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Classification of Maximal Subalgebras and Corresponding Reductive Pairs of Lie Algebra of All 2 × 2 Real Matrices

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Abstract

The purpose of the article is to describe all 3-dimensional subalgebras and all corresponding reductive pairs of Lie algebra of all 2 × 2 real matrices. This Lie algebra is 4-dimensional as a vector space, it's not simple, and it's not solvable. The evaluation procedure utilizes the canonical bases for subspaces that were introduced. In Part I of this article, all 3-dimensional subalgebras of the given Lie algebra *g* are classified. All reductive pairs {*h, m*} with 3-dimensional subalgebras *h* are found in Part II. Surprisingly, there is only one reductive pair {*h, m*} with special 3-dimensional subalgebra *h* and 1-dimensional complement *m*. Finally, all reductive pairs {*h, m*} with 1-dimensional subalgebras *h* of algebra *g* are classified in Part III of the article.

Keywords: Lie algebra; Subalgebras; Reductive pairs

Introduction

Reductive homogeneous spaces appeared for the first time in the fundamental manuscript [1,2] of Katsumi Nomizu, in which the author investigated invariant affine connections and Riemannian metrics on them. Sagle and Winter in their article [3] analyzed algebraic structures generated by reductive pairs of simple Lie algebras. The next problem studied by some authors was classification of subalgebras of some Lie algebras. For example, Patera and Winternitz have classified all subalgebras of real Lie algebras of dimensions *d*=3 and *d*=4 in their manuscript [4]. This classification of subalgebras of low dimensional real Lie algebras was done by a representative of each conjugacy class where the conjugacy was considered under the group of inner automorphisms of Lie algebras. The articles mentioned above have stimulated this research for all subalgebras and all reductive pairs at Lie algebra *g* of all real 2 × 2 matrices. In contrast to the article [4], this research is utilized a different method. Our method involves canonical bases for subspaces [1] that allow us to find all 3-dimensional subalgebras and the corresponding reductive pairs of the given Lie algebra *g*. Our classification of reductive pairs is done here for the first time. The classification of 2-dimensional subalgebras with its reductive pairs of the same Lie algebra will be done at the separate article. New knowledge concerning the structure of this Lie algebra is important for Algebra, Geometry, and Physics.

We start with standard definitions for the readers' convenience.

Definition 1

Let *g* be a vector space over a field *F*. Then *g* is called a Lie algebra over *F* if there exists a Lie bracket operation $\left[\vec{x},\vec{y}\right]\in g$ for any $\vec{x}\in g, \vec{y}\in g$ such that:

$$\left[a\vec{x},\vec{y}\right]=a\left[\vec{x},\vec{y}\right]=\left[\vec{x},a\vec{y}\right] \text{ for any } a\in F,$$

$$\left[\vec{x},\vec{y}+\vec{z}\right]=\left[\vec{x},\vec{y}\right]+\left[\vec{x},\vec{z}\right], \left[\vec{x},\vec{y}\right]=-\left[\vec{y},\vec{x}\right] \text{ and}$$

$$\left[\vec{x},\left[\vec{y},\vec{z}\right]\right]+\left[\vec{y},\left[\vec{z},\vec{x}\right]\right]+\left[\vec{z},\left[\vec{x}.\vec{y}\right]\right]=\vec{0}\ (\vec{x},\vec{y},\vec{z}\in g).$$

We call $\left[\vec{x},\vec{y}\right]$ a Lie product.

Definition 2

Let *g* be a Lie algebra. A subspace $h\subset g$ is called a (Lie) subalgebra of *g*, if $[h, h]\subset h$.

Definition 3

Let *g* be a Lie algebra, *h* be subalgebra of *g*. If there exists a subspace *m* of *g* such that $h \oplus m= g$ and $[h, m]\subset m$, then {*h*, *m*} is called a reductive pair of *g*, and { *g*,*h*, *m*} is called a reductive triple. We say also that subspace *m* is a reductive complement for *h*.

Lie algebra *g* and its standard basis:

This Lie algebra contains all 2 × 2 matrices over the field of all real numbers. The standard basis of this algebra consists of the next four matrices:

$$\vec{e_1}=\begin{bmatrix}1&0\\0&0\end{bmatrix}, \vec{e_2}=\begin{bmatrix}0&1\\0&0\end{bmatrix}, \vec{e_3}=\begin{bmatrix}0&0\\1&0\end{bmatrix}, \vec{e_4}=\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

It is well known that the Lie multiplication operation [*A*, *B*] for any two square matrices *A* and *B* of the same size is defined to be [*A*, *B*] = *ABBA*. According this rule, the fundamental products of the basic vectors (matrices) $\vec{e_1},\vec{e_2},\vec{e_3},\vec{e_4}$ can be computed:

$$\left[\vec{e_1},\vec{e_2}\right]=\vec{e_2}, \left[\vec{e_1},\vec{e_3}\right]=-\vec{e_3}, \left[\vec{e_2},\vec{e_3}\right]=\vec{e_1}-\vec{e_4}, \left[\vec{e_2},\vec{e_4}\right]=\vec{e_2}, \left[\vec{e_3},\vec{e_4}\right]=-\vec{e_3}. \quad (*)$$

All other products of basic vectors are zeros.

Let *h* be any 3-dimensional subspace of Lie algebra *g*. We can describe subspace *h* as $h = Span\{\vec{a},\vec{b},\vec{c}\}$ where $\vec{a}=a_1\vec{e_1}+a_2\vec{e_2}+a_3\vec{e_3}+a_4\vec{e_4}$, $\vec{b}=b_1\vec{e_1}+b_2\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}$ and $\vec{c}=c_1\vec{e_1}+c_2\vec{e_2}+c_3\vec{e_3}+c_4\vec{e_4}$ are 3 linearly independent vectors.

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According to the article [1], all canonical bases for 3-dimensional subspaces of 4-dimensional vector space are:

(1) $\vec{a}=\vec{e_1}+a_4\vec{e_4},\ \vec{b}=\vec{e_2}+b_4\vec{e_4},\ \vec{c}=\vec{e_3}+c_4\vec{e_4}$.

(2) $\vec{a}=\vec{e_1}+a_3\vec{e_3},\ \vec{b}=\vec{e_2}+b_3\vec{e_3},\ \vec{c}=\vec{e_4}$.

(3) $\vec{a}=\vec{e_1}+a_2\vec{e_2},\ \vec{b}=\vec{e_3},\ \vec{c}=\vec{e_4}$.

(4) $\vec{a}=\vec{e_2},\ \vec{b}=\vec{e_3},\ \vec{c}=\vec{e_4}$.

Part I. Maximal subalgebras of Lie algebra *g*

Now we start to determine that a 3-dimensional subspace $h=Span\{\vec{a},\vec{b},\vec{c}\}$ is a subalgebra of Lie algebra *g* when vectors $\vec{a},\vec{b},\vec{c}$ form one of the canonical bases (1), (2), (3), or (4) listed above. We have to check that the condition $[h, h]\subset h$ is true for each of these bases. The necessary evaluation procedure follows.

Let $\vec{a}=\vec{e_1}+a_4\vec{e_4},\ \vec{b}=\vec{e_2}+b_4\vec{e_4},\ \vec{c}=\vec{e_3}+c_4\vec{e_4}$ be the basis (1) for *h*. Evauate Lie products $\left[\vec{a},\vec{b}\right],\left[\vec{a},\vec{c}\right],\left[\vec{b},\vec{c}\right]$.

$$\left[\vec{a},\vec{b}\right]=\left[\vec{e_1}+a_4\vec{e_4},\vec{e_2}+b_4\vec{e_4}\right]=\vec{e_2}-a_4\vec{e_2}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c},\ \left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+a_4\vec{e_4},\vec{e_3}+c_4\vec{e_4}\right]=-\vec{e_3}+a_4\vec{e_3}$$
$$=x_2\vec{a}+y_2\vec{b}+z_2\vec{c},\ \left[\vec{b},\vec{c}\right]=\left[\vec{e_2}+b_4\vec{e_4},\vec{e_3}+c_4\vec{e_4}\right]=\vec{e_1}-\vec{e_4}+c_4\vec{e_2}+b_4\vec{e_3}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}.$$

So, $x_1=0$, $y_1=1-a_4$, $z_1=0$, $(1-a_4)b_4=0$; $x_2=0$, $y_2=0$, $z_2=a_4-1$, $(a_4-1)c_4=0$ and $x_3=1$, $y_3=c_4$, $z_3=b_4$, $a_4+c_4b_4+b_4c_4=-1$.

The system of 3 equations for a_4, b_4, c_4 is: $(1-a_4)b_4=0$, $(a_4-1)c_4=0$, $a_4+c_4b_4+b_4c_4=-1$. We have two different solutions: $b_4=0$, $c_4=0$, $a_4=-1$ and $a_4=1$, $c_4=\frac{-1}{b_4}$, $b_4\neq 0$. This means that the following 1-parameter set of subalgebras h_1 and one special subalgebra h_2 exist for this case:

$$h_1=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}-\frac{1}{b_4}\vec{e_4}\},\ h_2=Span\{\vec{e_1}-\vec{e_4},\vec{e_2},\vec{e_3}\}.$$

2. Let $\vec{a}=\vec{e_1}+a_3\vec{e_3},\ \vec{b}=\vec{e_2}+b_3\vec{e_3},\ \vec{c}=\vec{e_4}$, be the basis (2) for a possible subalgebra *h*. Evaluate Lie products $\left[\vec{a},\vec{b}\right],\left[\vec{a},\vec{c}\right],\left[\vec{b},\vec{c}\right]$:

$$\left[\vec{a},\vec{b}\right]=\left[\vec{e_1}+a_3\vec{e_3},\vec{e_2}+b_3\vec{e_3},\right]=\vec{e_2}-b_3\vec{e_3}-a_3(\vec{e_1}-\vec{e_4})=x_1\vec{a}+y_1\vec{b}+z_1\vec{c},$$
$$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+a_3\vec{e_3},\vec{e_4}\right]=-a_3\vec{e_3}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c},\ \left[\vec{b},\vec{c}\right]=\left[\vec{e_2}+b_3\vec{e_3},\vec{e_4}\right]=\vec{e_2}-b_3\vec{e_3}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}.$$

So, we have $x_1=-a_3$, $y_1=1$, $z_1=a_3$, $x_2=0$, $y_2=0$, $z_2=0$, $x_3=0$, $y_3=1$, $z_3=0$. Consequently, the following system of equations appears for components a_3, b_3:

$-a_3^{\,2}+b_3=-b_3$, $a_3=0$, $b_3=-b_3$.

This system of equations has only one solution $a_3=0$, $b_3=0$. This solution produces the following subalgebra:

$$h_3=Span\{\vec{e_1},\vec{e_2},\vec{e_4}\}.$$

Let $\vec{a}=\vec{e_1}+a_2\vec{e_2},\ \vec{b}=\vec{e_3},\ \vec{c}=\vec{e_4}$ be the basis (3) for *h*. Evaluate Lie products $\left[\vec{a},\vec{b}\right],\left[\vec{a},\vec{c}\right],\left[\vec{b},\vec{c}\right]$ for this case. We have:

$\left[\vec{a},\vec{b}\right]=\left[\vec{e_1}+a_2\vec{e_2},\vec{e_3}\right]=-\vec{e_3}+a_2(\vec{e_1}-\vec{e_4})=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}$. So, $x_1=a_2$, $y_1=-1$, $z_1=-a_2$, $a_2^{\,2}=0$.

$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+a_2\vec{e_2},\vec{e_4}\right]=a_2\vec{e_2}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}$. So, $x_2=0$, $y_{12}=0$, $z_2=0$, and $a_2=0$.

$\left[\vec{b},\vec{c}\right]=\left[\vec{e_3},\vec{e_4}\right]=-\vec{e_3}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}$. So, $x_3=0$, $y_3=-1$, $z_3=0$, and 0=0.

The system of equations has only one solution $a_2=0$. The corresponding subalgebra is $h_4=Span\{\vec{e_1},\vec{e_3},\vec{e_4}\}$.

Consider the last possible basis $\vec{a}=\vec{e_2},\ \vec{b}=\vec{e_3},\ \vec{c}=\vec{e_4}$. Evaluate Lie products $\left[\vec{a},\vec{b}\right],\left[\vec{a},\vec{c}\right],\left[\vec{b},\vec{c}\right]$ for this basis. We obtain:

$$\left[\vec{a},\vec{b}\right]=\left[\vec{e_2},\vec{e_3}\right]=\vec{e_1}-\vec{e_4}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}.$$

The vector $\vec{e_1}-\vec{e_4}$ doesn't belong to the subspace $h=Span\{\vec{a},\vec{b},\vec{c}\}$ at this case. So, this subspace is not subalgebra of algebra *g*.

The next statement describes all 3-dimensional subalgebras of Lie algebra *g*.

Theorem 1: All different 3-dimensional subalgebras of Lie algebra of all 2 × 2 real matrices are listed here:

$$h_1=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}-\frac{1}{b_4}\vec{e_4}\},\ b_4 0;\ h_2=Span\{\vec{e_1}-\vec{e_4},\vec{e_2},\vec{e_3}\};$$
$$h_3=Span\{\vec{e_1},\vec{e_2},\vec{e_4}\};\ h_4=Span\{\vec{e_1},\vec{e_3},\vec{e_4}\}.$$

Corollary: The subalgebras above are maximal for the given Lie algebra.

Part II. Reductive pairs with 3-dimensional subalgebras *h* of Lie algebra *g*

How many of 3-dimensional subalgebras *h* form reductive pairs {*h*, *m*} in this Lie algebra? To answer this question, we will use the conditions from the Definition 3, i.e. $[h, m]\subset m$, $g=h\oplus m$ where *m* is an appropriate 1-dimensional reductive complement for a given 3-dimensional subalgebra *h*. The list of all 3-dimensional subalgebras from Theorem 2 will be used to find all possible reductive complements. Let $m=Span\{\vec{d}=d_1\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\}$ be a possible 1-dimensional complement. To simplify our evaluation, we consider 2 possible cases for the generating vector $\vec{d}$:

$\vec{d}=\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$ ($d_1\neq 0$), and $\vec{d}=d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$ ($d_1\neq 0$).

Subalgebra h_1 Case 1: Multiply basic vectors $\vec{a}=\vec{e_1}+\vec{e_4},\ \vec{b}=\vec{e_2}+b_4\vec{e_4},\ \vec{c}=\vec{e_3}-\frac{1}{b_4}\vec{e_4}$ from h_1 by vector $\vec{d}=\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$. We have:

$\left[\vec{e_2}+b_4\vec{e_4},\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-\vec{e_2}+d_3\left(\vec{e_1}-\vec{e_4}\right)+d_4\vec{e_2}-b_4d_2\vec{e_2}+b_4d_3\vec{e_3}=y\vec{d}$ (it's an identity),

$$\left[\vec{e_2}+b_4\vec{e_4},\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-\vec{e_2}+d_3\left(\vec{e_1}-\vec{e_4}\right)+d_4\vec{e_2}-b_4d_2\vec{e_2}+b_4d_3\vec{e_3}=y\vec{d}$$
$$\left[\vec{e_3}-\frac{1}{b_4}\vec{e_4},\ \vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=\vec{e_3}-d_2\left(\vec{e_1}-\vec{e_4}\right)-d_4\vec{e_3}+\frac{1}{b_4}d_2\vec{e_2}+\frac{d_3}{b_4}\vec{e_3}$$
$$=-d_2e_1^{\cdot}+\frac{d_2}{b_4}e_2^{\cdot}+\left(1-d_4-\frac{d_3}{b_4}\right)e_3^{\cdot}+d_2e_4^{\cdot}=z\overline{d}.$$

From the vector equalities above, we obtain the following system of conditions for the components d_2, d_3, d_4 and coefficients *y*, *z*: $y=d_3$, $yd_2=-1+d_4-b_4d_2$, $yd_3=b_4d_3$, $yd_4=-d_3$; $z=-d_2$, $zd_2=\frac{d_2}{b_4}$, $zd_3=1-d_4-\frac{d_3}{b_4}$, $zd_4=d_2$. These equalities produce the system of 6 equations for d_2, d_3, d_4:

$d_2d_3=-1+d_4-b_4d_2$, $d_3^{\,2}=b_4d_3$, $d_3d_4=-d_3$, $-d_2^{\,2}=\frac{d_2}{b_4}$, $-d_2d_3=1-d_4-\frac{d_3}{b_4}$, $-d_2d_4=d_2$.

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From the equations $d_3d_4=-d_3$, $-d_2d_4=d_2$ we receive $d_4=-1$, $d_2\neq0$, $d_3\neq0$ or $d_2=0$,

$d_3=0$, and $d_4=1$. If $d_4=-1$ the $d_2=-\frac{1}{b_4}$, and $d_3=b_4$. The first solution is the vector $\dot{d}=e_1^{\cdot}-\frac{1}{b_4}e_2^{\cdot}+b_4e_3^{-}-e_4^{\cdot}$, and the subspace $m=Span\{\dot{d}=e_1^{\cdot}-\frac{1}{b_4}e_2^{\cdot}+b_4e_3^{-}-e_4^{\cdot}\}$ is a possible reductive complement for h_1. Unfortunately, the condition $g=h_1\oplus m$ is not satisfied because $\vec{d}=\vec{a}-\frac{1}{b_4}\vec{b}+b_4\vec{c}$, and m is a subspace of h_1, $m\subset h_1$. So, m is not reductive +e complement for h_1.

If $d_2=0$, $d_3=0$, and $d_4=1$ then vector $\vec{d}=\vec{e_1}+\vec{e_4}$ is the consequential result. In this case, $m\subset h_1$ again. This means that a reductive complement for subalgebra h_1 doesn't exist for this case.

Subalgebra h_1 Case 2: Multiply basic vectors from h_1 by vector $\vec{d}=d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$. We have:

$$\left[\vec{e_2}+b_4\vec{e_4},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=d_3(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}-b_4d_2\vec{e_2}+b_4d_3\vec{e_3}=y\vec{d}$$

(an identity),

$$\left[\vec{e_2}+b_4\vec{e_4},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=d_3(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}-b_4d_2\vec{e_2}+b_4d_3\vec{e_3}=y\vec{d},$$

$$\left[\vec{e_3}-\frac{1}{b_4}\vec{e_4},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-d_2\left(\vec{e_1}-\vec{e_4}\right)-d_4\vec{e_3}+\frac{1}{b_4}d_2\vec{e_2}+\frac{d_3}{b_4}\vec{e_3}$$

$$=-d_2e_1^{\cdot}+\frac{d_2}{b_4}e_2^{\cdot}+\left(1-d_4-\frac{d_3}{b_4}\right)e_3^{\cdot}+d_2e_4^{\cdot}=z\vec{d}.$$

From the vector equalities above, we obtain the following system of equations for the components d_2, d_3, d_4 and coefficients y, z:

$d_3=0$, $yd_2=d_4-b_4d_2$, $yd_3=b_4d_3$, $yd_4=-d_3$; $d_2=0$, $zd_2=\frac{d_2}{b_4}, zd_3=1-d_4-\frac{d_3}{b_4}$, $zd_4=d_2$. The last system of equations has just the zero solution for d2, d_3, d_4:$d_3=0$, $d_2=0$, $d_4=0$. The zero vectors $\vec{d}=\vec{0}$ is the only solution for the system. This means that a nonzero reductive complement for h_1 doesn't exist.

Subalgebra h_2. Case 1: Multiply basic vectors $\vec{a}=\vec{e_1}-\vec{e_4}$, $\vec{b}=\vec{e_2}$, $\vec{c}=\vec{e_3}$ from h_2 by vector $\vec{d}=\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$. We have:

$$\left\lfloor\vec{e_1}+\vec{e_4}\vec{e_1},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right\rfloor=d_2\vec{e_2}-d_3\vec{e_3}+d_2\vec{e_2}-d_3\vec{e_3}-2d_2\vec{e_2}-2d_3\vec{e_3}=x\vec{d},$$

$$\left\lfloor\vec{e_2}\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right\rfloor=-\vec{e_2}+d_3\left(\vec{e_1}-\vec{e_4}\right)+d_4\vec{e_2}=d_3\vec{e_1}+\left(d_4-1\right)\vec{e_2}-d_3\vec{e_4}=y\vec{d}$$

$$\left\lfloor\vec{e_2}\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right\rfloor=\vec{e_3}+d_2\left(\vec{e_1}-\vec{e_4}\right)-d_4\vec{e_3}=d_2\vec{e_1}+\left(1-d_4\right)\vec{e_3}-d_2\vec{e_4}=z\vec{d}.$$

From the last system of vector equalities, we obtain a system of equations for d_2, d_3, d_4 that has just one solution $d_2=0$, $d_3=0$, $d_4=1$. The subspace $m=Span\{\vec{e_1}+\vec{e_4}\}$ generated by vector $\vec{d}=\vec{e_1}+\vec{e_4}$ satisfies the conditions $\lfloor h_2, m\rfloor\subset m$ and g= $h\oplus m$, so $\{h, m\}$ is a reductive pair for this case where $h_2=Span\{\vec{e_1}-\vec{e_4},\vec{e_2},\vec{e_3}\}$, $m=Span\{\vec{e_1}+\vec{e_4}\}$.

Subalgebra h_2. Case 2: Multiply basic vectors of h_2 by vector $\vec{d}=d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$. We have:

$$\left\lfloor\vec{e_1}-\vec{e_4},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right\rfloor=d_2\vec{e_2}-d_3\vec{e_3}+d_2\vec{e_2}-d_3\vec{e_3}-2d_2\vec{e_2}-2d_3\vec{e_3}=x\vec{d},$$

$$\left\lfloor\vec{e_2},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right\rfloor=d_3\left(\vec{e_1}-\vec{e_4}\right)+d_4\vec{e_2}=d_3\vec{e_1}+d_4\vec{e_2}-d_3\vec{e_4}=y\vec{d},$$

$$\left\lfloor\vec{e_3},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right\rfloor=-d_2\left(\vec{e_1}-\vec{e_4}\right)-d_4\vec{e_3}=-d_2\vec{e_1}-d_4\vec{e_3}+d_2\vec{e_4}=z\vec{d}.$$

Transforming this system of vector equalities into a system of equations for the components d_2, d_3, d_4 and solving the system, the only zero solution $d_2=0$, $d_3=0$, $d_4=0$ is obtained. So, a nonzero reductive complement for h_2 doesn't exist for this case.

Subalgebras h_3 and h_4 don't produce any reductive pair. The details are similar for the cases of subalgebras h_1 and h_2, therefore they are omitted.

The total analysis conducted in this Part III establishes the following statement.

Theorem 2: The only one reductive pair with 3-dimensional subalgebra exists for Lie algebra g of all real 2 × 2 matrices; it is $\{h, m\}$ where, $h_2=Span\{\vec{e_1}-\vec{e_4},\vec{e_2},\vec{e_3}\}$, $m=Span\{\vec{e_1}+\vec{e_4}\}$.

Corollary: The subspace $m=Span\{\vec{e_1}+\vec{e_4}\}$ is a 1-dimensional ideal of Lie algebra g. Moreover, the subalgebra $h=Span\{\vec{e_1}-\vec{e_4},\vec{e_2},\vec{e_3}\}$ is a 3-dimensional ideal of Lie algebra g.

Part III. Reductive pairs {*h*, *m*} with 1-dimensional subalgebras *h* of Lie algebra *g*

It is well known that each 1-dimensional subspace h of algebra g is an abelian subalgebra of g. The corresponding reductive complements m for each h should be 3-dimensional subspaces m such that $\lfloor h_2,m\rfloor\subset m$ and $g=h\oplus m$. Therefore, the canonical bases for 3-dimensional subspaces that are found in the Part I can be utilized for m. The list of all canonical bases contains the next 4 bases:

1) $\vec{a}=\vec{e_1}+a_4\vec{e_4},\ \vec{b}=\vec{e_2}+b_4\vec{e_4},\vec{c}=\vec{e_3}+c_4\vec{e_4}$.

2) $\vec{a}=\vec{e_1}+a_3\vec{e_3},\vec{b}=\vec{e_2}+b_3\vec{e_3},\ \vec{c}=\vec{e_4}$.

3) $\vec{a}=\vec{e_1}+a_2\vec{e_2},\ \vec{b}=\vec{e_3},\ \vec{c}=\vec{e_4}$.

4) $\vec{a}=\vec{e_2},\vec{b}=\vec{e_3},\ \vec{c}=\vec{e_4}$.

Let $h=Span\{d_1\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\}$ be 1-dimensional subalgebra in algebra g. We will consider two cases for the generating vector $\vec{d}$:

(a) $\vec{d}=\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$, if $d_1\neq0$; (b) $\vec{d}=d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$, if $d_1=0$.

To determine if a subalgebra h forms a reductive pair with some complement m, we will check that the conditions $\lfloor h_2,m\rfloor\subset m$, $g=h\oplus m$ are satisfied.

1a. Consider the basis (1) for a complement m and case (a) for $\vec{d}$. Multiply vectors $\vec{a},\vec{b},\vec{c}$ by vector $\vec{d}$. We have:

$$\left\lfloor\vec{a},\vec{d}\right\rfloor=\left\lfloor\vec{e_1}+a_4\vec{e_4},\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right\rfloor=$$
$$d_2\vec{e_2}-d_3\vec{e_3}-a_4d_2\vec{e_2}+a_4d_3\vec{e_3}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}$$
$$=x_1\vec{e_1}+y_1\vec{e_2}+z_1\vec{e_3}+\left(x_1a_4+y_1b_4+z_1c_4\right)\vec{e_4}$$

So, $x_1=0$, $y_1=d_2-a_4d_2$, $z_1=a_4d_3-d_3$, $(d_2-a_4d_2)b_4+(a_4d_3-d_3)c_4=0$.

$$\left\lfloor\vec{b},\vec{d}\right\rfloor=\left\lfloor\vec{e_1}+b_4\vec{e_4},\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right\rfloor=-\vec{e_2}+d_3$$
$$\left(\vec{e_1}-\vec{e_4}\right)+d_4\vec{e_2}-b_4d_2\vec{e_2}+b_4d_3\vec{e_3}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}$$
$$=x_2\vec{e_1}+y_2\vec{e_2}+z_2\vec{e_3}+\left(x_2a_4+y_2b_4+z_2c_4\right)\vec{e_4}$$

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So, $x_2=d_3$, $y_2=-1+d_4-b_4d_2$, $z_2=b_4d_3$, $d_3a_4+(-1+d_4-b_4d_2)b_4+b_4d_3c_4=-d_3$.

$$\left[\vec{c},\vec{d}\right]=\left[\vec{e_3}+c_4\vec{e_4},\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=\vec{e_3}-d_2$$
$$\left(\vec{e_1}-\vec{e_4}\right)-d_4\vec{e_3}-c_4d_2\vec{e_2}+c_4d_3\vec{e_3}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}$$
$$=x_3\vec{e_1}+y_3\vec{e_2}+z_3\vec{e_3}+\left(x_3a_4+y_3b_4+z_3c_4\right)\vec{e_4}$$

So, $x_3=-d_2$, $y_3=-c_4d_2$, $z_3=1-d_3+c_4d_3$, $-d_2a_4-c_4d_2b_4+(1-d_4+c_4d_3)=d_2$.

The conditions found above produce the following system of 3 equations for components d_2, d_3, d_4:

$(d_2-a_4d_2)b_4+(a_4d_3-d_3)c_4=0$, $d_3a_4+(1+d_4-b_4d_2)b_4+b_4d_3c_4=d_3$,

$-d_2a_4-c_4d_2b_4+(1d_4+c_4d_3)c_4=d_2$.

The unknown variables are components d_2, d_3, d_4. All other components are supposed to be done. Solve the system starting with the first equation. We have:

$d_2(1-a_4)b_4+d_3(a_4-1)c_4=0$, and $(1-a_4)(d_2b_4-d_3c_4)=0$.

Two solutions are possible: $a_4=1$ and $d_2b_4=d_3c_4$.

1) If $a_4=1$ then from the second and third equations we obtain $d_3+(-1+d_4-b_4d_2)b_4+b_4d_3c_4=-d_3$, $-d_2-c_4d_2b_4+(1-c_4d_2b_4+(1-d_4+c_4d_3)c_4=d_2$. The last two equations produce two expressions for d_4:

$$d_4=1+b_4d_2-d_3(\frac{2}{b_4}+c_4)\text{, and } d_4=1+c_4d_3-d_2(\frac{2}{c_4}+b_4).$$

Comparing these two equalities for d_4, we obtain $b_4c_4=-1$ or $c_4d_3=b_4d_2$. Using these conditions, we obtain for this case $a_4=1$ the following solutions:

$$\vec{d}=\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+(1+b_4d_2-\frac{1}{b_4}d_3)\vec{e_4}\text{, and}$$

$$\vec{d}=\vec{e_1}+d_2\vec{e_2}+\frac{b_4}{c_4}d_3\vec{e_3}+(1-\frac{2}{c_4}d_2)\vec{e_4}\text{, } c_40.$$

These solutions produce the following two reductive pairs:

$$h=Span\{\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+(1+b_4d_2-\frac{1}{b_4}d_3)\vec{e_4}\},\quad m=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}-\frac{1}{b_4}\vec{e_4}\}\text{, } b_4\neq0;$$

$$h=Span\{\vec{e_1}+d_2\vec{e_2}+\frac{b_4}{c_4}d_2\vec{e_3}+(1-\frac{2}{c_4}d_2)\vec{e_4}\},\quad m=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}+c_4\vec{e_4}\}\text{, } c_4\neq0.$$

These two reductive pairs are particular cases of the pairs $\{h_1, m_1\}$, $\{h_2, m_2\}$ obtained in the next step 2.

1. If $b_4d_2=c_4d_3$ we can suppose that a_4 is any real number. Now, like in the previous step 1, we obtain the following system of three equations:

$d_2b_4=d_3c_4$, $a_4d_3+(-1+d_4-b_4d_2)b_4+b_4c_4d_3=-d_3$, $-a_4d_2+(1-d_4+c_4d_3)c_4-b_4d_2c_4=d_2$.

From the 2nd and 3rd equations above, we can find the 4th component d_4:

$$d_4=1+b_4d_2-d_3(\frac{a_4+1}{b_4}+c_4),\ d_4=1+c_4d_3-d_2(\frac{1+a_4}{c_4}+b_4)\cdot$$

Comparing the last two equalities for the same component d_4, we have $1+b_4d_2-d_3(\frac{a_4+1}{b_4}+c_4)=1+c_4d_3-d_2(\frac{a_4+1}{c_4}+b_4)$, and $d_2(\frac{a_4+1}{c_4}+2b_4)=d_3(\frac{a_4+1}{b_4}+2c_4)$.

The last equality produces two results: $d_3=\frac{b_4}{c_4}d_2$ ($c_4\neq0$), or $a_4+1+2b_4c_4=0$. Compute the component d_4 in terms of a_4, b_4, c_4, d_2 when $d_3=\frac{b_4}{c_4}d_2$. We have $d_4=1-\frac{a_4+1}{c_4}d_2$. So, we obtain the following reductive pair:

$$h_1=Span\{\vec{e_1}+d_2\vec{e_2}+\frac{b_4}{c_4}d_2\vec{e_3}+(1-\frac{a_4+1}{c_4}d_2)\vec{e_4}\},\ m_1=Span\{\vec{e_1}+a_4\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}+c_4\vec{e_4}\},$$

$c_4\neq0$.

For the condition $a_4+1+2b_4c_4=0$, we have $d_4=1+b_4d_2+c_4d_3$, and the corresponding reductive pair is:

$$h_2=Span\{\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+(1+b_4d_2+c_4d_3)\vec{e_4}\},\ m_2=Span\{\vec{e_1}-(1+2b_4c_4)\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}+c_4\vec{e_4}\}\cdot$$

Consider the special case when $c_4=0$. Then we have the following system of equations for d_2, d_3, d_4: $b_4d_2=0$, $d_3(a_4+1)=b_4(1-d_4+b_4d_2)$, $d_2(a_4+1)=0$. If a_4-1, then $d_2=0$, and $d_3=\frac{b_4}{a_4+1}(1-d_4)$, with any b_4. As the result, the following new reductive pair is obtained:

$$h_3=Span\{\vec{e_1}+\frac{b_4}{a_4+1}(1-d_4)\vec{e_3}+d_4\vec{e_4}\},\ m_3=Span\{\vec{e_1}+a_4\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}\}\text{, if}$$

$a_4\neq-1$.

If $a_4=-1$, then $b_4d_2=0$, $b_4(d_4-1)=0$, and we obtain the following new reductive pairs:

$$h_4=Span\{\vec{e_1}+d_3\vec{e_3}+\vec{e_4}\},\ m_4=Span\{\vec{e_1}-\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}\}\text{, b4}\neq0;$$

$$h_5=Span\{\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\},\ m_5=Span\{\vec{e_1}-\vec{e_4},\vec{e_2},\vec{e_3}\}\cdot$$

1b. Consider the basis (1) for a complement m and the case (b) for vector $\vec{d}=d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$.

Multiply vectors $\vec{a},\vec{b},\vec{c}$ by vector $\vec{d}$. We have:

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+a_4\vec{e_4},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=d_2\vec{e_2}-d_3\vec{e_3}-a_4d_2\vec{e_2}+a_4d_3\vec{e_3}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}\cdot$$

So, $x_1=0$, $y_1=d_2-a_4d_2$, $z_1=a_4d_3-d_3$, $d_2(1-a_4)b_4+d_3(a_4-1)c_4=0$.

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}+b_4\vec{e_4},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=d_3(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}-b_4d_2\vec{e_2}+b_4d_3\vec{e_3}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}.$$

So, $x_2=d_3$, $y_2=d_4-b_4d_2$, $z_2=b_4d_3$, $d_3a_4+(d_4-b_4d_2)b_4+b_4d_3c_4=-d_3$

$$\left[\vec{c},\vec{d}\right]=\left[\vec{e_3}+c_4\vec{e_4},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-d_2(\vec{e_1}-\vec{e_4})-d_4\vec{e_3}-c_4d_2\vec{e_2}+c_4d_3\vec{e_3}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}\cdot$$

So, $x_3=-d_2$, $y_3=-c_4d_2$, $z_3=c_4d_3-d_4$, $-a_4d_2-b_4c_4d_2+(c_4d_3-d_4)c_4=d_2$.

In this case, the following system of 3 equations for unknown components d_2, d_3, d_4 is obtained $d_2(1-a_4)b_4+d_3(a_4-1)c_4=0$, $d_3a_4+(d_4-b_4d_2)b_4+b_4d_3c_4=-d_3$, $-a_4d_2-c_4b_4d_2+(c_4d_3-d_4)c_4=d_2$.

Solve the system of equations starting with the first equation. We have the equality $d_2(1-a_4)b_4=d_3(1-a_4)c$, which produces $a_4=1$ or $d_3=\frac{b_4}{c_4}d_2$ ($c_4\neq0$).

1. If $a_4=1$, then we obtain $(d_4-b_4d_2)b_4+b_4c_4d_3=-2d_3$, $-b_4c_4d_2+(c_4d_3-d_4)c_4=2d_2$ from the 2nd and 3rd equations. Find d_4 from these two equalities:

$$d_4=b_4d_2-d_3(c_4+\frac{2}{b_4}),\ d_4=c_4d_3-d_2(b_4+\frac{2}{c_4})\cdot$$

Comparing two different expressions for the same component d_4, we find $d_3 = \frac{b_4}{c_4} d_2$, and $d_4 = -\frac{2}{c_4} d_2$ $c_4 \neq 0$ or we obtain $b_4c_4=-1$, and $d_4 = -\frac{d_2}{c_4}$ this means that two new reductive pairs are found:

$$h = Span\{d_2\vec{e_2} + \frac{b_4}{c_4}d_2\vec{e_3} - \frac{2}{c_4}d_2\vec{e_4}\},\ m = Span\{\vec{e_1}+\vec{e_4}, \vec{e_2}+b_4\vec{e_4}, \vec{e_3}+c_4\vec{e_4}\},$$

$c_4 \neq 0$;

$$h = Span\{d_2\vec{e_2} + d_3\vec{e_3} + (c_4d_3 - \frac{d_2}{c_4})\vec{e_4}\},\ m = Span\{\vec{e_1}+\vec{e_4}, \vec{e_2} - \frac{1}{c_4}\vec{e_4}, \vec{e_3}+c_4\vec{e_4}\},$$

$c_4 \neq 0$.

However, this pairs are particular cases of the reductive pair $\{h_6, m_6\}$ that will be found in the next subcase.

Suppose now that $b_4d_2=c_4d_3$, and equivalently $d_3 = \frac{b_4}{c_4}d_2$ if $c_4 \neq 0$. Utilizing this equality at the second and third equalities, we have the same result for d_4 from both of them:

$$d_4 = -\frac{d_2}{c_4}(a_4 + b_4c_4 + 1) + c_4d_3 = -\frac{d_2}{c_4}a_4 - b_4d_2 - \frac{d_2}{c_4} + b_4d_2 = -\frac{a_4+1}{c_4}d_2.$$

This produces the new reductive pair:

$$h_6 = Span\{d_2\vec{e_2} + \frac{b_4}{c_4}d_2\vec{e_3} - \frac{a_4+1}{c_4}d_2\vec{e_4}\}, m_6 = Span\{\vec{e_1}+a_4\vec{e_4}, \vec{e_2}+b_4\vec{e_4}, \vec{e_3}+c_4\vec{e_4}\},$$

$c_4 \neq 0$.

2. If $c_4=0$, then the corresponding system of equations for d_2, d_3, d_4 is: $(a_4-1)b_4d_2=0$, $d_3(a_4+1)+(d_4-b_4d_2)b_4=0$, $(a_4+1)d_2=0$.

If $a_4=-1$ then $b_4d_2=0$, $b_4d_4=0$, and $b_4=0$ or $d_2=d_4=0$. In the 1st case we obtain a reductive pair:

$$h_7 = Span\{d_2\vec{e_2} + d_3\vec{e_3} + d_4\vec{e_4}\},\ m_7 = Span\{\vec{e_1} - \vec{e_4}, \vec{e_2}, \vec{e_3}\}.$$

If $d_2=d_4=0$ then we obtain a nonreductive pair $h = Span\{\vec{e_3}\}, m = Span\{\vec{e_1} - \vec{e_4}, \vec{e_2} + b_4\vec{e_4}, \vec{e_3}\}$.

If a4≠–1, then $d_2=0$, $d_3 = -\frac{b_4}{a_4+1}d_4$, and a new reductive pair is $h_8 = Span\{-\frac{b_4}{a_4+1}d_4\vec{e_3} + d_4\vec{e_4}\}, m_8 = Span\{\vec{e_1}+a_4\vec{e_4}, \vec{e_2}+b_4\vec{e_4}, \vec{e_3}\}$, a_4–1.

2a. Consider the basis (2) for the complement m and the case (a) for vector $\vec{d} = \vec{e_1} + d_2\vec{e_2} + d_3\vec{e_3} + d_4\vec{e_4}$. Multiply vectors $\vec{a}, \vec{b}, \vec{c}$ by vector $\vec{d}$. We have:

$$\left[\vec{a},\vec{d}\right] = \left[\vec{e_1}+a_3\vec{e_3}, \vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right] = d_2\vec{e_2} - d_3\vec{e_3} + a_3\vec{e_3} - a_3d_2(\vec{e_1}-\vec{e_4}) - a_3d_4\vec{e_3} = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}.$$

So, $x_1=-a_3d_2$, $y_1=d_2$, $z_1=a_3d_2$, $x_1a_3+y_1b_3=a_3-d_3-a_3d_4$.

$$\left[\vec{b},\vec{d}\right] = \left[\vec{e_2}+b_3\vec{e_3}, \vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right] = -\vec{e_2} + d_3(\vec{e_1}-\vec{e_4}) + d_4\vec{e_2} + b_3\vec{e_3} - b_3d_2(\vec{e_1}-\vec{e_4}) - b_3d_4\vec{e_3} = x_2\vec{a} + +y_2\vec{b} + z_2\vec{c}.$$

So, $x_2=d_3-b_3d_2$, $y_2=-1d_4$, $z_2=-d_3+b_3d_2$, $x_2a_3+y_2b_3=b_3-b_3d_4$.

$$\left[\vec{c},\vec{d}\right] = \left[\vec{e_4}, \vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right] = -d_2\vec{e_2} + d_3\vec{e_3} = x_3\vec{a} + y_3\vec{b} + z_3\vec{c}.$$

So, $x_3=0$, $y_3=-d_2$, $z_3=0$, $x_3a_3+y_3b_3=d_3$.

These conditions generate the following system of equations for unknown components d_2, d_3, d_4:

$-a_3^2d_2 + b_3d_2 = a_3(1-d_4) - d_3$, $a_3(d_3-b_3d_2)+b_3(d_4-1)=b_3(1-d_4)$, $-b_3d_2=d_3$.

From the last equation, we have $d_3=-b_3d_2$. Substitute this value of d_3 into the first and the second equations: 0

$a_3^2d_2 = a_3(1-d_4)$, $a_3b_3d_2=b_3(1-d_4)$.

If $a_3 \neq 0$, $b_3 \neq 0$ then $d_4=1$, $d_2=0$, $d_3=0$, and the following reductive pair is obtained $h_9 = Span\{\vec{e_1}+\vec{e_4}\}$, $m_9 = Span\{\vec{e_1}+a_3\vec{e_3}, \vec{e_2}+b_3\vec{e_3}, \vec{e_4}\}$, $a_3$0.

If $a_3=0$, $b_3=0$ then $d_3=0$, and the following pair is obtained $h = Span\{\vec{e_1}+d_2\vec{e_2}+d_4\vec{e_4}\}$, $m = Span\{\vec{e_1}, \vec{e_2}, \vec{e_4}\}$.

This pair is not reductive because $h \subset m$.

If $a_3 \neq 0$, $b_3=0$ then $d_3=0$, $d_4=1-a_3d_2$, and the following reductive pair is obtained $h_{10} = Span\{\vec{e_1}+d_2\vec{e_2}+(1-a_3d_2)\vec{e_4}\}$, $m_{10} = Span\{\vec{e_1}+a_3\vec{e_3}, \vec{e_2}, \vec{e_4}\}$, $a_3 \neq 0$.

If $a_3=0$, $b_3 \neq 0$ then $d_3=-b_3d_2$, $d_4=1$, and the following reductive pair is obtained $h_{11} = Span\{\vec{e_1}+d_2\vec{e_2}-b_3d_2\vec{e_3}+\vec{e_4}\}$, $m_{11} = Span\{\vec{e_1}, \vec{e_2}+b_3\vec{e_3}, \vec{e_4}\}$, $b_3 \neq 0$.

2b. Consider the basis (2) for the complement m and the case (b) for vector $\vec{d} = d_2\vec{e_2} + d_3\vec{e_3} + d_4\vec{e_4}$. Multiply vectors $\vec{a}, \vec{b}, \vec{c}$ by vector $\vec{d}$. We have:

$$\left[\vec{a},\vec{d}\right] = \left[\vec{e_1}+a_3\vec{e_3}, d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right] = d_2\vec{e_2} - d_3\vec{e_3} - a_3d_2(\vec{e_1}-\vec{e_4}) - a_3d_4\vec{e_3} = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}.$$

So, $x_1=-a_3d_2$, $y_1=d_2$, $z_1=a_3d_2$, $x_1a_3+y_1b_3=-d_3-a_3d_4$.

$$\left[\vec{b},\vec{d}\right] = \left[\vec{e_2}+b_3\vec{e_3}, d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right] = d_3(\vec{e_1}-\vec{e_4}) + d_4\vec{e_2} - b_3d_2(\vec{e_1}-\vec{e_4}) - b_3d_4\vec{e_3} = x_2\vec{a} + y_2\vec{b} + z_2\vec{c}.$$

So, $x_2=d_3-b_3d_2$, $y_2=d_4$, $z_2=-d_3+b_3d_2$, $x_2a_3+y_2b_3=-b_3d_4$.

$$\left[\vec{c},\vec{d}\right] = \left[\vec{e_4}, d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right] = -d_2\vec{e_2} + d_3\vec{e_3} = x_3\vec{a} + y_3\vec{b} + z_3\vec{c}.$$

So, $x_3=0$, $y_3=-d_2$, $z_3=0$, $x_3a_3+y_3b_3=d_3$.

These conditions generate the next system of equations for unknown components d_2, d_3, d_4:

$-a_3^2d_2 + b_3d_2 = -a_3d_4 - d_3$, $a_3(d_3-b_3d_2)+b_3d_4=-b_3d_4$, $-b_3d_2=d_3$.

Simplifying this system of equations, we obtain $d_3=-b_3d_2$, $a_3^2d_2 = a_3d_4$, $a_3b_3d_2=b_3d_4$.

If $a_3 \neq 0$, $b_3 \neq 0$ then $d_3=-b_3d_2$, $d_4=a_3d_2$, and the following reductive pair is obtained $h_{12} = Span\{d_2\vec{e_2} - b_3d_2\vec{e_3} + a_3d_2\vec{e_4}\}$, $m_{12} = Span\{\vec{e_1}+a_3\vec{e_3}, \vec{e_2}+b_3\vec{e_3}, \vec{e_4}\}$.

If $a_3=0$, $b_3=0$ then a nonreductive pair $\{h, m\}$ is obtained because $h \subset m$. Similarly, if $a_3 \neq 0$, $b_3=0$ then a nonreductive pair $\{h, m\}$ is obtained because $h \subset m$.

If $a_3=0$, $b_3 \neq 0$ then the following reductive pair is obtained $h = Span\{d_2\vec{e_2} - b_3d_2\vec{e_3}\}$, $m = Span\{\vec{e_1}, \vec{e_2}+b_3\vec{e_3}, \vec{e_4}\}$ but this pair is a particular case of the pair $\{h_{12}, m_{12}\}$.

3a. Consider the basis (3) and the case (a) for the vector $\vec{d} = \vec{e_1} + \vec{d}_2\vec{e_2} + \vec{d}_3\vec{e_3} + \vec{d}_4\vec{e_4}$. Multiply vectors $\vec{a}, \vec{b}, \vec{c}$ by vector $\vec{d}$. We have:

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+a_2\vec{e_2},\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=d_2\vec{e_2}-d_3\vec{e_3}-a_2\vec{e_2}+a_2d_3(\vec{e_1}-\vec{e_4})+a_2d_4\vec{e_2}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}.$$

So, $x_1=a_2d_3$, $y_1=-d_3$, $z_1=-a_2d_3$, and $a_2^2d_3=d_2-a_2+a_2d_4$.

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_3},\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=\vec{e_3}-d_2(\vec{e_1}-\vec{e_4})-d_4\vec{e_3}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}\cdot$$

So, $x_2=-d_2$, $y_2=1-d_4$, $z_2=d_2$, $-a_2d_2=0$.

$$\left[\vec{c},\vec{d}\right]=\left[\vec{e_4},\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-d_2\vec{e_2}+d_3\vec{e_3}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}\cdot$$

So, $x_3=0$, $y_3=d_3$, $z_3=0$ and $d_2=0$.

These conditions imply $d_2=0$, $a_2^2d_3=a_2(d_4-1)$. If $a_2$0 then $d_4=1+a_2d_3$, and the following reductive pair is obtained $h_{13}=Span\{\vec{e_1}+d_3\vec{e_3}+(1+a_2d_3)\vec{e_4}\}$, $m_{13}=Span\{\vec{e_1}+a_2\vec{e_2},\vec{e_3},\vec{e_4}\}$ $a_2\neq0$.

If $a_2=0$ then the following pair appears: $h=Span\{\vec{e_1}+d_3\vec{e_3}+d_4\vec{e_4}\}$, $m=Span\{\vec{e_1},\vec{e_3},\vec{e_4}\}$. This pair is not reductive because $h\subset m$.

3b. Consider the basis (3) and the case (b) for the vector $\vec{d}=d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$. Multiply vectors $\vec{a},\vec{b},\vec{c}$ by vector $\vec{d}$. We have

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+a_2\vec{e_2},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=d_2\vec{e_2}-d_3\vec{e_3}+a_2d_3(\vec{e_1}-\vec{e_4})+a_2d_4\vec{e_2}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}.$$

So, $x_1=a_2d_3$, $y_1=-d_3$, $z_1=-a_2d_3$, and $a_2^2d_3=d_2+a_2d_4$.

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_3},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-d_2(\vec{e_1}-\vec{e_4})-d_4\vec{e_3}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}\cdot$$

So, $x_2=-d_2$, $y_2=-d_4$, $z_2=d_2$, and $-a_2d_2=0$.

$$\left[\vec{c},\vec{d}\right]=\left[\vec{e_4},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-d_2\vec{e_2}+d_3\vec{e_3}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}\cdot$$

So, $x_3=0$, $y_3=d_3$, $z_3=0$, and $-d_2=0$.

These conditions imply $d_2=0$, $a_2^2d_3=a_2d_4$. If $a_2$0 and $d_2=0$ then the pair $h=Span\{d_3\vec{e_3}+a_2d_3\vec{e_4}\}$, $m=Span\{\vec{e_1}+a_2\vec{e_2},\vec{e_3},\vec{e_4}\}$ is obtained but it is not reductive because $h\subset m$.

If $a_2=0$ and $d_2=0$, then we have a nonreductive pair $h=Span\{d_3\vec{e_3}+d_4\vec{e_4}\}$, $m=Span\{\vec{e_1},\vec{e_3},\vec{e_4}\}$ because hm.

4a. Consider the basis (4) and the case (a) for vector $\vec{d}=\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$. Multiply vectors $\vec{a},\vec{b},\vec{c}$ by vector $\vec{d}$. We have:

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_2},\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-\vec{e_2}+d_3(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}.$$

So, $x_1=d_4-1$, $y_1=0$, $z_1=-d_3$, and $d_3=0$.

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_3},\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=\vec{e_3}-d_2(\vec{e_1}-\vec{e_4})-d_4\vec{e_3}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}.$$

So, $x_2=0$, $y_2=1-d_4$, $z_2=d_2$, and $d_2=0$.

$$\left[\vec{c},\vec{d}\right]=\left[\vec{e_4},\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-d_2\vec{e_2}+d_3\vec{e_3}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}\cdot$$

So, $x_3=-d_2$, $y_3=d_3$, $z_3=0$, and 0=0 (it's an identity).

These equalities imply $d_3=0$, $d_2=0$. So, the following reductive pair is obtained $h_{14}=Span\{\vec{e_1}+d_4\vec{e_4}\}$, $m_{14}=Span\{\vec{e_2},\vec{e_3},\vec{e_4}\}$.

4b. Consider the basis (4) and the case (b) for vector $\vec{d}=d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$. Multiply vectors $\vec{a},\vec{b},\vec{c}$ by vector $\vec{d}$. We have:

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_2},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=d_3(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}\cdot$$

So, $x_1=d_4$, $y_1=0$, $z_1=-d_3$, and $d_3=0$.

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_3},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-d_2(\vec{e_1}-\vec{e_4})-d_4\vec{e_3}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}\cdot$$

So, $x_2=0$, $y_2=-d_4$, $z_1=d_2$, and $d_2=0$.

$$\left[\vec{c},\vec{d}\right]=\left[\vec{e_4},d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-d_2\vec{e_2}+d_3\vec{e_3}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}\cdot$$

So, $x_3=-d_2$, $y_3=d_3$, $z_3=0$, and 0=0.

These equalities imply $d_3=0$, $d_2=0$. So, the following pair is obtained $h=Span\{\vec{e_4}\}$, $m=Span\{\vec{e_2},\vec{e_3},\vec{e_4}\}$.

But this pair is not reductive because $h\subset m$.

Next statement describes all results that are found in the Part III.

Theorem 3: All different reductive pairs $\{h, m\}$ with 1-dimensional sub algebras h and 3-dimensiomal complements m of Lie algebra of all 2 × 2 real matrices are:

$h_1=Span\{\vec{e_1}+d_2\vec{e_2}+\frac{b_4}{c_4}d_2\vec{e_3}+(1-\frac{a_4+1}{c_4})d_2\vec{e_4}\}$,

$m_1=Span\{\vec{e_1}+a_4\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}+c_4\vec{e_4}\}$, $c_4\neq0$;

$h_2=Span\{\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+(1+b_4d_2+c_4d_3)\vec{e_4}\}$,

$m_2=Span\{\vec{e_1}-(1+2b_4c_4)\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}+c_4\vec{e_4}\}$;

$h_3=Span\{\vec{e_1}+\frac{b_4}{a_4+1}(1-d_4)\vec{e_3}+d_4\vec{e_4}\}$,

$m_3=Span\{\vec{e_1}+a_4\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}\}$, $a_4\neq-1$;

$h_4=Span\{\vec{e_1}+d_3\vec{e_3}+\vec{e_4}\}$, $m_4=Span\{\vec{e_1}-\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}\}$, $b_4\neq0$;

$h_5=Span\{\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\}$, $m_5=Span\{\vec{e_1}-\vec{e_4},\vec{e_2},\vec{e_3}\}$, $d_4\neq-1$;

$h_6=Span\{\vec{e_2}+\frac{b_4}{c_4}\vec{e_3}-\frac{a_4+1}{c_4}\vec{e_4}\}$, $m_6=Span\{\vec{e_1}+a_4\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}+c_4\vec{e_4}\}$, $c_4\neq0$.

$h_7=Span\{d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\}$, $m_7=Span\{\vec{e_1}-\vec{e_4},\vec{e_2},\vec{e_3}\}$, $d_4\neq0$;

$h_8=Span\{-\frac{b_4}{a_4+1}\vec{e_3}+\vec{e_4}\}$, $m_8=Span\{\vec{e_1}+a_4\vec{e_4},\vec{e_2}+b_4\vec{e_4},\vec{e_3}\}$, $a_4\neq-1$;

$h_9=Span\{\vec{e_1}+\vec{e_4}\}$, $m_9=Span\{\vec{e_1}+a_3\vec{e_3},\vec{e_2}+b_3\vec{e_3},\vec{e_4}\}$, $a_3\neq0$;

$m_{10}=Span\{e_1+a_3e_3,e_2,e_4\}$, $m_{10}=Span\{\vec{e_1}+a_3\vec{e_3},\vec{e_2},\vec{e_4}\}$, $a_3\neq0$;

$h_{11}=Span\{\vec{e_1}+d_2\vec{e_2}-b_3d_2\vec{e_3}+\vec{e_4}\}$, $m_{11}=Span\{\vec{e_1},\vec{e_2}+b_3\vec{e_3},\vec{e_4}\}$, $b_3\neq0$;

$h_{12}=Span\{\vec{e_2}-b_3\vec{e_3}+a_3\vec{e_4}\}$, $m_{12}=Span\{\vec{e_1}+a_3\vec{e_3},\vec{e_2}+b_3\vec{e_3},\vec{e_4}\}$, $b_3\neq0$;

$h_{13}=Span\{\vec{e_1}+d_3\vec{e_3}+(1+a_2d_3)\vec{e_4}\}$, $m_{13}=Span\{\vec{e_1}+a_2\vec{e_2},\vec{e_3},\vec{e_4}\}$, $a_2\neq0$;

$h_{14}=Span\{\vec{e_1}+d_4\vec{e_4}\}$, $m_{14}=Span\{\vec{e_2},\vec{e_3},\vec{e_4}\}$.

Remark

It's unknown what reductive pairs among $\{h_1, m_1\}$– $\{h_{14}, m_{14}\}$

are equivalent with respect to inner automorphisms of the given Lie algebra.

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