever $x, y \in X$ we are led to the concept of quasi-pseudo metric.

Definition 1.1: Let (X,m) be a metric space. Let $T: X\to X$ map is called a C-contraction if there exist, $0 \le k < \frac{1}{2}$ such that for all $x, y \in X$ the following inequality holds [10],

$$m(Tx,Ty) \le k\left[m(x,Tx)+m(y,Ty)\right]$$

Definition 1.2: Let (X,m) be a metric space. A map $T: X\to X$ is called a S-contraction if there exist $0 \le k < \frac{1}{3}$ such that for all $x, y \in X$ the following inequality holds [10]

$$m(Tx,Ty) \le k\left[m(x,Tx)+m(y,Ty)+m(x,y)\right]$$

Preliminaries

Now we recall some elementary definitions and terminology from the asymmetric topology which are necessary for a good understanding of the work below.

Definition 2.1: Let X be a non empty set. A function $d: X\times X \to [0,\infty)$ is called quasi pseudo metric on X if

$$d(x,x)=0, \forall x \in X$$

$$d(x,z) \le d(x,y)+d(y,z), \forall x,y,z \in X$$

Moreover if $d(x,y)=0=d(y,x) \Rightarrow x=y$ then d is said to be a T_0 quasi metric or di-metric. The latter condition is referred as the T_0 condition.

Example 2.1: On R×R, we define the real valued map d given by $d(x,y)=|x-y|=\max\{|x-y|,0\}$ then (R,d) is a di metric space.

Remark 2.1

Let d be quasi-pseudo metric on X, then the map d^{-1} defined by $d^{-1}(x, y)=d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudo metric on X, called the conjugate of d.

It is also denoted by d^t or d^-. It is easy to verify that the function d^s defined by $d^S = d \vee d^{-1}$

i.e. $d^S(x,y)=\max\{d(x,y),d(y,x)\}$

defines a metric on X whenever d is a T_0 quasi pseudo metric.

In some cases, we need to replace $[0, \infty)$ by $[0,\infty]$ (where for a d attaining the value ∞, the triangle inequality is interpreted in the obvious way). In such case we speak of extended quasi- pseudo metric.

Definition 2.2: The di metric space (X, d) is said to be bi complete if the metric space (R, d^s) is complete.

Example 2.2: Let $X=[0, \infty)$ define for each $x, y \in X$, $n(x,y)=x$ if $x>y$ and $n(x,y)=0$ if $x<y$. It is not difficult to check that (X,n) is a T_0 quasi pseudo metric space. Notice that, for $x,y \in [0,\infty)$, we have $n^s(x,y)=\max\{x,y\}$ if $x \ne y$ and $n^s(x,y)=0$ if $x=y$, the matrix n^s is complete on (X,d).

Definition 2.3: Let (X,d) be quasi pseudo metric space, for $x, \in X \,\&\, \in > 0$

$$B_d(x,\in)=\{y \in X : d(x,y) < \in\}$$

denotes the open $\in -$ ball at x**.** The collection of such balls is a base for a topology $\tau(d)$ induced by d on X. Similarly for $x, \in X \,\&\, \in \ge 0$

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$C_d(x,\in) = \{y \in X : d(x,y) \le \in\}$

denotes the closed $\in -$ ball at x.

Definition 2.4: Let (X,d) be quasi pseudo metric space Let $(x_i), i \in I$ be a family of points in X and let $(r_i), i \in I$ & $(s_i), i \in I$ be a family of non negative real numbers.

We say that $\left(C_d(x_i,r_i), C_{d^{-1}}(x_i,s_i)\right), i \in I$ has the mixed binary intersection property provided that

$\left(C_d(x_i,r_i), C_{d^{-1}}(x_i,s_i)\right) \ne \varphi$ for all $i, j \in I$

Definition 2.5: Let (X,d) be quasi pseudo metric space we say that (X,d) is Isbell complete provided that each family $\left(C_d(x_i,r_i), C_{d^{-1}}(x_i,s_i)\right), i \in I$ that has the mixed binary intersection property is such that

$$\bigcap_{i \in I} \left(c_d(x_i,r_i)\right) \cap \left(c_{d^{-1}}(x_i,s_i)\right) \ne \varphi$$

Proposition 2.1: If (X,d) is an extended Isbell-complete quasi-pseudo metric space then (X,d^s) is hyper complete. An interesting class of quasi pseudo metric space, for which, we investing a type of completeness are the ultra quasi pseudo metric.

Definition 2.6: Let X be a set & $d : X \times X \to [0,\infty)$ be function a function mapping into the set $[0,\infty)$ of non negative real's then d is ultra quasi pseudo metric on X if

$d(x,x) = 0$ for all x in X &

$d(x,z) \le \max\{d(x,y), d(y,z)\}$ whenever $x,y,z \in X$

The conjugate d^{-1} of d where $d^{-1}(x,y) = d(y,x)$ whenever $x, y \in X$ is also an ultra quasi pseudo metric on X.

If d also satisfies the T_0 – condition, then d is called a T_0- ultra quasi metric on X. Notice that $d^s = \sup\left\{d, \bar{d}\right\} = d \vee d^{-1}$ is an ultra-metric on X whenever d is a T_0- ultra quasi metric.

In a literature, T_0- ultra quasi metric spaces are also known as non Archimedean T_0- quasi metric.

q-spherically Completeness

In this section we shall recall some results about q- spherical completeness belonging mainly to [8].

Definition 3.1: Let (X,d) be an ultra –quasi pseudo metric space Let $(x_i), i \in I$ be a family of points in X and let $(r_i), i \in I$ & $(s_i), i \in I$ be a family of non negative real numbers we say that (X,d) is q- spherical complete provided that each family [2]

$\left(C_d(x_i,r_i)\ C_{d^{-1}}(x_i,s_i)\right), i \in I$

Satisfying $d(x_i,s_i) \le \max\{r_i, s_j\}$, whenever $i, j \in I$ is such that

$$\bigcap_{i \in I} \left(c_d(x_i,r_i)\right) \cap \left(c_{d^{-1}}(x_i,s_i)\right) \ne \varphi$$

Proposition 3.2: Each q- spherically complete T_0 ultra quasi metric space (*X, d*) is bi-complete[8].

Main Results

We recall the following interesting results respectively due to Chatterji [10] and to Shukla [11]

Theorem 4.1a A C- contraction on a complete metric space has a unique fixed point.

Theorem 4.1b A S- contraction on a complete metric space has a unique fixed point.

Following results generalizes the above theorem to setting of a bi-complete di-metric space.

Definition 4.1: Let (X,d) be a quasi pseudo metric space. A map $T : X \to X$ is called a c-pseudo contraction if there exist k, $0 \le k < \frac{1}{2}$ such that for all *x, y* ∈ *X* the following inequality holds.

$$d(Tx,Ty) \le k\left[d(Tx,x) + d(y,Ty)\right]$$

Definition 4.2: Let (X,d) be a quasi-pseudo metric space. A map $T : X \to X$ is called a S-pseudo contraction if there exist k, $0 \le k < \frac{1}{3}$ such that for all *x, y* ∈ *X* the following inequality holds.

$$d(Tx,Ty) \le k\left[d(Tx,x) + d(y,Ty) + d(x,y)\right]$$

Now we define following definitions

Definition 4.3: Let (X,d) be a quasi-pseudo metric space. A map $T : X \to X$ is called a generalized c-pseudo contraction if there exist k, $0 \le k < \frac{1}{4}$ such that for all *x, y* ∈ *X* the following inequality holds.

$$d(Tx,T) \le k\{d(,Tx,x) + d(y,Ty) + d(Tx,y) + d(x,Ty)\}$$

Definition 4.4: Let (X,d) be a quasi-pseudo metric space. A map $T : X \to X$ is called a generalized S-pseudo contraction if there exist k, $0 \le k < \frac{1}{8}$ such that for all *x, y* ∈ *X* the following inequality holds.

$$d(Tx,Ty) \le k\{d(x,Tx) + d(x,Ty) + d(y,Tx) + d(y,Ty) + d(Tx,x) + d(Ty,x) + d(Tx,y) + d(Ty,y)\}$$

Theorem 4.1:

Let (X,d) be a bi complete di metric space and let $T : X \to X$ be a generalized c- pseudo contraction then T has a unique fixed point.

Proof: Since $T : X \to X$ is a generalized c-pseudo contraction then there exist k, $0 \le k < \frac{1}{4}$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx,Ty) \le k\{d(,Tx,x) + d(y,Ty) + d(Tx,y) + d(x,Ty)\}$$

We shall first show that $T : \left(X, d^s\right) \to \left(X, d^s\right)$ is a generalized c-contraction.

Since for any *x, y* ∈ *X* we have

$$d^{-1}\left(Tx,Ty\right) = d(Ty,Tx)$$

$$\le k\{d(Ty,y) + d(x,Tx) + d(Ty,x) + d(y,Tx)\}$$

$$\le k\left\{d^{-1}(y,Ty) + d^{-1}(Tx,x) + d^{-1}d(x,Ty) + d^{-1}(Tx,y)\right\}$$

$$d^{-1}(Tx,Ty) \le k\left\{d^{-1}(y,Ty) + d^{-1}(Tx,x) + d^{-1}d(x,Ty) + d^{-1}(Tx,y)\right\}$$

We see that $T : \left(X, d^{-1}\right) \to \left(X, d^{-1}\right)$ is a generalized C-pseudo contraction therefore

$$d(Tx,Ty) \le k\{d(Tx,x) + d(y,Ty) + d(Tx,y) + d(x,Ty)\}$$

$$\le k\left\{d^s(x,Tx) + d^s(y,Ty) + d^s(Tx,y) + d^s(x,Ty)\right\}$$

and $d^{-1}(Tx,Ty) \le k\left\{d^{-1}(y,Ty) + d^{-1}(Tx,x) + d^{-1}(x,Ty) + d^{-1}(Tx,y)\right\}$

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$$\leq k\left\{d^s(y,Ty)+d^s(x,Tx)+d^s(x,Ty)+d^s(Tx,y)\right\} \text{ for all } x, y \in X$$

Hence $d^s(Tx,Ty) \leq k\left\{d^s(x,Tx)+d^s(y,Ty)+d^s(Tx,y)+d^s(x,Ty)\right\}$, for all $x, y \in X$

and so $T:\left(X,d^s\right)\to\left(X,d^s\right)$ is a generalized C- contraction.

By assumption $\left(X,d\right)$ is a bi complete. Hence $\left(X,d^s\right)$ is complete. There fore by theorem (4a) T has a unique fixed point. This completes the proof.

Corollary 4.1: Let $\left(X,d\right)$ be a T_0-Isbell-Complete quasi pseudo metric spaces and $T: X \to X$ be a generalized c – pseudo contraction then T has a unique fixed point.

The proof follows from the proposition 2.1

Corollary 4.2: Any generalized c- pseudo contraction on a q-spherically complete T_0 ultra quasi metric space has a unique fixed point.

The proof follows from the proposition 3.1

Theorem 4.2:

Let $\left(X,d\right)$ be a bi complete di metric space and let $T: X \to X$ be an generalized S pseudo contraction then T has a unique fixed point

Proof: As in the previous proof it is enough to prove that $T:\left(X,d^s\right)\to\left(X,d^s\right)$ is an generalized S –contraction.

Since $T: X \to X$ be a S –pseudo contraction then there exist k, $0 \leq k < \frac{1}{8}$ such that for all $x, y \in X$ the following inequality holds:

$$d\left(Tx,Ty\right) \leq k\left\{d(x,Tx)+d(x,Ty)+d(y,Tx)+d(y,Ty)+d(Tx,x)+d(Ty,x)+d(Tx,y)+d(Ty,y)\right\}$$

We shall first show that $T:\left(X,d^s\right)\to\left(X,d^s\right)$ is a generalized C-contraction.

Since for any $x, y \in X$ we have

$$d^{-1}\left(Tx,Ty\right)=d(Ty,Tx)$$

$$d\left(Ty,Tx\right) \leq k\left\{d(y,Ty)+d(y,Tx)+d(x,Ty)+d(x,Tx)+d(Ty,y)+d(Tx,y)+d(Ty,x)+d(Tx,x)\right\}$$

$$\leq k\left\{\begin{matrix} d^{-1}(Ty,y)+d^{-1}(Tx,y)+d^{-1}(Ty,x)+d^{-1}(Tx,x) \\ +d^{-1}(y,Ty)+d^{-1}(y,Tx)+d^{-1}(x,Ty)+d^{-1}(x,Tx)\end{matrix}\right\}$$

$$d^{-1}(Tx,Ty) \leq k\left\{\begin{matrix} d^{-1}(x,Tx)+d^{-1}(x,Ty)+d^{-1}(y,Tx)+d^{-1}(y,Ty) \\ +d^{-1}(Tx,x)+d^{-1}(Ty,x)+d^{-1}(Tx,y)+d^{-1}(Ty,y)\end{matrix}\right\}$$

We see that $T:\left(X,d^{-1}\right)\to\left(X,d^{-1}\right)$ is a pseudo contraction.

Therefore

$$d\left(Tx,Ty\right) \leq k\left\{d(x,Tx)+d(x,Ty)+d(y,Tx)+d(y,Ty)+d(Tx,x)+d(Ty,x)+d(Tx,y)+d(Ty,y)\right\}$$

$$d(Tx,Ty) \leq k\left\{d^s(Tx,x)+d^s(x,Ty)+d^s(y,Tx)+d^s(y,Ty)+d^s(Tx,x)+d^s(Ty,x)+d^s(Tx,y)+d^s(Ty,y)\right\}$$

and

$$d^{-1}(Tx,Ty) \leq k\left\{\begin{matrix} d^{-1}(x,Tx)+d^{-1}(x,Ty)+d^{-1}(y,Tx)+d^{-1}(y,Ty) \\ +d^{-1}(Tx,x)+d^{-1}(Ty,x)+d^{-1}(Tx,y)+d^{-1}(Ty,y)\end{matrix}\right\}$$

$$\leq k\left\{\begin{matrix} d^{n}(Tx,x)+d^{n}(Ty,x)+d^{n}(Tx,y)+d^{n}(Ty,y) \\ +d^{n}(x,Tx)+d^{n}(x,Ty)+d^{n}(y,Tx)+d^{n}(y,Ty)\end{matrix}\right\} \text{ For all } x, y \in X$$

Hence

$$d^{n}(Tx,Tx) \leq k\left\{\begin{matrix} d^{n}(Tx,x)+d^{n}(Ty,x)+d^{n}(Tx,y)+d^{n}(Ty,y) \\ +d^{n}(x,Tx)+d^{n}(x,Ty)+d^{n}(y,Tx)+d^{n}(y,Ty)\end{matrix}\right\}, \text{ for all } x, y \in X$$

and so $T:\left(X,d^s\right)\to\left(X,d^s\right)$ is a generalized s-contraction.

By assumption $\left(X,d\right)$ is a bi complete. Hence $\left(X,d^s\right)$ is complete. There fore by theorem (4a) T has a unique fixed point. This completes the proof.

Corollary 4.3: Let $\left(X,d\right)$ be a T_0-Isbell-Complete quasi pseudo metric spaces and $T: X \to X$ be a pseudo contraction then T has a unique fixed point.

The proof follows from the proposition 2.1

Corollary 4.4: Any s-pseudo contraction on a q-spherically complete T_0 ultra quasi metric space has a unique fixed point.

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