Certain New Subclasses of Uniformly P-Valent Star like and Convex Functions

Vandna Agnihotri* and Ran Singh²

1D-09, The LNM Institute of Information Technology, Jaipur-302031, Rajasthan, India
²Department of Mathematics, DAV College, CSJM University, Kanpur-208016, UP, India

Abstract

In this paper we introduce certain new subclasses of uniformly p-valent star like and convex functions. Sufficient coefficient conditions are obtained for functions in these classes. We provide geometrical properties of functions belonging to these classes. Hadamard product with convex functions and certain coefficient estimates are also obtained.

Keywords: Uniformly P-Valent functions; Star like functions; Convex functions; Hadamard Product; Jack’s Lemma

Introduction

Let \( A_p \) denote the class of functions \( f(z) \) of the form

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in \mathbb{N} = \{1, 2, \ldots\} \]

which are analytic in the open unit disc \( U = \{z \in \mathbb{C} : |z| < 1\} \).

A function \( f \in A_p \) is said to be \( p \)-valent star like of order \( \alpha \) \((0 \leq \alpha < p)\), if

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in U
\]

A function \( f \in A_p \) is said to be \( p \)-valent convex of order \( \alpha \) \((0 \leq \alpha < p)\), if

\[
\Re \left( \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in U
\]

Let \( S_p^*(\alpha) \) and \( K_p(\alpha) \) denote, respectively, the classes of \( p \)-valent star like and convex functions of order \( \alpha \) in \( U \).

Note that for \( p=1 \) the classes \( S_1^*(\alpha)=S^*(\alpha) \) and \( K_1(\alpha)=K(\alpha) \) are the classes of univalent star like and univalent convex functions of order \( \alpha \) \((0 \leq \alpha < 1)\) respectively. We know that \( f \in K_p(\alpha) \) if and only if \( zf''(z) \in S^*(\alpha) \).

The Subclasses \( S_p^*(\beta,\alpha) \) and \( K_p(\beta,\alpha) \)

We now introduce two new subclasses denoted by \( S_p^*(\beta,\alpha) \) and \( K_p(\beta,\alpha) \) of functions \( f(z) \in A_p \) as follows:-

**Definition 4.1** We say that a function \( f \in A_p \) is in the class \( S_p^*(\beta,\alpha) \) if

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \beta \frac{zf'(z)}{f(z)} - \beta + \alpha, \quad z \in U
\]

for some \( \beta \geq 0 \) and \( 0 \leq \alpha < p \).

**Definition 4.2** We say that a function \( f \in A_p \) is in the class \( K_p(\beta,\alpha) \) if

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta \left| \frac{zf''(z)}{f'(z)} - (p-1) \right| + \alpha, \quad z \in U,
\]

for some \( \beta \geq 0 \) and \( 0 \leq \alpha < p \).

Note that \( f(z) \in K_p(\beta,\alpha) \) if and only if \( zf''(z) \in S_p^*(\beta,\alpha) \).

For \( p=1 \) the subclasses \( S_1^*(\beta,\alpha)=S_1^*(\alpha) \) and \( K_1(\beta,\alpha)=K_1(\beta,\alpha) \) were introduced and studied by Shams, Kulkarni and Jahangiri in [1]. They obtained sufficient coefficient conditions for functions in the classes \( S(\beta,\alpha) \) and \( K(\beta,\alpha) \) along with geometric properties of functions in these classes. For \( p=1, \alpha=0 \) and \( \beta=1 \), we obtain the class \( K(1,1) \) of uniformly convex functions, defined by Goodman [2,3].

For \( p=1 \) and \( \alpha=1 \) the class \( K(1,1) \) of uniformly convex functions belonging to these classes. Hadamard product with convex functions and certain coefficient estimates are also obtained.

**Geometric Properties and Coefficient Inequalities**

Set \( w(z)=\frac{zf'(z)}{f(z)} \) and \( \Omega_{w,\alpha} = \{w : \Re(w) > \beta |w - \beta|^{-\alpha}\} \). If \( f(z) \in S_p(\beta,\alpha) \) then \( w(z) \) belongs to the region \( \Omega_{w,\alpha} \). If \( \beta=1 \) then \( \frac{zf'(z)}{f(z)} \) lies in the region \( \Omega_{w,1} \), which contains \( w=p \) and is bounded by the parabola \( v^2 = 2(p-\alpha) - \frac{p+\alpha}{2} \). Figure 1 shows the region \( \Omega_{w,1} \) for \( \alpha=0 \).

If \( \beta>1 \) then \( \frac{zf'(z)}{f(z)} \) lies in the region \( \Omega_{w,\beta} \), which contains \( w=p \) and is bounded by the ellipse

\[
\frac{u^2}{(p\beta^2-\alpha)} + \frac{v^2}{(p\beta^2-1)} = \frac{\beta^2(p-\alpha)}{(p\beta^2-1)^2}
\]

*Corresponding author: Vandna Agnihotri, aD-09, The LNM Institute of Information Technology, Jaipur-302031, Rajasthan, India, E-mail: vandnaiitk@yahoo.co.in

Received July 16, 2013; Accepted August 13, 2013; Published August 23, 2013


Copyright: © 2013 Agnihotri V, et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.
With vertices at the points

\[
\left( \frac{p^\beta - a}{\beta - 1}, \frac{p^\beta + a}{\beta + 1} \right), \left( \frac{p^{\beta^2} - a}{(\beta - 1)^2}, \frac{p^{\beta^2} + a}{(\beta + 1)^2} \right), \text{ and } \left( \frac{p^{\beta^3} - a}{(\beta - 1)^3}, \frac{p^{\beta^3} + a}{(\beta + 1)^3} \right)
\]

Since \( a < \frac{p^\beta + a}{\beta + 1} < p < \frac{p^\beta - a}{\beta - 1} \),

therefore, we obtain \( \Omega_{\beta,a} = \{ w : \Re(w) > a \} \). \( \Omega_{\beta,a} \) Hence, \( S_{\beta} \beta, \alpha \ast, \Omega_{\beta,a} \subset S' \alpha \).

We now give coefficient inequalities for functions belonging to the subclasses \( S_{\beta} \beta, \alpha \) and \( K_{\beta} \beta, \alpha \). Our first result is contained in

**Theorem 5.1:** If \( f(z) \in Ap \) satisfies

\[
\sum_{n=1}^{\infty} \left| n(1 + \beta) - (a + p\beta) \right| |a_n| \leq (1 - \alpha),
\]

then \( f(z) \in S_{\beta} \beta, \alpha \).

Proof. We know that \( f(z) \in S_{\beta} \beta, \alpha \) if

\[
\left| \frac{zf'(z)}{f(z)} \right| > \beta \left| \frac{zf'(z)}{f(z)} - p \right| + \alpha,
\]

or equivalently,

\[
|w(1 + \alpha) - \beta w - p| \leq |w(1 + \alpha) - \beta w - p| - |w(z) = \frac{zf'(z)}{f(z)}|.
\]

It is sufficient to show that \( R-L \geq 0 \), where

\[
R = |w(1 + \alpha) - \beta w - p| \quad \text{and} \quad L = |w(1 + \alpha) - \beta w - p|.
\]

Now,

\[
R = \left| \frac{zf'(z)}{f(z)} \right| + (1 - \alpha) - \beta \left| \frac{zf'(z)}{f(z)} - p \right|
\]

and

\[
L = \left| \frac{zf'(z)}{f(z)} \right| - (1 + \alpha) - \beta \left| \frac{zf'(z)}{f(z)} - p \right|
\]

thus

\[
L = \left| \frac{zf'(z)}{f(z)} \right| - (1 + \alpha) - \beta \left| \frac{zf'(z)}{f(z)} - p \right|
\]

\[
= \left| \frac{zf'(z)}{f(z)} \right| - (1 + \alpha) - \beta \left| \frac{zf'(z)}{f(z)} - p \right|
\]

\[
\leq \left| \frac{zf'(z)}{f(z)} \right| - (1 + \alpha) - \beta \sum_{n=1}^{\infty} (n - 1 - \alpha) |a_n||f'|^n + \beta \sum_{n=1}^{\infty} (n - p) |a_n||f'|^n
\]

\[
\leq \left| \frac{zf'(z)}{f(z)} \right| - (1 + \alpha) - \beta \sum_{n=1}^{\infty} (n - 1 - \alpha + n\beta - p\beta) |a_n|.
\]

From (3.2) and (3.3), we have

\[
R-L > \frac{1 - \alpha}{|f(z)|} \left[ 2(1-\alpha) - 2 \sum_{n=1}^{\infty} (n(1 + \beta) - (a + p\beta)) |a_n| \right]
\]

using (3.1), we get

\[
R-L > 0
\]

For \( p=1 \), as immediate consequence of Theorem 3.1 we obtain the following corollary due to Shams et al. [1]:

**Corollary 5.1:** If

\[
\sum_{n=1}^{\infty} \left| n(1 + \beta) - (a + p\beta) \right| |a_n| \leq (1 - \alpha),
\]

then \( f(z) \in S_{\beta} \beta, \alpha \).

For \( \beta=0 \) and \( p=1 \), we have

**Corollary 5.2:** If

\[
\sum_{n=1}^{\infty} (n - a) |a_n| \leq (1 - \alpha) \quad \text{then } f(z) \in S' \alpha \]

Here \( S' \alpha \) is the usual class of starlike functions of order \( \alpha \). Next, we state corresponding result for functions belonging to the subclass \( K_{\beta} \beta, \alpha \).

**Theorem 5.2:** If \( f(z) \in Ap \) satisfies

\[
\sum_{n=1}^{\infty} \left| n(1 + \beta) - (a + p\beta) \right| |a_n| \leq (1 - \alpha),
\]

then \( f(z) \in K_{\beta} \beta, \alpha \).

Proof: Proof follows from the proof of previous theorem and the fact that \( f(z) \in K_{\beta} \beta, \alpha \) if and only if \( zf'(z) \in S_{\beta} \beta, \alpha \).

Taking \( p=1 \) in Theorem 3.2, we obtain

**Corollary 5.3:** If

\[
\sum_{n=1}^{\infty} n(1 + \beta) - (a + p\beta) |a_n| \leq (1 - \alpha),
\]

then \( f(z) \in K(\beta, \alpha) \).

This was proved by Shams et al. [1]. For \( \beta=0 \) and \( p=1 \), we have

**Corollary 5.4:** If \( \sum_{n=1}^{\infty} n(1 + \beta) - (a + p\beta) |a_n| \leq (1 - \alpha) \) then \( f(z) \in K(\alpha) \).

It is easy to see that for \( a=0, \beta=0 \) and \( p=1 \) in Theorem 5.1 and 5.2 we obtain well known coefficient estimates for functions in the classes

**Figure 1:** Parabolic domain 1, for \( \Omega_{\beta,a} \) for \( \alpha=0 \).
of starlike and convex functions denoted by $S^*$ and $K$ respectively [6, 7].

**Hadamard Product**

Let $f(z)$, $g(z) \in A_p$ be given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$$

then the Hadamard product (convolution) of $f(z)$ and $g(z)$ is defined by

$$(f \ast g)(z) = z^p + \sum_{n=p+1}^{\infty} (a_n b_n) z^n$$

We now state a theorem, the proof of which follows using a convolution result of Shams et al. [1] and Theorem 5.1

**Theorem 6.1:** The classes $SD_p(\beta, \alpha)$ and $KD_p(\beta, \alpha)$ are closed under Hadamard product with convex functions in $U$.

As a consequence of the above theorem we have

**Corollary 6.1:** If $f(z)$ is in $SD_p(\beta, \alpha)$ and (or $KD_p(\beta, \alpha)$) then the function $g(z)$ defined by

$$g(z) = \frac{1 + \mu}{z^p} \int_0^z t^{p-\gamma} f(t) \text{d}t, \Re(\mu) \geq 0$$

is also in $SD_p(\beta, \alpha)$ (or $KD_p(\beta, \alpha)$).

Proof: Using definition of convolution of functions in $SD_p(\beta, \alpha)$ (or $KD_p(\beta, \alpha)$), $g(z)$ can be written as

$$g(z) = f(z) \ast \left( \sum_{n=p+1}^{\infty} \left( \frac{1 + \mu}{n + \mu} \right) z^n \right),$$

Where $\sum_{n=p+1}^{\infty} \left( \frac{1 + \mu}{n + \mu} \right) z^n$ is convex in $U$. The conclusion now follows from Theorem 6.1.

Similarly, we have

**Corollary 6.2:** If $f(z)$ is in $SD_p(\beta, \alpha)$ (or $KD_p(\beta, \alpha)$) then the function $h(z)$ defined by

$$h(z) = \frac{1 + \mu}{z^p} \int_0^z t^{p-\gamma} f(t) \text{d}t, \quad |\lambda| \leq 1, \lambda \neq 1$$

is also in $SD_p(\beta, \alpha)$ (or $KD_p(\beta, \alpha)$).

Proof: As earlier, we can write $h(z)$ as

$$h(z) = f(z) \ast \left( \sum_{n=p+1}^{\infty} \frac{1 - \lambda^n}{n(1 - \lambda^n)} z^n \right).$$

Note that the function $z^p + \sum_{n=p+1}^{\infty} \frac{1 - \lambda^n}{n(1 - \lambda^n)} z^n$ is convex in $U$. Therefore, using Theorem 6.1 the conclusion follows.

We now give a necessary and sufficient condition for functions to be in the class $SD_p(\beta, \alpha)$ (or $KD_p(\beta, \alpha)$). We know that if $f(z) \in SD_p(\beta, \alpha)$ and $\beta \geq 1$ then, where

$$z^p f(z) \subset \Omega_{p, \beta}, \quad \Omega_{p, \beta} \text{ is the region bounded by the ellipse }$$

$$\left( 1 - \frac{(p\beta^p - 1)}{(p^p - 1)} \right)^2 + \frac{\beta^p - 1}{(p^p - 1)} \gamma = \frac{\beta^p (p - 1)}{(p^p - 1)}.$$

In parametric form the equation of the ellipse becomes

$$w(t) = \left( \frac{(p\beta^p - 1)}{(p^p - 1)} \right) t + \frac{\beta^p - 1}{(p^p - 1)}, \quad 0 \leq t \leq 2\beta.$$

Hence, for $\beta > 1$, $z \in U\{0\}$, we have $f(z) \in SD_p(\beta, \alpha)$ if and only if $F(z)$ by

$$F(z) = \frac{1}{1 - w(t)} \left\{ \frac{z}{(1 - z^p)} - \frac{z}{(1 - z)} \right\}$$

Then using results from, we obtain $f(z) \in SD_p(\beta, \alpha)$ if and only if $F(z) \neq \frac{F(z)}{z} \neq 0$.

Conversely, if $F(z) \neq \frac{F(z)}{z} \neq 0$, then $\frac{z}{(1 - z^p)} \neq w(t), 0 \leq t \leq 2\beta$.

Therefore, $F(z) \neq \frac{F(z)}{z} \neq 0$, lies inside $\Omega_{p, \beta}$ or outside $\Omega_{p, \beta}$ for $z \in U$. But

$$F(z) = p \in \Omega_{p, \beta}, \text{thus } \frac{z}{(1 - z^p)} < \Omega_{p, \beta}. \text{ Hence, } f(z) \in SD_p(\beta, \alpha)$.

So, we have

**Theorem 6.2:** A function $f(z) \in SD_p(\beta, \alpha)$, $\beta > 1$ if and only if $F(z) \neq \frac{F(z)}{z} \neq 0$ where $F(z)$ is given by (4.2).

**Certain Sufficient Estimates**

In order to prove our results in this section we need the following lemma due to Jack [6].

**Lemma 7.1:** (Jack’s Lemma) Let $w(z)$ be (non-constant) analytic function in $U$ with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0$, then

$$z_0^p w(z_0) = c w(z_0),$$

where $c$ is real and $c \geq 1$.

Making use of Lemma 7.1 our first result is:

**Theorem 7.1:** Let $f(z) \in A_p$, then $f(z)$ is uniformly p-valent starlike of order $\alpha$ in $U$ if the inequality

$$\left| 1 + \frac{z^p f(z)}{f(z) - p} \right| > 1 + \frac{2\beta}{p(1 + 2\beta)},$$

holds.

Proof: Define $w(z)$ by

$$w(z) = \frac{2\beta}{p} \left( \frac{z}{f(z)} \right) - p, \quad p \in \mathbb{N}, z \in U.$$ (5.2)

Note that $w(z)$ is analytic in $U$ and $w(0)=0$. Differentiating (5.2) logarithmically we get

$$\frac{z^p f(z)}{f(z) - p} \frac{z^n w(z)}{w(z)} = \frac{z^p f(z)}{f(z)} \left( 1 - \frac{z^p f(z)}{f(z)} \right) \frac{z^n f(z)}{f(z)},$$

which leads to

$$1 + \frac{z^p f(z)}{f(z) - p} = \frac{p}{2\beta} w(z) + \frac{z^n w(z)}{2\beta + w(z)}, \quad p \in \mathbb{N}, z \in U.$$ (5.3)

Combining (5.2) and (5.3) we see that

$$1 + \frac{z^p f(z)}{f(z) - p} = \frac{p}{2\beta} w(z)(2\beta + w(z)), \quad p \in \mathbb{N}, z \in U.$$ (5.4)
Now assume that there exists a point \( z_0 \in U \) with \( |z_0| = 1 \) such that
\[
\max_{|w(z)| = 1} |w(z_0)| = 1, \quad \text{and let } w(z_0) = e^{i\theta} \neq -\pi. \quad \text{Now applying Jack's Lemma 7.1, we get,}
\]
\[
z_0 w(z_0) = c, \quad c \geq 1. \quad (5.5)
\]
From (5.4) and (5.5) we obtain
\[
\Re \left[ \frac{1 + z f(z_0) - p}{1 + \frac{z f(z_0) - p}{p + w(z_0)}} \right] = \Re \left[ \frac{1 + \frac{2\beta}{p} - \frac{1}{2\beta + w(z_0)}}{p} \right] = 1 + \frac{2\beta}{p} - \frac{1}{2\beta + e^{i\theta}}
\]
\[
\geq 1 + \frac{2\beta}{p} - \frac{1}{2\beta + e^{i\theta}} \geq 1 + \frac{\beta}{2\beta + 1 + 2\beta} \geq 1 + \frac{2\beta}{p + (1 + 2\beta)},
\]
which contradicts the hypothesis (5.1). Therefore, we conclude that \( |w(z)| < 1 \) for all \( z \in U \) and as in the proof of the previous theorem, (5.8) yields the inequality
\[
\frac{zf(z) - \alpha}{f(z)} > \beta \frac{zf(z) - (p - 1)}{f(z) - p}, \quad \text{i.e. } f(z) \text{ is uniformly p-valent starlike of order } \alpha \text{ in } U.
\]
Next, we determine the sufficient coefficient bound for uniformly p-valent convex functions of order \( \alpha \).

**Theorem 7.2:** Let \( f(z) \in Ap \), then \( f(z) \) is uniformly p-valent convex of order \( \alpha \) in \( U \) if the inequality holds.
\[
\Re \left[ \frac{1 + z f(z) - p}{1 + \frac{z f(z) - p}{p + w(z)}} \right] < 1 + \frac{2\beta}{p + (1 + 2\beta)}, \quad (5.7)
\]
Proof: Define \( w(z) \) by
\[
w(z) = \frac{2\beta}{p} \left( 1 + \frac{zf(z) - p}{f(z)} \right), \quad p \in \mathbb{N}, \quad z \in U. \quad (5.8)
\]
Note that \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). Differentiating (5.8) logarithmically we get
\[
\frac{1 + z f(z) - p}{f(z)} = p - 1 + \frac{p}{2\beta} w(z) + \frac{p}{2\beta} \frac{zw(z)}{w(z) - p - 1 + \frac{p}{2\beta} w(z)} \quad \text{and let } w(z_0) = e^{i\theta} \neq -\pi. \quad \text{Now applying Jack's Lemma 7.1, we get,}
\]
\[
z_0 w(z_0) = c, \quad c \geq 1. \quad (5.5)
\]
where \( p \in \mathbb{N}, \quad z \in U. \)

Suppose now that there exists a point \( z_0 \in U \) with \( |z_0| = 1 \) such that
\[
\max_{|w(z)| = 1} |w(z_0)| = 1, \quad \text{and let } w(z_0) = e^{i\theta} \neq -\pi. \quad \text{Applying Jack's Lemma 5.1, we obtain}
\]
\[
z_0 w(z_0) = c, \quad c \geq 1. \quad (5.11)
\]
Now (5.10) and (5.11) yield
\[
\Re \left[ \frac{1 + z f(z) - p}{1 + \frac{z f(z) - p}{p + w(z)}} \right] = \Re \left[ \frac{1 + \frac{2\beta}{p} - \frac{1}{2\beta + e^{i\theta}}}{1 + \frac{p}{2\beta} - \frac{1}{2\beta + w(z)}} \right]
\]
\[
\geq 1 + \frac{2\beta}{p} - \frac{1}{2\beta + e^{i\theta}} \geq 1 + \frac{\beta}{2\beta + 1 + 2\beta} \geq 1 + \frac{2\beta}{p + (1 + 2\beta)},
\]
which contradicts the hypothesis (5.7). Therefore, we conclude that \( |w(z)| < 1 \) for all \( z \in U \) and as in the proof of the previous theorem, (5.8) yields the inequality
\[
\frac{zf(z) - \alpha}{f(z)} > \beta \frac{zf(z) - (p - 1)}{f(z) - p}, \quad \text{i.e. } f(z) \text{ is uniformly p-valent convex of order } \alpha \text{ in } U.
\]

Taking \( \beta = 1 \) in Theorems 5.1 and 5.2 we get the following corollaries proved by Al-Kharsani and Al-Hajiry in [7,8]

**Corollary 7.1:** Let \( f(z) \in Ap \) satisfies the inequality
\[
\Re \left[ \frac{1 + z f(z) - p}{1 + \frac{z f(z) - p}{p + w(z)}} \right] < 1 + \frac{2}{3p},
\]
then \( f(z) \) is uniformly p-valent starlike in \( U. \)

**Corollary 7.2:** Let \( f(z) \in Ap \) satisfies the inequality
\[
\Re \left[ \frac{1 + z f(z) - p}{f(z) - p} \right] < 1 + \frac{2}{3p - 2},
\]
then $f(z)$ is uniformly $p$-valent convex in $U$.

In order to prove our next result we need the following definition:

**Definition 7.1:** A function $f(z) \in A_p$ is said to be uniformly $p$-valent close-to-convex (or uniformly close-to-convex when $p=1$) of order $\alpha$ in $U$ if it satisfies the inequality

$$\Re \left( \frac{zf'(z)}{f(z)} - p \right) \geq \frac{\beta}{1+2\beta}$$

for some $g(z) \in SDp(\alpha, \beta)$.

The following theorem gives the sufficient condition for uniformly $p$-valent close-to-convex functions.

**Theorem 7.3:** Let $f(z) \in A_p$ satisfies the inequality

$$\Re \left( \frac{zf'(z)}{f(z)} - p \right) < \frac{2\beta}{1+2\beta},$$

then $f(z)$ is uniformly $p$-valent close-to-convex of order $\alpha$ in $U$.

**Proof:** Let us define $w(z)$ by

$$w(z) = p - \frac{zf'(z)}{f(z)} - p,$$

Clearly, $w(z)$ is analytic in $U$ and $w(0)=0$. Moreover, logarithmic differentiation of (5.14) give rise to

$$\Re \left( \frac{zf'(z)}{f(z)} - p \right) = (p-1) + \frac{z^2w'(z)}{2\beta + w(z)}.$$

As earlier, using conditions of Jack’s Lemma and (5.13), we get

$$\Re \left( \frac{zf'(z)}{f(z)} - p \right) = (p-1) + c\Re \left( \frac{w(z)}{2\beta + w(z)} \right)$$

$$\geq (p-1) + c\Re \left( \frac{e^\theta}{2\beta + e^\theta} \right)$$

$$\geq (p-1) + c \frac{1}{1+2\beta}$$

$$\geq (p-1) + \frac{2\beta}{1+2\beta}$$

which contradicts the hypothesis (5.13). Therefore, we conclude that $|w(z)| < 1$ for all $z \in U$ and (5.14) yields the inequality

$$\left| \frac{f'(z)}{z^{p-1}} \right| < \frac{p}{2\beta}, \quad (p \in \mathbb{N}, z \in U)$$

which implies that $\frac{f'(z)}{z^{p-1}} \in \Omega_{\alpha, \beta}$ and therefore

$$\Re \left( \frac{f'(z)}{z^{p-1}} - \alpha \right) < \beta \left| \frac{f'(z)}{z^{p-1}} - p \right|$$

i.e. $f(z)$ is uniformly $p$-valent close-to-convex of order $\alpha$ in $U$.

If $\beta = 1$, then Theorem 5.3 gives us the corresponding result of Al-Kharsani and Al-Hajiry established in [2].

**Corollary 7.3:** Let $f(z) \in A_p$ satisfies the inequality

$$\Re \left( \frac{zf'(z)}{f(z)} - p \right) < \frac{2}{3},$$

then $f(z)$ is uniformly p-valent close-to-convex of order $\alpha$ in $U$.

For parametric values $p=1$, $\beta=1$ in Theorems 5.1, 5.2 and 5.3 we get

**Corollary 7.4:** Let $f(z) \in A$ satisfies the inequality

$$\Re \left( \frac{zf'(z)}{f(z)} - 1 \right) < \frac{5}{3},$$

then $f(z)$ is uniformly starlike in $U$.

**Corollary 7.5:** Let $f(z) \in A$ satisfies the inequality

$$\Re \left( \frac{zf'(z)}{f(z)} \right) < \frac{1}{3},$$

then $f(z)$ is uniformly convex in $U$.

**Corollary 7.6:** Let $f(z) \in A$ satisfies the inequality

$$\Re \left( \frac{zf'(z)}{f(z)} \right) < \frac{1}{3},$$

then $f(z)$ is uniformly p-valent close-to-convex in $U$.

**Remark 7.1:** For different values of the parameters $p, \alpha$ and $\beta$ in all our results of this chapter one can easily obtain many other interesting results that have been provided in [1,2,8] and references therein.

**References**