Research Article
Volume 14:2, 2020
DOI: 10.37421/glta.2020.14.300

Journal of
Generalized Lie Theory and Applications

ISSN: 1736-4337

Open Access

Coefficient Bounds of Functions over the Quaternions

Olubunmi A. Fadipe-Joseph*, Olanike R. Oluwaseyi and O. E. Opaleye
Department of Mathematics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria

Abstract

In this work, a new class of functions over the quaternions was defined. The initial coefficient bounds for the class defined were obtained. The work was concluded by establishing the Fekete-Szego˙ functional. AMS Mathematics Subject Classification (2010): 30C45, 30G45.

Keywords: Quaternions • Chebyshev polynomials • Modified sigmoid function

Introduction

Let a complex number z be defined as z = x + yi, where x, $y \in \mathbb{R}$ and $i^2=-1$. Let the quaternion field H be defined as

$H=\{q=x_1 + x_2 i + x_3 j + x_4 k : x_1, x_2, x_3, x_4 \in \mathbb{R}\}$,

where the imaginary units i,j,k $\notin \mathbb{R}$ satisfy

$i^2= j^2= k^2= -1$, ij=−ji=k, jk=−kj=i, ki=−ik=j.

The quaternion extends the class of complex numbers. Recall that

$|z| = \sqrt{x^2 + y}$

Similarly,

$$|q| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

We denote by B the open unit ball centered at the origin in H, i.e.,

$B=\{q \in H : |q| < 1\}$.

Let A be the class of functions of the form

$$f(q) = q + \sum_{m=2}^{\infty} q^m a_{m,} \ q \in B$$

that are holomorphic in the open unit ball $\{B=q \in H:|q| < 1\}$ [1-4].

The theory of functions over the complex field is very rich. These functions are very useful in the analysis of practical problems of hydrodynamics, aerodynamics, elasticity, electrodynamics and the natural sciences. The quaternions are four dimensional. Therefore, it is important to study the geometric theory for quaternionic functions.

The Chebyshev polynomials of the first kind $T_n(t)$, t[-1, 1] have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t) q^n = \frac{1-tq}{1-2tq+q^2} \qquad (q \in B)$$

and that of second kind is :

$$H(q,t) = \frac{1}{1-2tq+q^2} = 1 + \sum_{n=0}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} q^n \qquad (q \in B) \ |t| < 1.$$

Note that if t = cosα, α ∈ (−π/3 , π/3) then

$$H(q,t) = \frac{1}{1-2\cos\alpha q + q^2} = 1 + \sum_{n=0}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} q^n$$

Thus,

$H(q, t)=1+U_1(t)q+ U_2(t)q^2+\ldots \qquad (q \in B, t \in (-1, 1))$,

where

$$U_{n-1} = \frac{\sin(nar\cos t)}{\sqrt{1-t^2}} \ n \in N$$

are the Chebyshev polynomials of the second kind. Also,

$U_n(t) = 2tU_{n-1}(t)-U_{n-2}(t)$,

so that,

$U_1(t) = 2t, \ U_2(t)=4t^2-1, U_3(t)=8t^3-4t, \ldots$ (1.2)
[5,6].

Lemma 1.1

If $\omega(q)=b_1 q+b_2 q^2+b_1 \neq 0$ is analytic and satisfy $|\omega(q)|<1$ in the unit ball B, then for each 0<R<1, $|\omega'(q)|<1$ and $\omega(Re^{i\theta})<1$ unless $\omega(q) = e^{i\sigma}q$ for some real number σ

Lemma 1.2

Let $\omega \in \Omega=\{\omega \in A : |w(q)| \le |q| , \ q \in B\}$.

If $\omega \in \Omega$, $\omega(q) = \sum_{n=1}^{\infty} c_n q^n$ (q ∈ B), then

$|c_n| \le 1 \quad n = 1, 2, \ldots, \quad |c_2| \le 1 - |c_1|^2$ (1.3)

and

$$|c_2 - \mu c_1^2| \le \max\{1, |\mu|\} (\mu \in \mathbb{C}). \tag{1.4}$$

The result is sharp. The functions

$\omega(q) = q, \ \omega_a(q) = q\dfrac{q+a}{1+\bar{a}q}$ (q ∈ **B**, |a|<1)

are extremal functions.

Main Results

Definition 2.1

The modified sigmoid function is defined in series form as

$$g(q) = 1 + \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} q^n \right]^m \right) \qquad |q| < 1$$ [7].

***Address for Correspondence:** Olubunmi A. Fadipe-Joseph, Department of Mathematics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria, E-mail: famelov@unilorin.edu.ng; famelov@gmail.com*

Received 03 June, 2020; **Accepted** 15 June, 2020; **Published** 01 July, 2020

Theorem 2.1

Let $g(q)=1+\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}q^n\right]^m\right)$.

Then $\mathrm{Re}\left(g(q)+\frac{\alpha qg'(q)}{g(q)}\right)\geq\frac{1}{2}+\beta$ for some real α ≥ 0; $\beta=-\frac{1}{4}\left[\sum_{n=1}^{\infty}2\frac{(-1)^n}{n!}+n\alpha\right]$.

Proof: By definition,

$$g(q)=1+\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}q^n\right]^m\right)$$

$$\frac{\alpha qg'(q)}{g(q)}=\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}mn\alpha\left[q^{n-1}\right]^m\right)^{m-1}\cdot\left[1+\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}q^n\right]^m\right)\right]^{-1}$$

$$=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}q^n\right)^m+\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}mn\alpha\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}q^n\right)^{m-1} \quad (2.1)$$

$$-\sum_{m=1}^{\infty}\frac{(-1)^2m}{2^2m}mn\alpha\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}q^n\right)^{2m-1}$$

When m=1 in (2.1),

$$\frac{\alpha qg'(q)}{g(q)}=1+\frac{(-1)}{2}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}q^n\right)+\frac{(-1)}{2}n\alpha-\frac{(-1)}{4}n\alpha\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}q^n\right)$$

$$\frac{\alpha qg'(q)}{g(q)}=\frac{1}{2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}q^n\left(1+\frac{n\alpha}{2}\right)$$

$$\frac{\alpha qg'(q)}{g(q)}=\frac{1}{2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}q^n\left(1+\frac{n\alpha}{2}\right)$$

$$\Rightarrow\mathrm{Re}\left(g(q)+\frac{\alpha qg'(q)}{g(q)}\right)=\frac{1}{2}\left[g(q)+\frac{\alpha qg'(q)}{g(q)}+\overline{g(q)}+\overline{\frac{\alpha qg'(q)}{g(q)}}\right]$$

$$\Rightarrow\mathrm{Re}\left(\frac{1}{2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}q^n\left(1+\frac{n\alpha}{2}\right)\right)=\frac{1}{2}\left[1-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(q^n+\overline{q^n}\right)\left(1+\frac{n\alpha}{2}\right)\right]$$

Since q = Re$^{i\theta}$,

$$\frac{1}{2}\left[1-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(q^n+\overline{q^n}\right)\left(\frac{2+n\alpha}{2}\right)\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}2\cos n\alpha\left(\frac{2+n\alpha}{2}\right)\right]$$

$$=\frac{1}{2}-\frac{1}{4}\left[\sum_{n=1}^{\infty}2\frac{(-1)^n}{n!}n\alpha\right]$$

Hence,

$$\mathrm{Re}\left(g(q)+\frac{\alpha qg'(q)}{g(q)}\right)\geq\frac{1}{2}+\beta$$

Where $\beta=-\frac{1}{4}\left[\sum_{n=1}^{\infty}2\frac{(-1)^n}{n!}n\alpha\right]$.

Definition 2.2

A function f∈A is said to be in the class T(b, λ); 0 ∉ b∈ℂ, λ ≥ 1 if the following subordination holds.

$$1+\frac{1}{b}\left(\frac{q(f'(q))^{\lambda}}{f(q)}-1\right)\prec H(q,t)$$

Theorem 2.2

If f (q) as defined in (1.1) belongs to the class T(b, λ); 0 ∉ b∈ℂ, λ ≥ 1, then,

$$|a_2|\leq\frac{2|b|t}{(2\lambda-1)}$$

$$|a_3|\leq\frac{|b|(2t+(4t^2-1))}{(3\lambda-1)}$$

$$|a_4|\leq\frac{1}{(4\lambda-1)}\left(|b|K-\frac{4}{3}(\lambda^3-9\lambda^2+14\lambda-3)-\frac{M(6\lambda^2-11\lambda+2)}{(2\lambda-1)(3\lambda-1)}\right)$$

where

$$K=2t+2(4t^2-1)+(8t^3-4t)$$

and

$$M=\left[4b^2b^2+(8t^3-2t)\right]$$

Proof: Suppose f ∈ T (b, λ), then by definition

$$1+\frac{1}{b}\left(\frac{q(f'(q))^{\lambda}}{f(q)}-1\right)\prec H(q,t)$$

$$\Rightarrow 1+\frac{1}{b}\left(\frac{q(f'(q))^{\lambda}}{f(q)}-1\right)=H(\omega(q,t)) \quad (2.2)$$

Now f (q)=q + a_2q^2+ a_3q^3+ a_q^4 +...

and $(f'(q))^{\lambda}=(1+(2a_2q+3a_3q^2+4a_4q^3+\ldots))^{\lambda}$

Using binomial expansion,

$$(f'(q))^{\lambda}=1+\lambda\left(2a_2q+3a_3q^2+4a_4q^3+\ldots\right)+$$

$$\frac{\lambda(\lambda-1)}{2!}\left(4a_2^2q^2+\ldots+12a_2a_3q^3+\ldots\right)+\frac{\lambda(\lambda-1)(\lambda-2)}{3!}(8a_2^3q^3+\ldots)+\ldots$$

This implies that

$$\frac{q(f'(q))^{\lambda}}{f(q)}=1+(2\lambda-1)a_2q+((3\lambda-1)a_3+(2\lambda^2-4\lambda+1)a_2^2)q^2+((4\lambda-1)a_4+(6\lambda^2-11\lambda+2)a_2a_3$$

$$+\frac{4}{3}(\lambda^3-9\lambda^2+14\lambda-3))q^3+\ldots$$

Therefore,

$$1+\frac{1}{b}\left(\frac{q(f'(q))^{\lambda}}{f(q)}-1\right)=1+\frac{(2\lambda-1)}{b}a_2q+\frac{((3\lambda-1)a_3+(2\lambda^2-4\lambda+1))}{b}a_2^2q^2= \quad (2.3)$$

$$+\frac{\left((4\lambda-1)a_4+(6\lambda^2-11\lambda+2)a_2a_3+\frac{4}{3}(\lambda^3-9\lambda^2+14\lambda-3)\right)}{b}q^3+\ldots$$

Since $1+\frac{1}{b}\left(\frac{q(f'(q))^{\lambda}}{f(q)}-1\right)=H(\omega(q,t))$

$$1+\frac{(2\lambda-1)}{b}a_2q+\frac{((3\lambda-1)a_3+(2\lambda^2-4\lambda+1))}{b}a_2^2q^2$$

$$+\frac{\left((4\lambda-1)a_4+(6\lambda^2-11\lambda+2)a_2a_3+\frac{4}{3}(\lambda^3-9\lambda^2+14\lambda-3)\right)}{b}q^3+\ldots=H(\omega(q,t))$$

$$H(q,t)=1+U_1(t)q+U_2(t)q^2+U_3(t)q^3+U_4(t)q^4+\ldots \quad (2.4)$$

$$H(\omega(q,t))=1+U_1(t)\omega(q)+U_2(t)\omega(q)^2+U_3(t)\omega(q)^3+U_4(t)\omega(q)^4+\ldots \quad (2.5)$$

$$\omega(q)=c_1q+c_2q^2+c_3q^3+c_4q^4+\ldots \quad (2.6)$$

$$\omega^2(q)=c_1^2q^2+2c_1c_2q^3+(2c_1c_3+c_2^2)q^4+\ldots \quad (2.7)$$

$$\omega^3(q)=c_1^3q^2+3c_1^2c_2q^4+\ldots \quad (2.8)$$

substituting (2.6),(2.7) and (2.8) in (2.4) we have

$$H(\omega(q,t))=1+U_1(t)c_1(q)+(c_2U_1(t)+c_1^2U_2(t))q^2+(c_3U_1(t)+2c_1c_2U_2(t)+c_1^3U_3(t)q^3+\ldots \quad (2.9)$$

Equating (2.3) and (2.9) and comparing coefficients q, q^2 and q^3, applying equation (1.2) and Lemma 1.2 we have the result.

2.1 Fekete-Szegö Inequality

The Fekete-Szegö functional for the class T(b, λ); $0 \notin b \in \mathbb{C}$, $\lambda \geq 1$ is given here.

Theorem 2.3 If f (q) belongs to the class T(b, λ); $0 \notin b \in \mathbb{C}$, $\lambda \geq 1$ then

$$\left|a_3 - \mu a_2^2\right| \leq \left\{ \begin{array}{ll} \frac{2|b|t}{(3\lambda-1)}\left(\frac{4t^2-1}{2t}\right); & \mu = 0 \\ \frac{2|b|t}{(3\lambda-1)}\left(\frac{2bt(3\lambda-1)}{(2\lambda-1)^2} - \frac{4t^2-1}{2t}\right); & \mu = 1 \\ \frac{2|b|t}{(3\lambda-1)}\left(\frac{2\mu bt(3\lambda-1)}{(2\lambda-1)^2} - \frac{4t^2-1}{2t}\right); & 0 < \mu < 1 \end{array} \right\}$$

Conclusion

The classical geometric function theory was extended to functions of quaternionic variables in this work. A new class of function involving the Chebyshev polynomial of the second kind was defined. The coefficient bounds and the Fekete-Szegö functional for quaternionic functions were established using subordination principle.

References

1. Gal, Sorin G, J. Oscar Gonzalez-Cervantes and Irene Sabadini. "Univalence Results for Slice Regular Functions of a Quaternion Variable." *Complex Var Elliptic* 60 (2015): 1346-1365.
2. Ren, Guangbin and Xieping Wang. "Carathéodory Theorems for Slice Regular Functions." *Complex Anal Oper Th* 4 (2014): 1-15.
3. Qian, T. "Derivation of Monogenic Functions and Applications." *Studia Mathematica* 2 (2018): 118-126.
4. Xu, Zhenghua and Guangbin Ren. "Slice Starlike Functions over Quaternions." *J Math Anal Appl* 3 (2018): 25-38.
5. Fadipe-Joseph, Olubunmi A, Bilikis B. Kadir, Sunday E. Akinwumi and Esther O. Adeniran. "Polynomial Bounds for a Class of Univalent Function Involving Sigmoid Function." *Khayyam J Math* 4 (2018): 88-101.
6. Olatunji, Sunday O, and Olubunmi A. Fadipe-Joseph. "Initial Chebyshev Polynomial Class of Analytic Function based on Quasi-Subordination." *Palestine Journal of Mathematics* 8 (2019): 227-233.
7. Fadipe-Joseph, OA, A.T. Oladipo and A. Uzoamaka Ezeafulukwe. "Modified Sigmoid Function in Univalent Functions Theory." *International J of Math Sci & Engg Appls* 7 (2013): 313-317.

How to cite this article: Fadipe-Joseph, Olubunmi A, Olanike R. Oluwaseyi, and O.E. Opaleye. "Coefficient Bounds of Functions over the Quaternions." J Generalized Lie Theory Appl 14 (2020): 300. doi: 10.37421/GLTA.2020.14.300