Journal of
Applied & Computational Mathematics
ISSN: 2168-9679

Agnihotri and Singh, J Appl Computat Math 2013, 2:4
DOI: 10.4172/2168-9679.1000135

Research Article | Open Access

Certain New Subclasses of Uniformly P-Valent Star like and Convex Functions

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Abstract

In this paper we introduce certain new subclasses of uniformly p-valent star like and convex functions. Sufficient coefficient conditions are obtained for functions in these classes. We provide geometrical properties of functions belonging to these classes. Hadamard product with convex functions and certain coefficient estimates are also obtained.

Keywords: Uniformly P-Valent functions; Star like functions; Convex functions; Hadamard Product; Jack's Lemma

Introduction

Let A_p denote the class of functions f(z) of the form

$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, p \in \mathbb{N} = \{1,2,....\}$ which are analytic in the open unit disc U={z ∈ C : |z| < 1}.

A function f ∈ A_p is said to be p−valent star like of order α (0 ≤ α<p), if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in U$$

A function f ∈ A_p is said to be p-valent convex of order α (0 ≤ α<p), if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in U$$

Let S^*_p (α) and Kp(α) denote, respectively, the classes of p-valent star like and convex functions of order α in U.

Note that for p=1 the classes S^*_1 (α)=S*(α) and K1(α)=K(α) are the usual classes of univalent star like and univalent convex functions of order α (0 ≤ α < 1) respectively.

For p=1, α=0 the classes S^*_p (α) and Kp(α) reduces to S*(0)=S* and K(0)=K, which are the classes of star like and convex functions (univalent) with respect to the origin respectively. We know that f ∈ Kp(α) if and only if zf′(z) ∈ S^*_p (α).

The Subclasses SDp(β,α) and KDp(β, α)

We now introduce two new subclasses denoted by SD_p (β,α) and KDp (β,α) of functions f(z) ∈ A_p as follows:-

Definition 4.1) We say that a function f ∈ A_p is in the class SD_p (β,α) if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta\left|\frac{zf'(z)}{f(z)} - p\right| + \alpha,\ z \in U$$

for some β ≥ 0 and 0 ≤ α < p.

Definition 4.2) We say that a function f ∈ A_p is in the class KD_p (β,α) if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta\left|\frac{zf''(z)}{f'(z)} - (p-1)\right| + \alpha, z \in U,$$

for some β ≥ 0 and 0 ≤ α<p.

Note that f(z) ∈ KD_p(β,α) if and only if zf′(z) ∈ SD_p(β,α).

For p=1 the subclasses SD1(β,α)=SD(β,α) and KD1(β,α)=KD (β,α) were introduced and studied by Shams, Kulkarni and Jahangiri in [1]. They obtained sufficient coefficient conditions for functions in the classes SD(β,α) and KD(β,α) along with geometric properties of functions in these classes. For p=1, α=0 and β=1, we obtain the class KD1(1, 0) of uniformly convex functions, defined by Goodman [2,3]. For p=1 and α=1 the class KD1(1, α) of uniformly convex functions of order β was investigated by Rønning [4,5]. In this paper we shall study the geometric properties, coefficient bounds and convolution properties for functions in the classes SD_p(β,α) and KD_p(β,α). We will show that these classes are closed under certain integral operators.

Geometric Properties and Coefficient Inequalities

Set $w(z) = \frac{zf'(z)}{f(z)}$ and $\Omega_{\beta,\alpha} = \{w : \Re(w) > \beta|w-p| + \alpha\}$. If $f(z) \in SD_p(\beta\alpha)$ then w(z) belongs to the region $\Omega_{\beta\alpha}$. If β=1 then $\frac{zf'(z)}{f(z)}$ lies in the region Ω_1, α which contains w=p and is bounded by the parabola $v^2 = 2(p-\alpha)\left(u - \frac{p+\alpha}{2}\right)$. Figure 1 shows the region 1, α for α=0.

If β>1 then $\frac{zf'(z)}{f(z)}$ lies in the region $\Omega_{\beta,\alpha}$ which contains w=p and is bounded by the ellipse

$$\left(u - \frac{(p\beta^2-\alpha)}{(\beta^2-1)}\right)^2 + \frac{\beta^2}{(\beta^2-1)}v^2 = \frac{\beta^2(p-\alpha)^2}{(\beta^2-1)^2}$$

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Received July 16, 2013; **Accepted** August 13, 2013; **Published** August 23, 2013

Citation: Agnihotri V, Singh R (2013) Certain New Subclasses of Uniformly P-Valent Star like and Convex Functions. J Appl Computat Math 2: 135. doi:10.4172/2168-9679.1000135

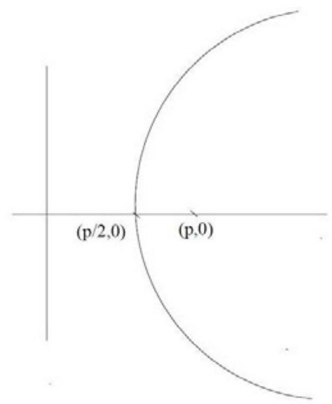

Figure 1: Parabolic domain 1, for $\Omega_{1\alpha}$ for α=0.

With vertices at the points

$$\left(\frac{p\beta-\alpha}{\beta-1},0\right),\left(\frac{p\beta+\alpha}{\beta+1},0\right),\left(\frac{(p\beta^2-\alpha)}{(\beta^2-1)},\frac{(p-\alpha)}{\sqrt{(\beta^2-1)}}\right), and \left(\frac{(p\beta^2-\alpha)}{(\beta^2-1)},\frac{(\alpha-p)}{\sqrt{(\beta^2-1)}}\right).$$

Since

$$\alpha<\frac{p\beta+\alpha}{\beta+1}<p<\frac{p\beta-\alpha}{\beta-1},$$

therefore, we obtain $\Omega_{\beta,\alpha}\subset\{w:\Re(w)>\alpha\}$. $\Omega_{\beta,\alpha}$ Hence, $SD_p(\beta,\alpha)\subset S^*(\alpha)$.

We now give coefficient inequalities for functions belonging to the subclasses SD_p (β,α) and KD_p(β,α). Our first result is contained in

Theorem 5.1: If f(z) ∈ Ap satisfies

$$\sum_{n=p+1}^{\infty}\{n(1+\beta)-(\alpha+p\beta)\}|\alpha_n|\le(1-\alpha),$$

then $f(z)\in SD_p(\beta,\alpha)$.

Proof. We know that $f(z)\in SD_p(\beta,\alpha)$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\beta\left|\frac{zf'(z)}{f(z)}-p\right|+\alpha,$$

or equivalently,

$$|w-(1+\alpha)-\beta|w-p||<|w+(1-\alpha)-\beta|w-p||,\ where\ \ w(z)=\frac{zf'(z)}{f(z)}.$$

It is sufficient to show that R–L>0, where

$$R=|w+(1-\alpha)-\beta|w-p||\ \ and\ \ L=|w-(1+\alpha)-\beta|w-p||.$$

Now,

$$R=\left|\frac{zf'(z)}{f(z)}+(1-\alpha)-\beta\left|\frac{zf'(z)}{f(z)}-p\right|\right|$$

$$=\frac{1}{|f(z)|}\left|zf'(z)+(1-\alpha)f(z)-\beta\frac{f(z)}{|f(z)|}|zf'(z)-pf(z)|\right|$$

$$=\frac{1}{|f(z)|}\left|(p+1-\alpha)z^p+\sum_{n=p+1}^{\infty}(n+1-\alpha)a_nz^n-\beta e^{i\theta}\left|\sum_{n=p+1}^{\infty}(n-p)a_nz^n\right|\right|$$

$$\ge\frac{1}{|f(z)|}\left[(p+1-\alpha)|z|^p-\sum_{n=p+1}^{\infty}(n+1-\alpha)|a_n||z|^n-\beta\sum_{n=p+1}^{\infty}(n-p)|a_n||z|^n\right]$$

$$\ge\frac{|z|^p}{|f(z)|}\left[(p+1-\alpha)-\sum_{n=p+1}^{\infty}(n+1-\alpha+n\beta-p\beta)|a_n|\right],\qquad(3.2)$$

and

$$L=\left|\frac{zf'(z)}{f(z)}-(1+\alpha)-\beta\left|\frac{zf'(z)}{f(z)}-p\right|\right|$$

$$=\frac{1}{|f(z)|}\left|zf'(z)-(1+\alpha)f(z)-\beta\frac{zf'(z)}{|f(z)|}|zf'(z)-pf(z)|\right|$$

$$\le\frac{1}{|f(z)|}\left[(p-1+\alpha)|z|^p+\sum_{n=p+1}^{\infty}(n-1-\alpha)|a_n||z|^n+\beta\sum_{n=p+1}^{\infty}(n-p)|a_n||z|^n\right]$$

$$<\frac{|z|^p}{|f(z)|}\left[(p-1+\alpha)+\sum_{n=p+1}^{\infty}(n-1-\alpha+n\beta-p\beta)|a_n|\right].\ (3.3)$$

From (3.2) and (3.3), we have

$$R-L>\frac{|z^p|}{|f(z)|}\left[2(1-\alpha)-2\sum_{n=p+1}^{\infty}2\{n(1+\beta)-(\alpha+p\beta)\}|a_n|\right]$$

using (3.1), we get

R–L>0

For p=1, as immediate consequence of Theorem 3.1 we obtain the following corollary due to Shams et al. [1]:

Corollary 5.1: If

$$\sum_{n=p+1}^{\infty}\{n(1+\beta)-(\alpha+\beta)\}|a_n|\le(1-\alpha),$$

then f(z) ∈ SD (β,α).

For β=0 and p=1, we have

Corollary 5.2: If

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\le(1-\alpha)\ \ then\ f(z)\in S^*(\alpha)$$

Here S^* (α) is the usual class of starlike functions of order α. Next, we state corresponding result for functions belonging to the subclass KD_p(β,α).

Theorem 5.2: If f(z) ∈ Ap satisfies

$$\sum_{n=p+1}^{\infty}n\{n(1+\beta)-(\alpha+p\beta)\}|a_n|\le(1-\alpha),\qquad(3.4)$$

then f(z) ∈ SD_p(β,α).

Proof: Proof follows from the proof of previous theorem and the fact that f(z) ∈ KD_p(β,α) if and only if zf'(z) ∈ SD_p(β,α).

Taking p=1 in Theorem 3.2, we obtain

Corollary 5.3: If

$$\sum_{n=p+1}^{\infty}n\{n(1+\beta)-(\alpha+\beta)\}|a_n|\le(1-\alpha),$$

then f(z)∈ KD(β,α).

This was proved by Shams et al. [1]. For β=0 and p=1, we have

Corollary 5.4: If $\sum_{n=2}^{\infty}n(n-\alpha)|a_n|\le(1-\alpha)\ then\ f(z)\in K(\alpha)$.

It is easy to see that for α=0, β=0 and p=1 in Theorem 5.1 and 5.2 we obtain well known coefficient estimates for functions in the classes

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of starlike and convex functions denoted by S* and K respectively [6, 7].

Hadamard Product

Let f(z), g(z) ∈ Ap be given by

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n,\ g(z)=z^p+\sum_{n=p+1}^{\infty}b_nz^n$$

then the Hadamard product (convolution) of f(z) and g(z) is defined by

$$(f*g)(z)=z^p+\sum_{n=p+1}^{\infty}b_na_nz^n$$

We now state a theorem, the proof of which follows using a convolution result of Shams et al. [1] and Theorem 5.1

Theorem 6.1: The classes $SD_p(\beta,\alpha)$ and $KD_p(\beta,\alpha)$ are closed under Hadamard product with convex functions in U.

As a consequence of the above Theorem, we have

Corollary 6.1: If f(z) is in $SD_p(\beta,\alpha)$ and (or $KD_p(\beta,\alpha)$) then the function g(z) defined by

$$g(z)=\frac{1+\mu}{z^{\mu}}\int_0^z t^{\mu-p}f(t)dt,\ \Re(\mu)\geq 0$$

is also in $SD_p(\beta,\alpha)$(or $KD_p(\beta,\alpha)$).

Proof: Using definition of convolution of functions in $SD_p(\beta,\alpha)$(or $KD_p(\beta,\alpha)$), g(z) can be written as

$$g(z)=f(z)*\sum_{n=p+1}^{\infty}\left(\frac{1+\mu}{n+\mu}\right)z^n,$$

Where $\sum_{n=p+1}^{\infty}\left(\frac{1+\mu}{n+\mu}\right)z^n$ is convex in U. The conclusion now follows from Theorem 6.1.

Similarly, we have

Corollary 6.2: If f(z) is in $SD_p(\beta,\alpha)$(or $KD_p(\beta,\alpha)$) then the function h(z) defined by

$$h(z)=\int_0^z\frac{f(t)-f(\lambda t)}{t^p(1-\lambda)}dt,\ |\lambda|\leq 1,\lambda\neq 1$$

is also in $SD_p(\beta,\alpha)$(or $KD_p(\beta,\alpha)$). Proof: As earlier, we can write h(z) as

$$h(z)=f(z)*\left(z^p+\sum_{n=p+1}^{\infty}\frac{1-\lambda^n}{n(1-\lambda)}z^n\right).$$

Note that the function $z^p+\sum_{n=p+1}^{\infty}\frac{1-\lambda^n}{n(1-\lambda)}z^n$ is convex in U. Therefore, using Theorem 6.1 the conclusion follows.

We now give a necessary and sufficient condition for functions to be in the class $SD_p(\beta,\alpha)$(or $KD_p(\beta,\alpha)$).We know that if f(z) ∈ $SD_p(\beta,\alpha)$ and β>1 then, where $\frac{zf'(z)}{f(z)}\subset\Omega_{\beta,\alpha}$, $\Omega_{\beta,\alpha}$ is the region bounded by the ellipse

$$\left(u-\frac{(p\beta^2-\alpha)}{(\beta^2-1)}\right)^2+\frac{\beta^2}{(\beta^2-1)}v^2=\frac{\beta^2(p-\alpha)^2}{(\beta^2-1)^2}.$$

In parametric form the equation of the ellipse becomes

$$w(t)=\left(\frac{(p\beta^2-\alpha)}{(\beta^2-1)}+\frac{\beta(p-\alpha)}{(\beta^2-1)}\cos t,\frac{(p-\alpha)}{\sqrt{\beta^2-1}}\sin t\right),\ 0\leq t\leq 2\lambda. \tag{4.1}$$

Hence, for β>1, z ∈ U\{0}, we have f(z) ∈ $SD_p(\beta,\alpha)$ if and only if. Define F(z) by

$$F(z)=\frac{1}{1-w(t)}\left\{\frac{z}{(1-z)^2}-\frac{z}{(1-z)}w(t)\right\} \tag{4.2}$$

Then using results from, we obtain f(z) ∈ SDp(β,α) if and only if $F(z)*\frac{F(z)}{z}\neq 0$.

Conversely, if $F(z)*\frac{F(z)}{z}\neq 0$, then $\frac{zf'(z)}{f(z)}\neq w(t),0\leq t\leq 2\lambda$. Therefore, $\frac{zf'(z)}{f(z)}$ lies inside $\Omega_{\beta,\alpha}$ or outside $\Omega_{\beta,\alpha}$ for z ∈ U. But $\left.\frac{zf'(z)}{f(z)}\right|_{z=0}=p\in\Omega_{\beta,\alpha}$, thus $\left.\frac{zf'(z)}{f(z)}\right|_{z\in U}\subset\Omega_{\beta,\alpha.}$ Hence, f(z) ∈ $SD_p(\beta,\alpha)$.

So, we have

Theorem 6.2: A function f(z) ∈ $SD_p(\beta,\alpha)$, β>1 if and only if $f(z)*\frac{F(z)}{z}\neq 0$ where F(z) is given by (4.2).

Certain Sufficient Estimates

In order to prove our results in this section we need the following lemma due to Jack [6].

Lemma 7.1: (Jack's Lemma) Let w(z) be (non-constant) analytic function in U with w(0)=0. If |w(z)| attains its maximum value on the circle |z|=r<1 at a point z_0, then

$$z_0w'(z_0)=cw(z_0)$$

where c is real and c ≥ 1.

Making use of Lemma 7.1 our first result is:

Theorem 7.1: Let f(z) ∈ Ap, then f(z) is uniformly p-valent starlike of order α in U if the inequality

$$\Re\left(\frac{1+\frac{zf''(z)}{f'(z)}-p}{\frac{zf'(z)}{f(z)}-p}\right)<1+\frac{2\beta}{p(1+2\beta)}, \tag{5.1}$$

holds.

Proof: Define w(z) by

$$w(z)=\frac{2\beta}{p}\left(\frac{zf'(z)}{f(z)}-p\right),\ p\in\mathbb{N},z\in U. \tag{5.2}$$

Note that w(z) is analytic in U and w(0)=0. Differentiating (5.2) logarithmically we get

$$\left(\frac{zf'(z)}{f(z)}-p\right)\frac{zw'(z)}{w(z)}=\frac{zf'(z)}{f(z)}\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right),$$

which leads to

$$\left(1+\frac{zf''(z)}{f'(z)}-p\right)=\frac{p}{2\beta}w(z)+\frac{zw'(z)}{2\beta+w(z)}. \tag{5.3}$$

Combining (5.2) and (5.3) we see that

$$\left(\frac{1+\frac{zf''(z)}{f'(z)}-p}{\frac{zf'(z)}{f(z)}-p}\right)=1+\frac{2\beta}{p}\frac{zw'(z)}{w(z)(2\beta+w(z))},\ p\in\mathbb{N},z\in U \tag{5.4}$$

Now assume that there exists a point $z_0 \in U$ with $|z_0|=1$ such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1, \quad w(z_0\neq 1),$$

and let $w(z_0)=e^{i\theta}\neq -\pi$. Now applying Jack's Lemma 7.1, we get,

$$\frac{z_0 w'(z_0)}{w(z_0)}=c, \;\; c\ge 1. \tag{5.5}$$

From (5.4) and (5.5) we obtain

$$\Re\left(\frac{1+\frac{z_0 f''(z_0)}{f'(z_0)}-p}{\frac{z_0 f'(z_0)}{f(z_0)}-p}\right)=\Re\left(1+\frac{2\beta}{p}\frac{z_0 w'(z_0)}{w(z_0)(2\beta+w(z_0))}\right)=\Re\left(1+\frac{2\beta c}{p}\frac{1}{2\beta+w(z_0)}\right)$$

$$=1+\frac{2\beta c}{p}\Re\left(\frac{1}{2\beta+w(z_0)}\right)=1+\frac{2\beta c}{p}\Re\left(\frac{1}{2\beta+e^{i\theta}}\right)$$

$$\ge 1+\frac{2\beta c}{p}\Re\left|\frac{1}{2\beta+e^{i\theta}}\right|\ge 1+\frac{c}{p}\left(\frac{2\beta}{1+2\beta}\right)\ge 1+\frac{2\beta}{p(1+2\beta)},$$

which contradicts the hypothesis (5.1). Therefore, we conclude that $|w(z)|<1$ for all $z \in U$ and (5.2) yields the inequality

$$\left|\frac{zf'(z)}{f(z)}-p\right|<\frac{p}{2\beta}, \;(p\in\mathbb{N}, z\in U)$$

which implies that $\frac{zf'(z)}{f(z)}$ lies in a circle centered at p and with radius $\frac{p}{2\beta}$. This amounts to say that lies in a circle centered at p and with radius $\frac{zf'(z)}{f(z)}\in\Omega_{\alpha,\beta}$ and so,

$$\Re\left(\frac{zf'(z)}{f(z)}-\alpha\right)>\beta\left|\frac{zf'(z)}{f(z)}-p\right|$$

i.e. f(z) is uniformly p-valent starlike of order α in U.

Next, we determine the sufficient coefficient bound for uniformly p-valent convex functions of order α.

Theorem 7.2: Let f(z) ∈ Ap, then f(z) is uniformly p-valent convex of order α in U if the inequality holds.

$$\Re\left(\frac{1+\frac{zf'''(z)}{f''(z)}-p}{1+\frac{zf''(z)}{f''(z)}-p}\right)<1+\frac{2\beta}{p(1+2\beta)-2'} \tag{5.7}$$

Proof: Define w(z) by

$$w(z)=\frac{2\beta}{p}\left(1+\frac{zf''(z)}{f'(z)}-p\right), \;\; p\in\mathbb{N}, z\in U. \tag{5.8}$$

Note that w(z) is analytic in U and w(0)=0. Differentiating (5.8) logarithmically we get

$$\left(1+\frac{zf'''(z)}{f''(z)}-p\right)=\; p-1+\frac{p}{2\beta}w(z)+\frac{p}{2\beta}\frac{zw'(z)}{\left(p-1+\frac{p}{2\beta}w(z)\right)}. \tag{5.9}$$

Combining (5.8) and (5.9) we get

$$\left(\frac{1+\frac{zf'''(z)}{f''(z)}-p}{1+\frac{zf''(z)}{f''z)}-p}\right)=1+\frac{p-1}{\frac{p}{2\beta}w(z)}+\frac{zw'(z)}{w(z)\left(p-1+\frac{p}{2\beta}w(z)\right)}, \tag{5.10}$$

where $p \in \mathbb{N}$, $z \in U$.

Suppose now that there exists a point $z_0 \in U$ with $|z_0|=1$ such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1, \quad (w(z_0)\neq 1),$$

and let $w(z_0)=e^{i\theta}, \theta\neq -\pi$. Applying Jack's Lemma 5.1, we obtain

$$\frac{z_0 w'(z_0)}{w(z_0)}=c, c\ge 1. \tag{5.11}$$

Now (5.10) and (5.11) yield

$$\Re\left(\frac{1+\frac{z_0 f'''(z_0)}{f''(z_0)}-p}{1+\frac{z_0 f''(z_0)}{f''z_0)}-p}\right)=\Re\left(1+\frac{2\beta}{p}\frac{(p-1)}{w(z_0)}+\frac{2\beta}{p}\frac{z_0 w'(z_0)}{w(z_0)\left(2\beta(p-1)+pw(z_0)\right)}\right)$$

$$=1+\frac{2\beta}{p}(p-1)\Re\left(\frac{1}{w(z_0)}\right)+\frac{2\beta c}{p}\Re\left(\frac{1}{(2\beta(p-1)+pw(z_0))}\right)$$

$$=1+\frac{2\beta}{p}(p-1)\Re\left(\frac{1}{e^{i\theta}}\right)+\frac{2\beta c}{p}\Re\left(\frac{1}{(2\beta(p-1)+pe^{i\theta}}\right)$$

$$\ge 1+\frac{2\beta}{p}(p-1)\left|\frac{1}{e^{i\theta}}\right|+\frac{2\beta c}{p}\left|\frac{1}{(2\beta(p-1)+pe^{i\theta}}\right|$$

$$\ge 1+\frac{2\beta}{p}(p-1)+\frac{2\beta c}{p}\left(\frac{1}{p(1+2\beta)-2\beta}\right)$$

$$\ge 1+\frac{2\beta}{p}(p-1)+\frac{2\beta}{p}\left(\frac{1}{p(1+2\beta)-2\beta}\right)$$

$$\ge 1+\frac{2\beta}{p(1+2\beta)-2\beta}$$

which contradicts the hypothesis (5.7). Therefore, we conclude that $|w(z)|<1$ for all $z \in U$ and as in the proof of the previous theorem, (5.8) yields the inequality

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|<\frac{p}{2\beta}, \;(p\in\mathbb{N}, \; z\in U)$$

which implies that $1+\frac{zf''(z)}{f'(z)}$ lies in a circle centered at p and with radius $\frac{p}{2\beta}$. This implies that $1+\frac{zf''(z)}{f'(z)}\in\Omega_{\alpha,\beta}$, and therefore

$$\Re\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)>\beta\left|\frac{zf''(z)}{f'(z)}-(p-1)\right| \tag{5.12}$$

i.e. f(z) is uniformly p-valent convex of order α in U.

Taking β=1 in Theorems 5.1 and 5.2 we get the following corollaries proved by Al-Kharsani and Al-Hajiry in [7,8]

Corollary 7.1: Let f(z) ∈ Ap satisfies the inequality

$$\Re\left(\frac{1+\frac{zf''(z)}{f'(z)}-p}{\frac{zf'(z)}{f(z)}-p}\right)<1+\frac{2}{3p},$$

then f(z) is uniformly p-valent starlike in U.

Corollary 7.2: Let f(z) ∈ Ap satisfies the inequality

$$\Re\left(\frac{1+\frac{zf'''(z)}{f''(z)}-p}{\frac{zf''(z)}{f'(z)}-p}\right)<1+\frac{2}{3p-2},$$

then f(z) is uniformly p-valent convex in U.

In order to prove our next result we need the following definition:

Definition 7.1: A function f(z) ∈ Ap is said to be uniformly p-valent close-to-convex (or uni-formly close-to-convex when p=1) of order α in U if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{g(z)}\right) \geq \beta\left|\frac{zf'(z)}{g(z)} - p\right| + \alpha,$$

for some g(z) ∈ SDp(α,β).

The following theorem gives the sufficient condition for uniformly p-valent close-to-convex functions.

Theorem 7.3: Let f(z) ∈ A_p satisfies the inequality

$$\Re\left(\frac{zf''(z)}{f'(z)}\right) < p - \frac{2\beta}{1+2\beta}, \tag{5.13}$$

then f(z) is uniformly p-valent close-to-convex of order α in U.

Proof Let us define w(z) by

$$w(z)\frac{2\beta}{p}\left(\frac{f'(z)}{z^{p-1}} - p\right), \quad p \in \mathbb{N}, z \in U. \tag{5.14}$$

Clearly, w(z) is analytic in U and w(0)=0. Moreover, logarithmic differentiation of (5.14) give rise to

$$\frac{zf''(z)}{f'(z)} = (p-1) + \frac{zw'(z)}{2\beta + w(z)}. \tag{5.15}$$

As earlier, using conditions of Jack's Lemma and (5.13), we get

$$\Re\left(\frac{z_0 f''(z_o)}{f'(z_0)}\right) = (p-1) + c\Re\left(\frac{w(z_0)}{2\beta + w(z_0)}\right)$$

$$= (p-1) + c\Re\left(\frac{e^{i\theta}}{2\beta + e^{i\theta}}\right)$$

$$\geq (p-1) + c\frac{1}{1+2\beta}$$

$$\geq (p-1) + \frac{1}{1+2\beta}$$

$$\geq p - \frac{2\beta}{1+2\beta}$$

which contradicts the hypothesis (5.13). Therefore, we conclude that |w(z)| < 1 for all z ∈ U and (5.14) yields the inequality

$$\left|\frac{f'(z)}{z^{p-1}}\right| < \frac{p}{2\beta}, (p \in \mathbb{N}, z \in U)|$$

which implies that $\frac{f'(z)}{z^{p-1}} \in \Omega_{\alpha,\beta}$ and therefore

$$\Re\left(\frac{f'(z)}{z^{p-1}} - \alpha\right) < \beta\left|\frac{f'(z)}{z^{p-1}} - p\right| \tag{5.16}$$

i.e. f(z) is uniformly p-valent close-to-convex of order α in U.

If β= 1, then Theorem 5.3 gives us the corresponding result of Al-Kharsani and Al-Hajiry established in [2].

Corollary 7.3: Let f(z) ∈ A_p satisfies the inequality

$$\Re\left(\frac{zf''(z)}{f'(z)}\right) < p - \frac{2}{3},$$

then f(z) is uniformly p-valent close-to-convex of order α in U.

For parametric values p=1, β=1 in Theorems 5.1, 5.2 and 5.3 we get

Corollary 7.4: Let f(z) ∈ A satisfies the inequality

$$\Re\left(\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)} - 1}\right) < \frac{5}{3},$$

then f(z) is uniformly starlike in U.

Corollary 7.5: Let f(z) ∈ A satisfies the inequality

$$\Re\left(\frac{\frac{zf'''(z)}{f''(z)}}{\frac{zf''(z)}{f'(z)}}\right) < 3,$$

then f(z) is uniformly convex in U.

Corollary 7.6: Let f(z) ∈ Ap satisfies the inequality

$$\Re\left(\frac{zf''(z)}{f''(z)}\right) < \frac{1}{3},$$

then f(z) is uniformly p-valent close-to-convex in U.

Remark 7.1: For different values of the parameters p, α and β in all our results of this chapter one can easily obtain many other interesting results that have been provided in [1,2,8] and references therein.

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