Classification of Maximal Subalgebras and Corresponding Reductive Pairs of Lie Algebra of All 2 × 2 Real Matrices

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Abstract

The purpose of the article is to describe all 3-dimensional subalgebras and all corresponding reductive pairs of Lie algebra of all 2 × 2 real matrices. This Lie algebra is 4-dimensional as a vector space, it's not simple, and it’s not solvable. The evaluation procedure utilizes the canonical bases for subspaces that were introduced. In Part I of this article, all 3-dimensional subalgebras of the given Lie algebra g are classified. All reductive pairs (h, m) with 3-dimensional subalgebras h are found in Part II. Surprisingly, there is only one reductive pair (h, m) with special 3-dimensional subalgebra h and 1-dimensional complement m. Finally, all reductive pairs (h, m) with 1-dimensional subalgebras h of algebra g are classified in Part III of the article.

Keywords: Lie algebra; Subalgebras; Reductive pairs

Introduction

Reductive homogeneous spaces appeared for the first time in the fundamental manuscript [1,2] of Katsumi Nomizu, in which the author investigated invariant affine connections and Riemannian metrics on them. Sagle and Winter in their article [3] analyzed algebraic structures generated by reductive pairs of simple Lie algebras. The next problem studied by some authors was classification of subalgebras of some Lie algebras. For example, Patera and Winternitz have classified all subalgebras of real Lie algebras of dimensions $d=3$ and $d=4$ in their manuscript [4]. This classification of subalgebras of low dimensional real Lie algebras was done by a representative of each conjugacy class where the conjugacy was considered under the group of inner automorphisms of Lie algebras. The articles mentioned above have stimulated this research for all subalgebras and all reductive pairs at the separate article. New knowledge concerning the structure of this Lie algebra is important for the conjugacy class where the conjugacy was considered under the group of inner automorphisms of Lie algebra.

We start with standard definitions for the readers’ convenience.

Definition 1

Let $g$ be a vector space over a field $F$. Then $g$ is called a Lie algebra over $F$ if there exists a Lie bracket operation $[x, y] \in g$ for any $x, y \in g$ such that:

\[
\begin{align*}
[a(x, y)] &= a[x, y], \\
[x, y + z] &= x[y, z] + [x, y], \\
[x, y^+] &= [x, y] + [y, x]
\end{align*}
\]

We call $[x, y]$ a Lie product.

Definition 2

Let $g$ be a Lie algebra. A subspace $h \subseteq g$ is called a (Lie) subalgebra of $g$, if $[h, h] \subseteq h$.

Definition 3

Let $g$ be a Lie algebra, $h$ be subalgebra of $g$. If there exists a subspace $m$ of $g$ such that $h \oplus m = g$ and $[h, m] = m$, then $[h, m]$ is called a reductive pair of $g$, and $[g, h, m]$ is called a reductive triple. We say also that subspace $m$ is a reductive complement for $h$.

Lie algebra $g$ and its standard basis:

This Lie algebra contains all $2 \times 2$ matrices over the field of all real numbers. The standard basis of this algebra consists of the next four matrices:

\[
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

It is well known that the Lie multiplication operation $[A, B]$ for any two square matrices $A$ and $B$ of the same size is defined to be $[A, B] = ABA$. According this rule, the fundamental products of the basic vectors (matrices) can be computed:

\[
\begin{align*}
e_1e_1 &= e_1^0, & e_1e_2 &= e_1^0, & e_2e_1 &= e_2^0, \\
e_1e_2 &= e_1^0, & e_2e_1 &= e_1^0, & e_2e_2 &= e_2^0, \\
e_2e_1 &= e_2^0, & e_2e_2 &= e_2^0, & e_3e_3 &= e_3^0, \\
e_3e_1 &= e_3^0, & e_3e_2 &= e_3^0, & e_3e_3 &= e_3^0
\end{align*}
\]

All other products of basic vectors are zeros.

Let $h$ be any 3-dimensional subspace of Lie algebra $g$. We can describe subspace $h$ as $h = \text{Span}(a, b, c)$ where $a = a_1e_1 + a_2e_2 + a_3e_3$, $b = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4$ and $c = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4$ are 3 linearly independent vectors.

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According to the article [1], all canonical bases for 3-dimensional subspaces of 4-dimensional vector space are:

1. \( \bar{a} = e_1 + a_1 e_2, \bar{b} = e_1 + b_1 e_2, \bar{c} = e_2 + c_1 e_1 \).
2. \( \bar{a} = e_1 + a_1 e_2, \bar{b} = e_1 + b_1 e_2, \bar{c} = e_2 + c_1 e_1 \).
3. \( \bar{a} = e_1 + a_1 e_2, \bar{b} = e_1 + b_1 e_2, \bar{c} = e_2 + c_1 e_1 \).
4. \( \bar{a} = e_1, \bar{b} = e_1, \bar{c} = e_1 \).

**Part I. Maximal subalgebras of Lie algebra**

Now we start to determine that a 3-dimensional subspace \( h = Span(a, b, c) \) is a subalgebra of Lie algebra \( g \) when vectors \( \bar{a}, \bar{b}, \bar{c} \) form one of the canonical bases (1), (2), (3), or (4) listed above. We have to check that the condition \([h, h] = h\) is true for each of these bases. The necessary evaluation procedure follows.

Let \( \bar{a} = e_1 + a_1 e_2, \bar{b} = e_1 + b_1 e_2, \bar{c} = e_2 + c_1 e_1 \) be the basis (1) for \( h \).

Evaluate Lie products \( [\bar{a}, \bar{b}], [\bar{a}, \bar{c}], [\bar{b}, \bar{c}] \):
\[
[\bar{a}, \bar{b}] = [e_1 + a_1 e_2, e_1 + b_1 e_2] = e_1 + a_1 e_2, e_1 + b_1 e_2, e_2 = e_2 + (a_1 - b_1) e_1.
\]

So, \( x_1 = y_1 = z_1 = 0, y_2 = 0, x_2 = a_1 - b_1, a_1 = 0, b_1 = 0 \). This means that the following 1-parameter set of subalgebras \( h_1 \) and one special subalgebra \( h_1 \) exist for this case:

\[
h_1 = Span(e_1, e_2, e_3 - \frac{1}{b_1} e_1), h_1 = Span(e_1 - e_2, e_3).
\]

2. Let \( \bar{a} = e_1 + a_1 e_2, \bar{b} = e_1 + b_1 e_2, \bar{c} = e_3, \) be the basis (2) for a possible subalgebra \( h \). Evaluate Lie products \( [\bar{a}, \bar{b}], [\bar{a}, \bar{c}], [\bar{b}, \bar{c}] \):
\[
[\bar{a}, \bar{b}] = [e_1 + a_1 e_2, e_1 + b_1 e_2] = e_1 + a_1 e_2, e_1 + b_1 e_2, e_2 = e_2 + (a_1 - b_1) e_1.
\]

So, we have \( x_1 = -a_1, y_1 = 1, z_1 = a_1, x_2 = 0, y_2 = 0, x_2 = 0, y_2 = -a_1, z_2 = -a_1. \) Consequently, the following system of equations appears for components \( a_2, b_2 \):
\[
-a_2^2 + b_2 = -a_1, a_2 = 0, b_2 = -b_1.
\]

This system of equations has only one solution \( a_2 = 0, b_2 = 0 \). This solution produces the following subalgebra:

\[
h_2 = Span(e_1, e_2, e_3) .
\]

Let \( \bar{a} = e_1 + a_1 e_2, \bar{b} = e_1 + b_1 e_2, \bar{c} = e_3 \) be the basis (3) for \( h \). Evaluate Lie products \( [\bar{a}, \bar{b}], [\bar{a}, \bar{c}], [\bar{b}, \bar{c}] \) for this case. We have:
\[
[\bar{a}, \bar{b}] = [e_1 + a_1 e_2, e_1 + b_1 e_2] = e_1 + a_1 e_2, e_1 + b_1 e_2, e_2 = e_2 + (a_1 - b_1) e_1.
\]

So, \( x_1 = -a_2, y_2 = -1, z_1 = a_2, a_2 = 0, b_2 = 0 \).

Evaluate the system of equations has only one solution \( a_2 = 0, b_2 = 0 \). The corresponding subalgebra is
\[
h_3 = Span(e_1, e_2, e_3) .
\]

Consider the last possible basis \( \bar{a} = e_1, \bar{b} = e_1, \bar{c} = e_1 \). Evaluate Lie products \( [\bar{a}, \bar{b}], [\bar{a}, \bar{c}], [\bar{b}, \bar{c}] \) for this basis. We obtain:
\[
[\bar{a}, \bar{b}] = [e_1, e_1] = e_1 + e_1 = e_1 + e_1 = e_1 + c_1 e_1.
\]

The vector \( e_1 - e_1 \) doesn’t belong to the subspace \( h = Span(\bar{a}, \bar{b}, \bar{c}) \) at this case. So, this subspace is not subalgebra of Lie algebra.

The next statement describes all 3-dimensional subalgebras of Lie algebra \( g \).

**Theorem 1:** All different 3-dimensional subalgebras of Lie algebra of all 2 × 2 real matrices are listed here:

\[
h_1 = Span(e_1 + e_2, e_2, e_3 - \frac{1}{b_1} e_1), h_2 = Span(e_1 - e_2, e_3) ;
\]

\[
h_3 = Span(e_1, e_2, e_3) .
\]

**Corollary:** The subalgebras above are maximal for the given Lie algebra.

**Part II. Reductive pairs with 3-dimensional subalgebras**

How many of 3-dimensional subalgebras \( h \) form reductive pairs \( [h, m] \) in this Lie algebra? To answer this question, we will use the conditions from the Definition 3, i.e. \([h, m] = 0\) where \( m \) is an appropriate 1-dimensional reductive component for a given 3-dimensional subalgebra \( h \). The list of all 3-dimensional subalgebras from Theorem 2 will be used to find all possible reductive components. Let \( m = Span(d, e_1, e_2, e_3) \) be a possible 1-dimensional component. To simplify our evaluation, we consider 2 possible cases for the generating vector \( \bar{d} \):

\[
\bar{d} = e_1 + d_1 e_1 + d_2 e_2 + d_3 e_3, \quad \bar{d} = e_1 + d_1 e_1 + d_2 e_2 + d_3 e_3
\]

(\( d \neq 0 \)).

**Subalgebra** \( h_1 \) **Case 1:** Multiply basic vectors \( a = e_1 + e_2, b = e_1 + e_2, c = e_1 - \frac{1}{b_1} e_1 \) from \( h_1 \) by vector \( \bar{d} = e_1 + d_1 e_1 + d_2 e_2 + d_3 e_3 \). We have:

\[
[e_1, e_1 - d_1 e_1 + d_2 e_2 + d_3 e_3] = -e_1 - d_1 (e_1 - e_1) + d_2 e_2 + d_3 e_3 = -d_1 e_1 + d_2 e_2 + d_3 e_3
\]

(\( \text{it's an identity} \)).

From the vector equalities above, we obtain the following system of conditions for the coefficients \( d_1, d_2, d_3 \) and coefficients \( y, z = y, z \), \( yd_1 = -1 + d_2 - b_1 d_3, yd_2 = b_1 d_3, yd_3 = -d_3 \), \( zd_1 = \frac{z}{b_1}, zd_2 = \frac{z}{b_1}, zd_3 = -d_3 \). These equalities produce the system of 6 equations for \( d_1, d_2, d_3 \).

\[
\bar{d} = -d_1 e_1 + d_2 e_2 + d_3 e_3 \quad \bar{d} = e_1 - d_1 e_1 + d_2 e_2 + d_3 e_3
\]

We have:

\[
[e_1, e_1 - d_1 e_1 + d_2 e_2 + d_3 e_3] = -e_1 - d_1 (e_1 - e_1) + d_2 e_2 + d_3 e_3 = -d_1 e_1 + d_2 e_2 + d_3 e_3
\]

(\( \text{it's an identity} \)).

From the vector equalities above, we obtain the following system of conditions for the coefficients \( d_1, d_2, d_3 \) and coefficients \( y, z = y, z \), \( yd_1 = -1 + d_2 - b_1 d_3, yd_2 = b_1 d_3, yd_3 = -d_3 \), \( zd_1 = \frac{z}{b_1}, zd_2 = \frac{z}{b_1}, zd_3 = -d_3 \). These equalities produce the system of 6 equations for \( d_1, d_2, d_3 \).
Subalgebra \( h \), Case 2: Multiply basic vectors from \( h \) by vector \( \vec{d} \):

\[
\vec{d} = \vec{e}_1 + \vec{e}_2 + \vec{d}_1 + \vec{d}_2 + \vec{d}_3.
\]

We have:

\[
\begin{align*}
\vec{e}_1 + \vec{e}_2 + \vec{d}_1 &= y \vec{d} \\
\vec{e}_1 + \vec{d}_1 + \vec{d}_2 + \vec{d}_3 &= y \vec{d} \\
\vec{e}_1 + \vec{d}_1 + \vec{d}_2 + \vec{d}_3 &= y \vec{d} \\
\vec{e}_1 + \vec{d}_1 + \vec{d}_2 + \vec{d}_3 &= y \vec{d} \\
\vec{e}_1 + \vec{d}_1 + \vec{d}_2 + \vec{d}_3 &= y \vec{d}
\end{align*}
\]

From the vector equalities above, we obtain the following system of equations for the components \( d_1, d_2, d_3 \) and coefficients \( y, z \):

\[
\begin{align*}
d_1 &= 0, \quad y d_2 &= -d_1 \\
d_2 &= 0, \quad y d_3 &= -d_1 \\
z d_3 &= 0.
\end{align*}
\]

The last system of equations has just the zero solution for \( d_2, d_3, d_4 = 0 \), \( d_1 = 0 \). The zero vectors \( \vec{d} = \vec{0} \) is the only solution for the system. This means that a nonzero reductive complement for \( h \) doesn’t exist.

Subalgebra \( h \), Case 1: Multiply basic vectors \( \vec{a} = -\vec{e}_1, \vec{b} = \vec{e}_2, \vec{c} = \vec{e}_3 \) from \( h \) by vector \( \vec{d} = \vec{e}_1 + \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4 \).

We have:

\[
\begin{align*}
\vec{e}_1 + \vec{d}_1 &= x \vec{d} \\
\vec{e}_1 + \vec{d}_2 &= y \vec{d} \\
\vec{e}_1 + \vec{d}_3 &= z \vec{d} \\
\vec{e}_1 + \vec{d}_4 &= y \vec{d}
\end{align*}
\]

From the last system of vector equalities, we obtain a system of equations for \( d_1, d_2, d_3 \) that has just one solution \( d_1 = 0, d_2 = 0, d_3 = 1 \). The subspace \( m = \text{Span}(\vec{e}_1, \vec{e}_2) \) generated by vector \( \vec{d} = \vec{e}_1 + \vec{e}_2 \) satisfies the conditions \([h, m] \subseteq m\) and \(g = h \oplus m\), so \([h, m] \) is a reductive pair for this case where \( h = \text{Span}(\vec{e}_1, \vec{e}_2, \vec{e}_5), m = \text{Span}(\vec{e}_1 + \vec{e}_2) \).

Subalgebra \( h \), Case 2: Multiply basic vectors of \( h \) by vector \( \vec{d} = \vec{e}_1 + \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4 \).

We have:

\[
\begin{align*}
\vec{e}_1 - \vec{d}_1 &= x \vec{d} \\
\vec{e}_1 - \vec{d}_2 &= y \vec{d} \\
\vec{e}_1 - \vec{d}_3 &= z \vec{d} \\
\vec{e}_1 - \vec{d}_4 &= y \vec{d}
\end{align*}
\]

Transforming this system of vector equalities into a system of equations for the components \( d_1, d_2, d_3 \), and solving the system, the only zero solution \( d_1 = 0, d_2 = 0, d_3 = 0 \) is obtained. So, a nonzero reductive complement for \( h \) doesn’t exist for this case.

Subalgebras \( h_1 \) and \( h_2 \) don’t produce any reductive pair. The details are similar for the cases of subalgebras \( h_1 \) and \( h_2 \), therefore they are omitted.

The total analysis conducted in this Part III establishes the following statement.

**Theorem 2:** The only one reductive pair with 3-dimensional subspace \( m \) for Lie algebra \( g \) of real all \( 2 \times 2 \) matrices; it is \([h, m]\) where \( h_2 = \text{Span}(\vec{e}_1, \vec{e}_2, \vec{e}_3), m = \text{Span}(\vec{e}_2 + \vec{e}_3) \).

**Corollary:** The subspace \( m = \text{Span}(\vec{e}_2 + \vec{e}_3) \) is a 1-dimensional ideal of Lie algebra \( g \). Moreover, the subalgebra \( h = \text{Span}(\vec{e}_1, \vec{e}_2, \vec{e}_3) \) is a 3-dimensional ideal of Lie algebra \( g \).

**Part III. Reductive pairs \([h, m]\) with 1-dimensional subalgebras \( h \) of Lie algebra \( g \)**

It is well known that each 1-dimensional subspace \( h \) of algebra \( g \) is an abelian subalgebra of \( g \). The corresponding reductive complements \( m \) for each \( h \) should be 3-dimensional subspaces \( m \) such that \([h, m] \subseteq m\) and \(g = h \oplus m\). Therefore, the canonical bases for 3-dimensional subspaces that are found in the Part I can be utilized for \( m \). The list of all canonical bases contains the next 4 bases:

1. \( a = \vec{e}_1 + \vec{e}_2, b = \vec{e}_3, c = \vec{e}_1 + \vec{e}_3 \).
2. \( \vec{a} = \vec{e}_1 + \vec{e}_2, b = \vec{e}_3, c = \vec{e}_1 + \vec{e}_3 \).
3. \( \vec{a} = \vec{e}_1 + \vec{e}_2, b = \vec{e}_3, c = \vec{e}_1 + \vec{e}_3 \).
4. \( \vec{a} = \vec{e}_1, b = \vec{e}_2, c = \vec{e}_3 \).

Let \( h = \text{Span}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3 + \vec{e}_4) \) be 1-dimensional subalgebra in algebra \( g \). We will consider two cases for the generating vector \( \vec{d} \):

- \( \vec{a} = \vec{e}_1 + \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4, \) if \( d_1 \neq 0 \);
- \( \vec{a} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4, \) if \( d_1 = 0 \).

To determine if a subalgebra \( h \) forms a reductive pair with some complement \( m \), we will check that the conditions \([h, m] \subseteq m\), \(g = h \oplus m\) are satisfied.

1a. Consider the basis (1) for a complement \( m \) and case (a) for \( \vec{d} \).

Multiply vectors \( \vec{d}, \vec{b}, \vec{c} \) by vector \( \vec{d} 
\] \[
\vec{e}_1 - \vec{d}_1 \vec{e}_2 + \vec{e}_3 + \vec{d}_2 \vec{e}_1 + a \vec{d}_3 \vec{e}_1 + d_3 \vec{e}_3 + d_4 \vec{e}_4 = \vec{e}_1 + d_1 \vec{e}_2 + \vec{e}_3 + a \vec{d}_3 \vec{e}_1 + d_4 \vec{e}_4
\]

The corresponding reductive pairs with 3-dimensional subalgebra \( h \) in algebra \( g \). We will consider two cases for the generating vector \( \vec{d} \):

- \( \vec{a} = \vec{e}_1 + \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4, \) if \( d_1 \neq 0 \);
- \( \vec{a} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4, \) if \( d_1 = 0 \).

To determine if a subalgebra \( h \) forms a reductive pair with some complement \( m \), we will check that the conditions \([h, m] \subseteq m\), \(g = h \oplus m\) are satisfied.

1a. Consider the basis (1) for a complement \( m \) and case (a) for \( \vec{d} \).

Multiply vectors \( \vec{d}, \vec{b}, \vec{c} \) by vector \( \vec{d} 
\] \[
\vec{e}_1 - \vec{d}_1 \vec{e}_2 + \vec{e}_3 + \vec{d}_2 \vec{e}_1 + a \vec{d}_3 \vec{e}_1 + d_3 \vec{e}_3 + d_4 \vec{e}_4 = \vec{e}_1 + d_1 \vec{e}_2 + \vec{e}_3 + a \vec{d}_3 \vec{e}_1 + d_4 \vec{e}_4
\]
Comparing these two equalities for $a_1=1$ then from the second and third equations we obtain $d_2(-1+a_1)b+bd_4=0$, $d_2a_1+(1+a_1)b+bd_4=0$, $d_2a_1+(1+a_1)b+bd_4=0$. The last two equations produce two expressions for $d_4$:

\[ d_4 = 1 + b_1d_2 - \frac{(2 - c_i)}{\frac{1}{a_i}} \quad \text{and} \quad d_4 = 1 + c_4d_1 - \frac{(2 - c_i)}{\frac{1}{a_i}} + b_4. \]

Comparing these two equalities for $d_4$, we obtain $b \neq -1$ or $c_i \neq b_4d_2$. Using these conditions, we obtain for this case $a_1=1$ the following solutions:

\[ \bar{a} = a_1 + \frac{1}{b_4} + b_1d_4, \quad \bar{d} = a_1 + \frac{1}{c_4} + b_4d_2, \quad \text{and} \quad \bar{d} = a_1 + \frac{1}{c_4} + b_4d_2 + \frac{(2 - c_i)}{\frac{1}{a_i}}. \]

These solutions produce the following two reductive pairs:

\[ h = \text{Span} [\bar{a}, \bar{c}], \quad m = \text{Span} [c_i, c_i, c_i, c_i, c_i, c_i], \quad b \neq 0; \]

\[ h = \text{Span} [\bar{a}, \bar{c}], \quad m = \text{Span} [c_i, c_i, c_i, c_i, c_i, c_i], \quad c_i \neq 0. \]

These two reductive pairs are particular cases of the pairs $[h, m]$, $[h, m]$ obtained in the next step 2.

1. If $b, c_1 = 0$, we can suppose that $a_1$ is any real number. Now, like in the previous step 1, we obtain the following system of three equations:

\[ d_2d_4 = \frac{a_1 + 1}{b_4} + c_i, \quad d_1 = d_1(1 + c_4d_1 - \frac{(1 + a_1)}{c_i}). \]

From the 2nd and 3rd equations above, we can find the 4th component $d_4$:

\[ d_4 = 1 + b_1d_2 - d_1(\frac{(a_1 + 1)}{b_4} + c_i), \quad d_4 = 1 + c_4d_1 - d_1(\frac{(1 + a_1)}{c_i}). \]

Comparing the last two equalities for the same component $d_4$, we have

\[ d_4 = b_1d_2 - d_1(\frac{(a_1 + 1)}{b_4} + c_i), \quad d_4 = c_4d_1 - d_1(\frac{(1 + a_1)}{c_i}). \]

The last equality produces two results: $d_4 = \frac{b_1}{b_4}d_2 \quad (c_i \neq 0)$, or $a_1 + 1 + 2b_4 = 0$. Compute the component $d_4$ in terms of $a_1$, $b_4$, $c_i$, $d_4$, when $d_4 = \frac{b_1}{b_4}d_2$. We have $d_4 = \frac{1 - a_1}{c_i} = d_4$. So, we obtain the following reductive pair:

\[ h = \text{Span} [\bar{a}, \bar{c}], \quad m = \text{Span} [c_i, c_i, c_i, c_i, c_i, c_i], \quad c_i \neq 0. \]

For the condition $a_1 + 1 + 2b_4 = 0$, we have $d_4 = b_1d_2 + c_i$, and the corresponding reductive pair is:

\[ h = \text{Span} [\bar{a}, \bar{c}], \quad m = \text{Span} [c_i, c_i, c_i, c_i, c_i, c_i], \quad b_4 \neq 0; \]

\[ h = \text{Span} [\bar{a}, \bar{c}], \quad c_i \neq 0. \]

1b. Consider the basis (1) for a complement $m$ and the case (b) for vector $ar{a}$.

Multiply vectors $\bar{a}, \bar{b}, \bar{c}$ by vector $\bar{d}$. We have

\[ \bar{a}d_1 = a_1 + \frac{1}{b_1} + b_2d_4, \quad \bar{b}d_2 = b_1 + \frac{1}{b_1} + b_2d_4, \quad \bar{c}d_3 = c_4d_1 - \frac{1}{c_4} + b_4d_2. \]

So, $x_0 = 0, y_0 = -d_1b_2d_4, z_0 = a_1 + \frac{1}{b_1} + b_2d_4$. Therefore, $d_1 = d_1(a_1) = 0$. From $d_1 = 0$, we obtain $d_1 = \frac{b_1}{b_4}d_2$.}

In this case, the following system of 3 equations for unknown components $d_2$, $d_4$, $d_4$ is obtained $d_2(1 - a_1)b_2 + d_2(a_1) = 0, d_2(1 - a_1)b_2 + d_2(a_1) = 0$. Solve the system of equations starting with the first equation. We have the equality $d_2(1 - a_1)b_2 + d_2(a_1) = 0$, which produces $a_1 = 1$ or $b_4 = d_2$ ($c_i \neq 0$).

1. If $a_1 = 1$, then we obtain $d_2(1 - a_1)b_2 + d_2(a_1) = 0$, $d_2 = -b_2d_4 + (c_i - d_i)c_i = 2d_4$ from the 2nd and 3rd equations. Find $d_4$ from these two equalities:

\[ d_4 = b_1d_2 - d_1(c_i + \frac{2}{b_4}), \quad d_4 = c_4d_1 - d_1(b_4 + \frac{2}{c_i}). \]
Comparing two different expressions for the same component $d_i$, we find $d_i = \frac{b_i}{c_i} d_j$, and $d_i = -\frac{2}{c_i} d_j$, or we obtain $b \neq 0$, and $c \neq 0$.

This means that two new reductive pairs are found:

$h = \text{Span}(c_i, e_i, c_i - 2 d_i)$, $m = \text{Span}(c_i + e_i, c_i + b_i e_i, c_i + c_i e_i)$, $c \neq 0$.

$h = \text{Span}(c_i, e_i + c_i d_i - \frac{d_i}{c_i} e_i)$, $m = \text{Span}(c_i + e_i, c_i + \frac{1}{c_i} e_i, c_i + c_i e_i)$, $c \neq 0$.

However, these pairs are particular cases of the reductive pair $[h, m_i]$ that will be found in the next subsection.

Suppose now that $b d_i = c d_i$, and equivalently $d_i = \frac{b}{c} d_j$ if $c \neq 0$.

Utilizing this equality at the second and third equalities, we have the same result for $d_i$ from both of them:

$d_i = -\frac{d_j}{c_i} (a_i + b c_i + 1) + c_i d_i = -\frac{d_j}{c_i} a_i - b d_i - \frac{d_j}{c_i} + b d_i = -\frac{a_i + 1}{c_i} d_j$.

This produces the new reductive pair:

$h_i = \text{Span}(c_i, e_i - d_i c_i, m_i = \text{Span}(c_i + e_i, c_i + b_i e_i, c_i + c_i e_i))$.

If $c = 0$, then the corresponding system of equations for $d_1, d_2, d_4$ is: $(a_i - 1)b d_i = 0$, $(a_i + 1) + (a_i - 1) b d_i = 0$, $(a_i + 1) d_i = 0$.

If $a_i = -1$ then $b d_i = 0$, $b d_i = 0$, and $b d_i = 0$.

1a. Consider the basis (2) for the complement $m$ and the case (a) for the vector $\bar{d} = e_i + d_i e_i + d_i c_i + d_i e_i$.

Multipliy vectors $\bar{a}, \bar{b}, \bar{c}$ by vector $\bar{d}$.

We have:

$[\bar{a}, \bar{d}] = \left[ \begin{array}{c} e_i + a_i e_i, e_i + e_i c_i + e_i c_i + e_i e_i + d_i e_i + d_i e_i + d_i c_i + d_i e_i \end{array} \right] = d_i e_i - d_i e_i + a_i e_i - a_i d_i e_i - a_i d_i c_i = x_i a_i + y_i b + z_i c$.

So, $x_i = -a_i d_i, y_i = y_i, z_i = a_i d_i, x_i a_i + y_i b + z_i c = d_i e_i$.

Similarly, if $a_i = 1$ then $b d_i = 0$, $b d_i = 0$, and $b d_i = 0$.

1b. Consider the basis (2) for the complement $m$ and the case (b) for the vector $\bar{d} = e_i + e_i e_i + b_i e_i + c_i e_i$.

Multipliy vectors $\bar{a}, \bar{b}, \bar{c}$ by vector $\bar{d}$.

We have:

$[-a_i \bar{d}] = \left[ \begin{array}{c} e_i + a_i e_i, e_i + e_i c_i + e_i c_i + e_i e_i + d_i e_i + d_i e_i + d_i c_i + d_i e_i \end{array} \right] = -d_i e_i + d_i e_i + a_i e_i + a_i d_i c_i + a_i d_i e_i = x_i a_i + y_i b + z_i c$.

So, $x_i = -d_i, y_i = a_i d_i, x_i a_i + y_i b + z_i c = d_i e_i$.

These conditions generate the following system of equations for unknown components $d_1, d_2, d_4$:

$-a_i^2 d_i + b_i d_i = a_i(d_i - d_j), a_i(d_i - b_i d_i + b_i(d_i - 1)) = b_i(1 - d_i) - b_i d_i$.

From the last expression, we have $d_i = -b_i d_i$. Substitute this value of $d_i$ into the first and the second equations: $0$

$-a_i^2 d_i = a_i(1 - d_j), a_i b_i d_i = b_i(1 - d_j)$.

If $a_i \neq 0, b_i \neq 0$ then $d_i = 1, d_i = 0$, and the following reductive pair is obtained $h_1 = \text{Span}(e_i, e_i + (1 - a_i d_i) e_i)$, $m_1 = \text{Span}(e_i, a_i e_i, e_i + b_i e_i, c_i e_i), a_i \neq 0$.

If $a_i = 0, b_i \neq 0$ then $d_i = -b_i d_i$, and the following reductive pair is obtained $h_2 = \text{Span}(c_i + d_i e_i)$,

$m_2 = \text{Span}(c_i + e_i, c_i + b_i e_i, c_i + c_i e_i)$.

This is not reductive because $h \neq m$.

If $a_i \neq 0, b_i \neq 0$ then $d_i = 1, d_i = 1 - a_i d_i$, and the following reductive pair is obtained $h_3 = \text{Span}(e_i + d_i e_i + (1 - a_i d_i) e_i)$, $m_3 = \text{Span}(e_i + a_i e_i, e_i + b_i e_i, c_i e_i), a_i \neq 0$.

If $a_i = 0, b_i = 0$ then $d_i = -b_i d_i$, and the following reductive pair is obtained $h_4 = \text{Span}(c_i + d_i c_i + d_i e_i)$, $m_4 = \text{Span}(c_i + e_i, c_i + b_i e_i, c_i + c_i e_i)$.
If \( m \) pair

These equalities imply \( d = 0 \) and \( d = 0 \).

These conditions imply \( d = 0 \), \( a^2 d = a (d - 1) \). If \( a \neq 0 \)

Then \( d = 1 + a d \) and the following reductive pair is obtained

\[ h_a = \text{Span}(e_1 + d e_2), \quad m_a = \text{Span}(e_3 + e_4 e_2). \]

Next statement describes all results that are found in the Part III.

**Theorem 3:** All different reductive pairs \( [h, m] \) with 1-dimensional sub algebras \( h \) and 3-dimensional complements \( m \) of Lie algebra of all \( 2 \times 2 \) real matrices are:

\[ h_1 = \text{Span}(e_1 + e_2), \quad m_1 = \text{Span}(e_3 + e_4 e_2), \quad a = 0; \]

\[ h_2 = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_2 = \text{Span}(e_3 + e_4 e_2), \quad c = 0; \]

\[ h_3 = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_3 = \text{Span}(e_3 + e_4 e_2), \quad c = 0; \]

\[ h_4 = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_4 = \text{Span}(e_3 + e_4 e_2), \quad c = 0; \]

\[ h_5 = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_5 = \text{Span}(e_3 + e_4 e_2), \quad d = 0; \]

\[ h_6 = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_6 = \text{Span}(e_3 + e_4 e_2), \quad d = 0; \]

\[ h_7 = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_7 = \text{Span}(e_3 + e_4 e_2), \quad c = 0; \]

\[ h_8 = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_8 = \text{Span}(e_3 + e_4 e_2), \quad a = 0; \]

\[ h_9 = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_9 = \text{Span}(e_3 + e_4 e_2), \quad a = 0; \]

\[ h_{10} = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_{10} = \text{Span}(e_3 + e_4 e_2), \quad a = 0; \]

\[ h_{11} = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_{11} = \text{Span}(e_3 + e_4 e_2), \quad b = 0; \]

\[ h_{12} = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_{12} = \text{Span}(e_3 + e_4 e_2), \quad b = 0; \]

\[ h_{13} = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_{13} = \text{Span}(e_3 + e_4 e_2), \quad b = 0; \]

\[ h_{14} = \text{Span}(e_1 + e_2 + e_3 + e_4), \quad m_{14} = \text{Span}(e_3 + e_4 e_2). \]

**Remark**

It’s unknown what reductive pairs among \( [h_1, m_1] - [h_{14}, m_{14}] \)
are equivalent with respect to inner automorphisms of the given Lie algebra.

References