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Collocation Method Based on Bernoulli Polynomial and Shifted Chebychev for Solving the Bratu Equation

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Abstract

In this work, Bernoulli-collocation method is proposed for solving nonlinear Bratu's type equations. The operational matrix of derivative of Bernoulli is introduced. The matrix together with the collocation method are then utilized to reduce the problem into a system of nonlinear algebraic equations. Also, a reliable approach for solving this nonlinear system is discussed. Numerical results and comparisons with other existing methods provided in the literature are made.

Keywords: Bratu's problem; Bernoulli polynomial; Nonlinear; Shifted chebychev

Mathematics Subject Classification Primary: 34B15; Secondary 65N35.

Introduction

Nonlinear boundary value problems (BVPs) of ordinary type plays an important role in all branches of science and engineering especially the two point BVPs. These types of equations appears in a wide variety of problems including but not limited to chemical reactions, hear transfer and solution of optimal control problems. Therefore, the need for fast and efficient methods for solving this type of equations is a must.

In this work we will develop a collocation approach based on Bernoulli polynomials for solving the famous Bratu's equation in the form

$$u''(x)+\lambda e^{\pm u(x)=0}\ , \qquad x\in(0,1)$$
$$u(0)=u(1)=0 \tag{1}$$

The closed form for the exact solution to eqn. (1) is

$$u(x)=-2\text{In}\left[\frac{-\cosh((x-\frac{1}{2})\frac{\theta}{2}}{\cos\left(\frac{\theta}{4}\right)}\right] \tag{2}$$

where θ satisfies $\theta=\sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right)$. Eqn. (1) has zero, one or two solutions when $\lambda>\lambda_c;\lambda=\lambda_c$ and $\lambda<\lambda_c$ respectively, where the critical value λ_c satisfies the following equation $\lambda_c=8\sec^2\left(\frac{\theta_c}{4}\right)$. It was evaluated from that the critical value of λ_c has a value of λ_c=3.513830719.

This type of equation has many applications including the modelling of a combustion in a numerical slab, the ignition of fuel of the thermal theory, the thermal reaction process modelling, the expansion of the universe model and open questions regarding this theory, chemical reaction theory and nanotechnology [1].

There has been numerous analytic and numerical methods that has been applied for solving Bratu's equation with different forms of exact solutions [2,3]. For instance, B-spline method [4], parametric spline [5], non-polynomial spline [6], quintic spline [7], cubic spline [8,9], Sinc-Galerkin [10,11], lie group shooting [12], Adomian decomposition method [13-17], Homotopy perturbation method [18-20], optimal perturbation [21], successive differentiation method [22,23], Chebychev wavelet [24,25], Legendre wavelet [26], variation iteration [27-29], iterative finite difference [30], genetic algorithm based methods [31], multi step iterative [32], neural network [33,34], particle swarm shooting [35] and pseudospectral method [36,37]. In addition to the standard Bratu problem, there are other Bratu-type problems which will be introduced and examined later.

Bernoulli polynomials have gained increasing importance in numerical analysis since they are straightforward and need less computational errors. Many researchers have been working on proving the efficiency of this method [38-43].

The organization of the paper is as follows. We recall the basic concepts of Bernoulli polynomials and their relevant properties needed hereafter. Bernoulli method is presented for solving the general Bratu's type equations. Some numerical examples are presented along with a comparison with other techniques. Finally, the closing stage which provides the conclusions of the study [44-47].

Fundamental Relations

Bernoulli polynomials play an important role in different areas of mathematics, including number theory and the theory of finite differences. They are also can be found in the integral representation of the differentiable periodic functions, since they are employed for approximating such functions in term of polynomials. The classical Bernoulli polynomials $B_n(x)$ is usually defined by means of exponential generating functions [40].

$$\frac{te^{tx}}{e^{t-1}}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!}$$

from which we can find the following known expansion

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$$\sum_{k=0}^{n}\binom{n+1}{k}B_k(x)=(n+1)x^n$$

which is the most known expansion of the Bernoulli polynomials and from we can generate the first few polynomials as Figure 1

$$B_0(x)=1,\quad B_1(x)=x-\frac{1}{2},$$
$$B_2(x)=x^2-x+\frac{1}{6},\quad B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x,$$
$$B_4(x)=x^4-2x^3+x^2-\frac{1}{30}$$

these polynomials have many interesting properties from which the following Figure 1

$$\frac{dB_n(x)}{dx}=nB_{n-1}(x),\quad (n\geq 1),\int_0^1 B_n(x)dx=\frac{B_{n+1}(1)-B_{n+1}(0)}{n+1},$$
$$B_0(x)=1\quad B_n(x+1)-B_n(x)=nx^{n-1},$$
$$B_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}B_k(x),\quad B_n(1-x)=(-1)^n B_n(x)$$
$$(-1)^n B_n(-x)=B_n(x)+nx^{n-1}$$

In the next we will introduce Bernoulli matrix of differentiation that will be needed later.

Bernoulli operational matrix of differentiation

We will use Bernoulli approximation technique to approximate the solution of eqn. (1) expressed in the truncated Bernoulli series in the form

$$u_N(x)=\sum_{n=0}^{N}c_nB_n(x)=\mathbf{B}(x)\mathbf{c} \tag{3}$$

where $\{c_n\}_{n=0}^{N}$ are the unknown Bernoulli coe cients, N is any chosen positive integer such that N ≥2, and $B_n(x)$,n=0; 1; : : : ; N are the Bernoulli polynomial of the rst kind which are constructed according to equation (2), the Bernoulli coeffcient vector c and the Bernoulli vector **B**(x) are given by

$c^t=[c_0; c_{1,\ldots..,}c_N]$; $\mathbf{B}(x)=[B_0(x); B_1(x); : : : ; B_N(x)]$.

According to eqn. (3)

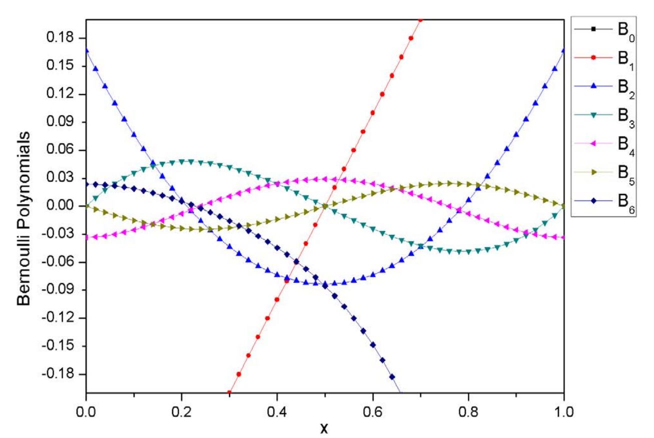

Figure 1: Shows the first few Bernoulli polynomials on the interval [0,1].

$$\underbrace{\begin{bmatrix}B_0(x)\\ B_1(x)\\ \vdots\end{bmatrix}}_{B^t(x)}=\underbrace{\begin{bmatrix}B_0(0) & 0 & \cdots & 0\\ \binom{1}{1}B_1(0) & \binom{1}{0}B_0(0) & \cdots & 0\\ \binom{2}{2}B_2(0) & \binom{2}{1}B_1(0) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \binom{N}{N}B_N(0) & \binom{N}{N-1}B_{N-1}(0) & \cdots & \binom{N}{0}B_0(0)\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}1\\ x\\ \vdots\\ x^N\end{bmatrix}}_{\Omega^t(x)}$$

Since **D** is a lower triangular matrix with nonzero diagonal elements and det (**D**)=1, so **D** is an invertible matrix. Thus, the Bernoulli vector can be given directly from

$$B(x)=\Omega(x)\,\mathbf{D}^t \tag{4}$$

note that $[]^t$, denotes transpose of the matrix [] and $\mathbf{B}^t(x)$ and Ω(x) be the (N+1)×1 and D is the (N+1)×(N+1) operational matrix whose elements are

$$\{D\}_{i,j=1}^{N+1}=\begin{cases}\binom{i-1}{j-1}B_{i-j}, & i\geq j\\ 0, & i<j\end{cases} \tag{5}$$

Now, the matrix forms of the solution functions as

$$y_N=\Omega(x)\,\mathbf{D}^t\,c \tag{6}$$

According to the eqn. (5) the following formula is concluded evidently. Also, the relation between Ω(x) and it's ith derivative $\Omega^{(1)}(x)$ is

$$\Omega^{(1)}(x)=\underbrace{\left[1,x,x^2,\ldots..,x^N\right]}_{\Omega(x)}\underbrace{\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & N\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}}_{M} \tag{7}$$

and the following formula holds as

$$\Omega^{(k)}(x)=\Omega(x)M^k,\quad k=1,2 \tag{8}$$

where $\Omega^{(i)}(x)$ is denoting the it's derivative of Ω(x), we have

$$u_N^{(k)}=\Omega(x)M^kD^tc,\quad k=1,2 \tag{9}$$

Application of the Proposed Method

First, we need to treat the nonlinear term in eqn. (1) by expanding it using Taylor series expansion in the form

$$e^{\pm u(x)}\approx 1\pm u(x)+\frac{u^2(x)}{2!}\pm\frac{u^3(x)}{3!}+\frac{u^4(x)}{4!}\pm\ldots \tag{10}$$

By substituting the expanded term from eqn. (10) into eqn. (1), the equation becomes

$$u''(x)+\lambda\left[1\pm u(x)+\frac{u^2(x)}{2!}\pm\frac{u^3(x)}{3!}+\frac{u^4(x)}{4!}\pm\ldots\right]=0 \tag{11}$$

Second, we will use the shifted chebychev defined on the interval [0,1] in the form

$$x_k=\frac{1}{2}\left[1-\cos\left(\frac{k\pi}{N}\right)\right],\quad k=0,1,\ldots..,N$$

After substituting those collocation points into eqn. (11), we reach the following system

$$u''(x)+\lambda\left[1\pm u(x)+\frac{u^2(x)}{2!}\pm\frac{u^3(x)}{3!}+\frac{u^4(x)}{4!}\pm\ldots\right]=0 \qquad (12)$$

from eqn. (12), we need to approximate the term $u^v(x_k)$,k =0,1, 2, 3; : : : . We will need the following theorem.

Theorem

The approximation of the function $u^v(x_k)$, k=0,1,…., N can be represented according to the following relation

$$\begin{pmatrix} u^v(x_0) \\ u^v(x_1) \\ \vdots \\ u^v(x_N) \end{pmatrix} = \begin{pmatrix} u(x_0) & 0 & 0 & \cdots & 0 \\ 0 & u(x_1) & 0 & \cdots & 0 \\ 0 & 0 & u(x_2) & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & u(x_N) \end{pmatrix}^{v-1} \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_N) \end{pmatrix} \qquad (13)$$

$$=\left(\tilde{U}\right)^{v-1}U$$

$$=\left(\tilde{B}\tilde{C}\right)^{v-1}(BC)$$

Where

$$\tilde{B}=\begin{pmatrix} B(x_0) & 0 & 0 & \cdots & 0 \\ 0 & B(x_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & B(x_N) \end{pmatrix},\ \tilde{C}=\begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & c \end{pmatrix},$$

$$B=\begin{pmatrix} B_0(x_0) & B_1(x_0) & B_2(x_0) & \cdots & B_N(x_0) \\ B_0(x_1) & B_1(x_1) & B_2(x_1) & \cdots & B_N(x_1) \\ B_0(x_2) & B_1(x_2) & B_2(x_2) & \cdots & B_N(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_0(x_N) & B_1(x_N) & B_2(x_N) & \cdots & B_N(x_N) \end{pmatrix}$$

By substituting the above theorem into eqn. (12), we reach the following theorem.

Theorem

If the assumed approximate solution of the problem eqn. (12) is eqn. (8), then the discrete Bernoulli system is

$$u''(x_k)+\lambda\left[u(x_k)+\frac{u^v(x_k)}{n!}\right]=f(x_k),\ \ 0\le k\le N,\ n,v=2,3,4.... \qquad (14)$$

Proof: If we replace each term of (12) with its corresponding approximation given

by eqns.(2), (9) and (13) and substituting $x=x_k$ collocation points.

The matrix form for this system is

$$\Theta c=F \qquad (15)$$

Where

$$\Theta=BM^2+\lambda\left[\mathbf{B}+\left(\tilde{\mathbf{B}}\tilde{\mathbf{C}}\right)^{v-1}\mathbf{B}\right],$$

And

$$F=\begin{pmatrix} -\lambda \\ -\lambda \\ \vdots \\ -\lambda \end{pmatrix},\ \lambda=\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

The matrix form of the boundary conditions represented in eqn. (1) will be in the form

$$B(0)=[B_0(0),B_1(0),B_2(0),...,B_N(0)]=0$$

$$B(1)=[B_0(1),B_1(1),B_2(1),...,B_N(1)]=0 \qquad (16)$$

By replacing two rows of the augmented matrix [Θ: **F**] with the boundary conditions defined from eqn. (16), we have

$$\tilde{\Theta}_c=\tilde{\mathbf{F}} \qquad (17)$$

Now we have a nonlinear system of N+1 equation in N+1 unknown coefficient c. We can obtain these coefficients by solving the above nonlinear system using the following algorithm.

Algorithm

1. Input (integer) N.

Input (double) tol.

Input (array) $c_{old}=c_0$, (initial approximation, c_0 with N +1 dimension, are chosen so that the boundary conditions are satis ed).

2. $\tilde{\Theta}=(c_{old}):c_{new}=\tilde{\mathbf{F}}$ is a linear algebraic equation system which is solved and c_{new} is found.

Go to (2).

2.1 If $|C_{old}-C_{new}|$<tol then c_{new}=c, break (the program is finished).

2.2 Else then $c_{old}\leftarrow c_{new}$.

3. Go to (2).

Numerical Examples

To illustrate the ability, reliability and the performance of the proposed method for Bratu's problem, some examples are provided. The results reveal that the method is very effective and simple. All computations were carried out using Matlab 2014a on a personal computer. The absolute error can be calculated according to the following

$$||\mathbf{E}_N(x)||=^{\max}|^{u}\text{Exact}\,(x_k)-u_{\text{Bernoulli}}(x_k)|,\ k=0;1;2;::::$$

Example 1

First, we consider the initial value problem in the form [31,9,25,21,15,46]

$$u''(x)"-2e^{u(x)}=0;\quad x\in(0,1)$$

subject to the initial conditions

$$u'(0)=u(0)=0$$

which has the exact solution given by

u(x)=−2ln[cos(x).

Table 1 exhibits the maximum absolute error at different values of N along with the elapsed CPU time in seconds. Also, in Table 2 a comparison between the reported results in [31, 9, 25, 21, 15, 46] along with our method at N=18. This table indicates that our method provide better results than the other methods. Figure 2 demonstrates the Bernoulli approximate solution versus the exact solution for x 2 [0; 1].

Example 2

Next, we apply our method for the solution of the special form of Bratu equation [4-12,17,25,28,30,33-34,46]

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$$u''(x)+\lambda e^{u(x)}=0,\ x\in(0,1)$$

subject to the boundary conditions

u(0)=u(1)=0.

with the exact solution

N	$\|\|E_N(x)\|\|$	CPU time (sec)
4	1.39189E-02	1.264
6	8.70130E-04	1.329
8	5.39038E-05	1.670
10	3.33058E-06	2.310
12	2.05470E-07	2.868
14	1.26651E-08	3.674
16	7.80110E-10	4.469
18	4.80861E-11	11.98

Table 1: Maximum absolute error and CPU time for Example 4.1.

Presented method, N=18	4.809E-11
Optimal spline method [9]	3.450E-06
Restarted Adomian decomposition [15]	5.745E-04
Optimal perturbation iteration [21]	4.230E-04
Chebychev wavelet method[25]	1.528E-05
ASM method [31]	6.220E-07
Taylor wavelet [46]	7.801E-08

Table 2: Comparison of maximum absolute error for Example 1.

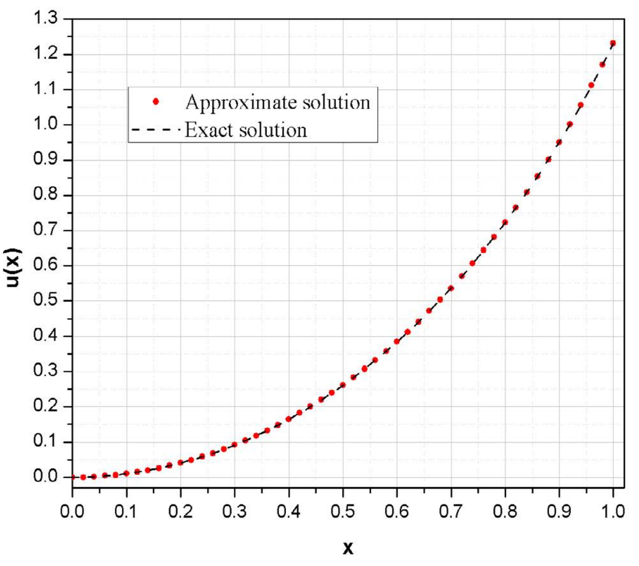

Figure 2: Bernoulli solution and exact solution for example 1.

$$u(x)=-2In\left[\frac{-\cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right] \quad (18)$$

The form of the above equation is very familiar and has a tremendous work for solving it as mentioned in the literature. We applied our method with various (Figure 2) values for=0:1; 0:5; 1; 2; 3 and 3:51. The computed maximum absolute errors at different values of N and are tabulated in Table 3. A comparison with the other methods reported in the literature are presented in Table 4 shows that our method is computationally effective even for=3:51 which is near the critical value. The graph of the approximate solutions for different values of has been plotted in Figure 3.

Example 3

Now, we turn our attention to the BVP of Bratu's equation in the form [21,31]

$$u''("x)+\pi^2 e^{-u(x)}=0,\ \ x\in(0,1)$$

subject to the boundary conditions

u(0)=u(1)=0.

This equation is a standard Bratu equation for which $\lambda=\pi^2>\lambda_c$ which has two possible solutions in the form u(x)=ln(1±sin(x)). This solution with the negative sign blows up at x=0:5 so we will use the solution of positive sign which is convergent and bounded in the form

u(x)=ln [1+sin(x)]

Maximum absolute error is tabulated in Table 5 along with the CPU time and a comparison is made in Table 6 between our method along with the other methods in [21,31]. From Table 6, we noticed that our method is more accurate than the other existing methods. Figure 4 demonstrates the Bernoulli approximate solution and the exact which appears to be in good agreement with each other.

Example 4

Finally, consider another form of Bratu's equation

$$u''("x)+\pi^2 e^{-u(x)}=0,\ \ x\in(0,1)$$

subject to the boundary conditions

u(0)=u(1)=0.

with the exact solution

u(x)=-log [1-cos ((0:5+x)π)].

The maximum absolute error for this problem is tabulated in Table 7

N	λ=0:1	λ=0:5	λ=1:0	λ=2:0	λ=3:0	λ=3:51
4	2.2697E-08	3.4346E-06	3.60159E-05	5.8666E-04	7.4020E-03	2.8391E-02
6	7.2250E-12	5.3149E-09	1.06201E-07	2.8664E-06	5.3078E-05	7.7256E-03
8	3.9100E-15	2.4657E-11	1.12485E-09	8.6920E-08	3.1672E-06	4.1393E-04
10	2.3922E-15	4.7454E-13	1.15237E-11	1.5694E-09	8.8354E-08	9.0424E-06
12	2.3939E-15	4.4039E-13	3.3323E-12	1.2423E-10	4.4392E-09	1.1319E-06
14	-	-	3.4237E-12	6.4105E-12	3.1209E-10	7.8326E-08
16	-	-	-	5.9817E-12	1.2622E-10	7.0247E-09
18	-	-	-	-	-	5.5903E-10

Table 3: Maximum absolute error at different λ Example 2.

Method	λ=1:0	λ=2:0	λ=3:51
Presented method, N=14	3.4237E-12	6.4105E-12	5.5903E-10
Sinc galerkin [10]	2.010E-10	1.801E-11	1.4528E-07
Optimal spline [9]	1.810E-07	3.940E-11	×
Chebychev wavelet [25]	×	5.487E-06	×
Spline method [8]	8.773E-05	8.971E-04	6.890E-06
Lie group shooting [12]	1.018E-06	5.220E-06	7.305E-05
B-spline method [4]	8.892E-06	5.561E-05	1.350E-01
Restarted Adomian [17]	9.410E-07	2.300E-04	×
Adomian with Taylor [17]	4.900E-06	6.800E-04	×
Spline method [6]	2.178E-06	7.264E-07	×
Variational iteration [28]	4.210E-05	1.416E-03	×
Parametric spline [5]	5.870E-10	3.530E-08	×
Finite di erence [30]	5.703E-10	2.096E-09	6.21E-07
Neural network [34]	3.20E-03	4.95E-03	1.76E-02
Particle swarm shooting [35]	1.150E-08	6.71E-09	3.58E-06
Mexican Hat wavelet [33]	1.90E-08	3.75E-08	1.72E-05
Taylor wavelet [46]	7.760E-12	1.190E-09	1.31E-06
Matlab routine bvp4c [47]	2.47E-07	1.33E-06	1.43E-02
Mathematica routine NDsolve	5.71E-09	5.88E-09	3.01E-07

Table 4: Comparison of maximum absolute error at different λ for Example 2.

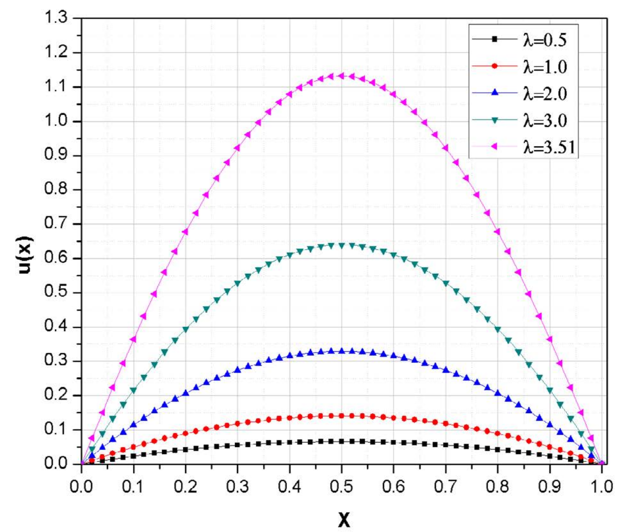

Figure 3: For λ = 0.5,1.0,2.0,3.0 and λ = 3.51, the approximate solution of the purposed method for N = 14.

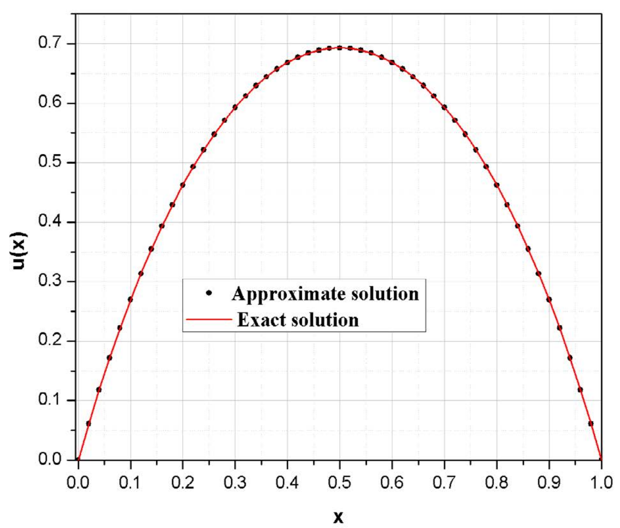

Figure 4: Bernoulli solution and exact solution for example 3.

N	$kE_N(x)k$	CPU time (sec)
4	2.20480E-03	1.264
6	6.78106E-05	1.329
8	2.64416E-06	1.670
10	1.13839E-07	2.310
12	5.62457E-09	2.868
14	2.64951E-10	3.674
16	4.76705E-11	4.469
18	9.04055E-13	11.98

Table 5: Maximum absolute error and CPU time for Example 3.

Presented method, N=18	9.04055E-13
Optimal perturbation iteration [21]	8.886E-07
ASM method [31]	8.034E-06
GA-ASM method [31]	3.449E-05
GA method [31]	2.088E-02

Table 6: Comparison of maximum absolute error for Example 3.

N	$\|\|E_N(x)\|\|$	CPU time (sec)
4	2.20480E-03	1.065
6	6.78106E-05	1.269
8	2.64416E-06	1.696
10	1.13839E-07	2.266
12	5.62457E-09	2.859
14	2.64951E-10	3.624
16	4.76705E-11	4.794
18	7.46625E-13	5.573

Table 7: Maximum absolute error and CPU time for Example 4.

at different values of N along with the CPU time. A comparison is made along with other methods which is given in the literature in Table 8 which proves the applicability of the purposed algorithm of providing good results. Figure 5 demonstrates the Bernoulli approximate solution and the exact which appears to be in good agreement with each other.

Presented method, N=18	7.46625E-13
Sinc-Galerkin [10]	8390E-12
Optimal Spline method [9]	4.341E-10
GA method [31]	1.293E-06
Taylor Wavelet [46]	1.434E-07

Table 8: Comparison of maximum absolute error for Example 4.

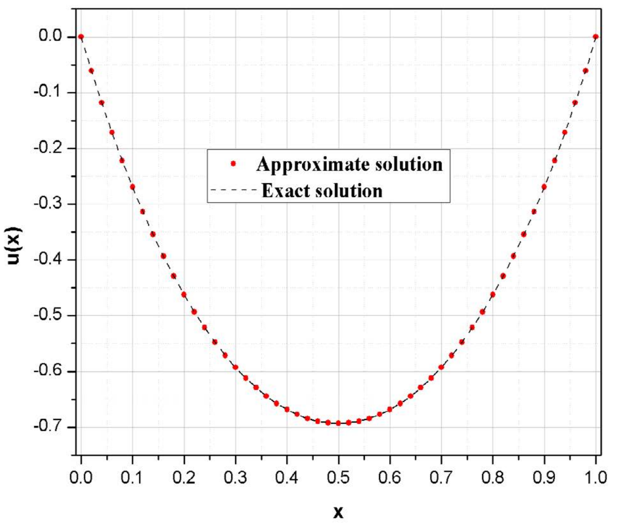

Figure 5: Approximate solution versus exact solution for example 4.

Conclusion

In this paper, we showed that Bernoulli-collocation method can be utilized to and an approximate solution of the nonlinear Bratu's type equations. The method reduces the problem into a system of nonlinear algebraic equations and this system is solved using a novel technique. Also, the efficiency of the method with respect to the other method was shown. In comparison to other methods, we illustrated that Bernoulli-collocation method has very high accuracy.

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