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Coefficient Inequalities for Uniformly P-Valent Starlike and Convex Functions

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Abstract

In this paper we obtain some coefficient inequalities for subclasses of uniformly p−valent starlike and convex functions in the open unit disk denoted by SD_p (β, α) and KD_p (β, α). Growth bounds and distortion bounds are discussed for functions in these classes. For different values of the parameters p, α and β our results of this paper generalize those obtained by several authors in the literature.

Keywords: P-valent functions; Starlike functions; Convex functions

Introduction

Let $U=\{z\in\mathbb{C}:|z|<1\}$ denote the open unit disk and let A_p be the class of functions f(z) of the form $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n,\ p\in\mathbb{N}=\{1,2,...\}$ which are analytic in the open unit disk U. A function f $\in A_p$ is said to be p−valent starlike of order α (0 ≤ α<p), if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\ z\in U.$$

The class of all such functions is denoted by $S_p^*(\alpha)$. A function f $\in A_p$ is said to be p−valent convex of order α (0 ≤ α<p), if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\ z\in U$$

Let $K_p(\alpha)$ denote the class of all such functions. For p=1 we write A1=A. Note that for p=1 the classes $S_1^*(\alpha)$ and $K_1(\alpha)$ are the usual classes of starlike and convex functions of order α(0 ≤ α<1) respectively, and will be denoted by S* (α) and K(α) respectively. For p=1 and α=0, the classes $S_p^*(\alpha)$ and Kp(α) reduces to S*(0)=S*and K(0)=K respectively, which are the classes of starlike (with respect to the origin) and convex functions.

The Subclasses SD_p (β,α) and KD_p (β,α)

We begin this Section by remark that this article is motivated by the work of Owa et al. [1]. We now recall the definitions of the subclasses SD_p (β, α) and KD_p (β, α) of uniformly p−valent function introduced and studied by Agnihotri and Singh [2].

A function f∈Ap is said to be in the class SD_p (β, α) if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\beta\left|\frac{zf'(z)}{f(z)}-p\right|+\alpha,\ z\in U,$$

for some β ≥ 0 and α (0 ≤ α<p).

A function f ∈Ap is said to be in the class KDp (β,α) if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\beta\left|\frac{zf''(z)}{f'(z)}-(p-1)\right|+\alpha,\ z\in U,$$

for some β ≥ 0 and α (0 ≤ α<p). Note that f(z) ∈ KD_p (β, α) if and only if zf′(z) ∈ SD_p (β, α). Agnihotri and Singh [2] have shown some sufficient conditions for f to be in the classes SD_p (β, α) and KD_p (β, α).

The subclasses SD_1(β, α) and KD_1(β, α) which will also be denoted by SD(β, α) and KD(β, α) respectively were studied by Shams, Kulkarni and Jahangiri in [3]. They have obtained sufficient conditions for f to be in the classes SD (β, α) and KD (β, α).

Coefficient Inequalities

We now give coefficient inequalities for functions belonging to the subclasses SD_p (β, α) and KD_p (β, α). Our first result is contained in

Theorem 3.1: If $f\in SD_p(\beta,\alpha)$ with $0\le p\beta\le\alpha<p$, then $f\in s_p^*\left(\frac{\alpha-p\beta}{1-\beta}\right)$ and if $\beta>\frac{p+\alpha}{2p}$ then $f\in s_p^*\left(\frac{\alpha-p\beta}{\beta-1}\right)$

Proof: We know that $\Re(z)\le|z|$ for any complex number z. Therefore $f\in SD_p(\beta,\alpha)$ gives us

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\beta\Re\left(\frac{zf'(z)}{f(z)}-p\right)+\alpha \qquad (3.1)$$

From this we get

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\frac{\alpha-p\beta}{1-\beta}(z\in U). \qquad (3.2)$$

Now, if $0\le p\beta\le\alpha<p,$ then it follows that

$$0\le\frac{\alpha-p\beta}{1-\beta}<p,$$

and if $\beta>\frac{p+\alpha}{2p}$, then we have

$$-p<\frac{p\beta-\alpha}{\beta-1}\le 0.$$

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Thus, $0 \le \frac{\alpha - p\beta}{\beta - 1} < p,$

For p=1, we obtain the following corollary due to Owa, Polatoˇglu, and Yuvaz [1].

Corollary 3.1: If f ∈ SD (β,α) with $0 \le pB \le \alpha$ then $f \in S^*\left(\frac{\alpha-\beta}{1-\beta}\right)$.

Next, we state the corresponding result for functions belonging to the subclass $KD_p(\beta,\alpha)$.

Theorem 3.2: If $f \in KD_p(\beta,\alpha)$ with $0 \le pB \le \alpha < p$ then $f \in K_p\left(\frac{\alpha - p\beta}{1-\beta}\right)$ and if $\beta > \frac{p+\alpha}{2p}$, then $f \in K_p\left(\frac{\alpha - p\beta}{\beta - 1}\right)$

Proof: Proof is similar to the proof of Theorem 3.1.

The following corollary is due to Owa, Polatoˇglu, and Yuvaz [1] for p=1.

Corollary 3.2: If $f \in KD(\beta,\alpha)$ with $0 \le \beta \le \alpha$ then $f \in K\left(\frac{\alpha-\beta}{1-\beta}\right)$

We now state the main theorem of this paper.

Theorem 3.3: If $f \in SD_p(\beta,\alpha)$ then $|a_p + 1| \le \frac{2(p-\alpha)}{|1-\beta|}$ (3.3)

and $|a_p + n| \le \frac{2(p-\alpha)}{n|1-\beta|}\prod_{j=1}^{n-1}\left(1 + \frac{2(p-\alpha)}{j|1-\beta|}\right) (n \ge 2)$ (3.4)

Proof: We know that if $f \in SD_p(\beta,\alpha)$, then $R\left(\frac{zf'(z)}{f(z)}\right) > \frac{\alpha - p\beta}{1-\beta}\ (z \in U)$

Define a function q(z) by

$$q(z) = \frac{(1-\beta)\left(\frac{zf'(z)}{f(z)}\right) - (\alpha - p\beta)}{(p-\alpha)}\ (z \in U) \qquad (3.5)$$

Note that q is analytic in U with $q(0) = 1$ and $R(q(z)) > 0$ If $q(z) = 1 + q_1 z + q_2 z^2 + \ldots,$

then we can write $\frac{zf'(z)}{f(z)} = \frac{\alpha - p\beta}{1-\beta} + \frac{p-\alpha}{1-\beta}\sum_{n=0}^{\infty} q_n z^n$

or $$zf'(z) = f(z)\left(p + \left(\frac{p-\alpha}{1-\beta}\right)\sum_{n=1}^{\infty} q_n z^n\right),\ (q_0 = 1). \qquad (3.6)$$

From this, we obtain

$$na_{p+n} = \left(\frac{p-\alpha}{1-\beta}\right)(q_n + a_p + 1q_{n-1} + a_p\,\_2q_{n-2} + \ldots + a_p + n - 1q_1) \qquad (3.7)$$

From the coefficient estimates for Carath´eodory functions [4], we know that $|q_n| \le 2$ for all $n \ge 1$

Making use of it in (3.7) we see that

$$|a_{p+n}| \le \frac{2(p-\alpha)}{n|1-\beta|}(1 + |a_{p+1}| + |a_{p+2}| + \ldots + |a_{p+n-1}|). \qquad (3.8)$$

Therefore, for n=1, we have

$$|a_{p+1}| \le \frac{2(p-\alpha)}{|1-\beta|}, \qquad (3.9)$$

which proves (3.3). Now for n=2, we obtain $|a_{p+2}| \le \frac{2(p-\alpha)}{2|1-\beta|},(1 + |a_{p+1}|)$ $\le \frac{2(p-\alpha)}{2|1-\beta|}\left(1 + \frac{2(p-\alpha)}{|1-\beta|}\right)$.

This shows that (3.4) is true for n=2. For n=3, we see that

$$|a_{p+3}| \le \frac{2(p-\alpha)}{3|1-\beta|}(1 + |a_{p+1}| + |a_{p+2}|) \quad \le \frac{2(p-\alpha)}{3|1-\beta|}\left(1 + \frac{2(p-\alpha)}{|1-\beta|} + \frac{2(p-\alpha)}{2|1-\beta|} + \frac{2^2(p-\alpha)^2}{2|1-\beta|^2}\right)$$

Thus, (3.4) holds for n=3. Next, we assume that (3.4) is true for n=k and therefore

$$|a_{p+k+1}| \le \frac{2(p-\alpha)}{(k+1)|1-\beta|}\left(1 + \frac{2(p-\alpha)}{|1-\beta|} + \frac{2(p-\alpha)}{|1-\beta|}\left(1 + \frac{2(p-\alpha)}{|1-\beta|}\right) + \ldots \frac{2(p-\alpha)}{k|1-\beta|}\prod_{j=1}^{k-2}\left(1 + \frac{2(p-\alpha)}{j|1-\beta|}\right)\right)$$
$$\le \frac{2(p-\alpha)}{(k+1)|1-\beta|}\prod_{j=1}^{k-1}\left(1 + \frac{2(p-\alpha)}{j|1-\beta|}\right).$$

This shows that (3.4) is true for n=k+1. Hence, by using the principle of mathematical induction, (3.4) holds for all n ≥ 2.

Remark 3.1: Taking p=1 in Theorem 3.3, we obtain

$$|a_{p+1}| \le \frac{2(1-\alpha)}{n|1-\beta|}\prod_{j=1}^{n-1}\left(1 + \frac{2(p-\alpha)}{j|1-\beta|}\right) (n \ge 2) \qquad (3.10)$$

which was given by Owa, Polatoˇglu and Yavuz [1]

Remark 3.2: Taking p=1 and β=0 in Theorem 3.3, we have

$|a_{n+1}| \le \frac{1}{n!}\prod_{j=2}^{n+1}(j - 2\alpha)\ (n \ge 1),$ which was proven by Robertson [5].

We know that $f \in KD_p(\beta,\alpha)$ if and only if $zf' \in SD_p(\beta,\alpha)$ [2]. Thus, we have

Theorem 3.4: If $f(z) \in KD_p(\beta,\alpha)$ then

$$|a_{p+1}| \le \frac{2p(p-\alpha)}{(p+1)|1-\beta|} \qquad (3.12)$$

and $$|a_{p+1}| \le \frac{2p(p-\alpha)}{(p+1)|1-\beta|}\prod_{j=1}^{n-1}\left(1 + \frac{2(p-\alpha)}{j|1-\beta|}\right) (n \ge 2). \qquad (3.13)$$

Proof: For $f \in KD_p(\beta,\alpha)$ we know $zf'(z) + pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1} \in SD_p(\beta,\alpha)$.

Therefore

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{\alpha - p\beta}{1-\beta}\ (z \in U).$$

Define a function r(z) by

$$r(z)\frac{(1-\beta)\left(\frac{zf''(z)}{f'(z)}\right) - (\alpha - p\beta)}{(p-\alpha)}\ (z \in U) \qquad (3.14)$$

Note that r is analytic in U with r(0)=1 and $\Re(r(z)) > 0$ If $r(z) = 1 + q_1 z + q_2 z^2 + \ldots$, then we can write

$$1 + \frac{zf''(z)}{f'(z)} = \frac{\alpha - p\beta}{1-\beta} + \left(\frac{p-\alpha}{1-\beta}\right)\sum_{n=0}^{\infty} r_n z^n, \text{ or}$$

$$zf''(z) = f'(z)\left(p - 1 + \left(\frac{p-\alpha}{1-\beta}\right)\sum_{n=1}^{\infty} r_n z^n\right),\ (r_0 = 1). \qquad (3.15)$$

From this, we obtain

$$n(p+n)a_{p+n} = \left(\frac{p-\alpha}{1-\beta}\right)(pr_n + (p+1)a_{p+1}r_{n-1} + (p+2)a_{p+2}r_{n-2} + \ldots + (p+n-1)a_{p+n-1}r_1). \qquad (3.16)$$

From the coefficient estimates for Carath´eodory functions [4] , we know that $|r_n| \le 2$ for all n ≥ 1.

Making use of it in (3.16) we see that

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$$|a_{p+n}| \le \frac{2(p-\alpha)}{n(p+n)|1+\beta|}(p+(p+1))|a_{p+1}|+(p+2)|a_{p+1}|+...+(p+n-1)|a_{p+n-1}| \quad (3.17)$$

Therefore, for n=1, we have

$$|a_{p+1}| \le \frac{2p(p-\alpha)}{(p+1)|1-\beta|}, \quad (3.18)$$

which proves (3.12). Now for n=2, we obtain

$$|a_{p+2}| \le \frac{2(p-\alpha)}{2(p+2)|1-\beta|}(p+(p+1)|a_{p+1}|$$

$$|a_{p+3}| \le \frac{2(p-\alpha)}{3(p+3)|1-\beta|}(p+(p+1)|a_{p+1}|+(p+2)|a_{p+2}|)$$

$$\le \frac{2p(p-\alpha)}{2(p+2)|1-\beta|}\left(1+\frac{2(p-\alpha)}{|1-\beta|}\right)$$

This shows that (3.12) is true for n=2. For n=3, we see that

$$|a_{p+3}| \le \frac{2(p-\alpha)}{3(p+3)|1-\beta|}(p+(p+1)|a_{p+1}|+(p+2)|a_{p+2}|)$$

$$\le \frac{2(p-\alpha)}{3(p+3)|1-\beta|}(1+\frac{2(p-\alpha)}{|1-\beta|}+\frac{2(p-\alpha)}{2|1-\beta|}+\frac{2^2(p-\alpha)^2}{|1-\beta|^2})$$

Thus, (3.12) holds for n=3. Next, we assume that (3.12) is true for n=k and therefore

$$|a_{p+k+1}| \le \frac{2p(p-\alpha)}{(k+1)(p+k+1)|1-\beta|}\left(1+\frac{2(p-\alpha)}{|1-\beta|}+\frac{2(p-\alpha)}{2|1-\beta|}\left(1+\frac{2(p-\alpha)}{|1-\beta|}\right)+...\right.$$

$$\left.+\frac{2(p-\alpha)}{k|1-\beta|}\prod_{j=1}^{k-2}\left(1+\frac{2(p-\alpha)}{j|1-\beta|}\right)\right)$$

$$\le \frac{2p(p-\alpha)}{(k+1)(p+k+1)|1-\beta|}\prod\left(\quad+\frac{2(p-\alpha)}{j|1-\beta|}\right)$$

This shows that (3.12) is true for n=k + 1. Hence, by using the principle of mathematical induction, (3.12) holds for all n ≥ 2.

Remark 3.3: If we take p=1 in Theorem 3.4, then

$|a_{n+1}| \le \frac{2(1-\alpha)}{(n+1)n|1-\beta|}\prod_{j=1}^{n-1}\left(1+\frac{2(p-\alpha)}{j|1-\beta|}\right) (n \ge 2)$ which was proven by Owa et al. [1].

Remark 3.4: Taking p=1 and β= 0 in Theorem 3.4, we get

$|a \quad| \le \frac{}{(\quad 1)!}\prod(j-2 \quad) \, (n \ge 1)$, which was proven by Robertson [5]

Theorem 3.5: If $f \in SD_p(\beta,\alpha)$

$$\max\left\{0,|z|^p-\frac{2(p-\alpha)}{|1-\beta|}|z|^{p+1}-\sum_{n=2}^{\infty}\frac{2(p-\alpha)}{n|1-\beta|}\left(\prod_{j=1}^{n-1}\frac{2(p-\alpha)}{j|1-\beta|}\right)\right)|z|^{p+n}\right\} \le |f(z)|$$

$$\le |z|^p+\frac{2(p-\alpha)}{|1-\beta|}|z|^{p+1}\sum_{n=2}^{\infty}\frac{2(p-\alpha)}{n|1-\beta|}\left(\prod_{j=1}^{n-1}\left(\frac{2(p-\alpha)}{j|1-\beta|}\right)\right)|z|^{p+n}$$

and $\max\left\{0,p|z|^{p-1}-\frac{2(p+1)(p-\alpha)}{|1-\beta|}|z|^p-\sum_{n=2}^{\infty}\frac{2(p+n)(p-\alpha)}{n|1-\beta|}\left(\prod_{j=1}^{n-1}\left(1+\frac{2(p-\alpha)}{j|1-\beta|}\right)\right)\right.$

$$\left.\times|z|^{p+n-1}\right\}|f'(z)| \le p|z|^{p-1}+\frac{2(p+1)(p-\alpha)}{|1-\beta|}|z|^p$$

Proof: Proof follows from the fact that

$f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k,\ p=1,2,...$ and using Theorem 3.3.

Corollary 3.3: If $f \in KD_p(\beta,\alpha)$ then

$$\max\left\{0,|z|^p-\frac{2p(p-\alpha)}{(p+1)|1-\beta|}|z|^{p+1}-\sum_{n=2}^{\infty}\frac{2p(p-\alpha)}{n(p+n)|1-\beta|}\left(\prod_{j=1}^{n-1}\left(1+\frac{2(p-\alpha)}{j|1-\beta|}\right)\right)|z|^{p+n}\right\} \le |f(z)|$$

$$\le |z|^p+\frac{2p(p-\alpha)}{(p+1)|1-\beta|}|z|^{p+1}+\sum_{n=2}^{\infty}\frac{2p(p-\alpha)}{n(p+n)|1-\beta|}\left(\prod_{j=1}^{n-1}\left(1+\frac{2(p-\alpha)}{j|1-\beta|}\right)\right)|z|^{p+n}$$

and $\max\left\{0,p|z|^{p-1}-\frac{2p(p-\alpha)}{|1-\beta|}|z|^p-\sum_{n=2}^{\infty}\frac{2p(p-\alpha)}{n|1-\beta|}\left(\prod_{j=1}^{n-1}\left(1+\frac{2(p-\alpha)}{j|1-\beta|}\right)\right)|z|^{p+n-1}\right\} \le |f'(z)|$

$$\times|z|^{p+n-1}\}|f'(z)| \le p|z|^{p-1}+\frac{2(p+1)(p-\alpha)}{|1-\beta|}|z|^p$$

Proof: Proof follows from the fact that

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k,\ p=1,2,...$$

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