Bruck decomposition for endomorphisms of quasigroups

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Abstract

In 1944, R. H. Bruck has described a very general construction method which he called the extension of a set by a quasigroup. We use it to construct a class of examples for *LF*-quasigroups in which the image of the map $e(x)=x\backslash x$ is a group. More generally, we consider the variety of quasigroups which is defined by the property that the map e is an endomorphism and its subvariety where the image of the map e is a group. We characterize quasigroups belonging to these varieties using their Bruck decomposition with respect to the map e.

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1 Introduction

A binary algebra $(Q,\cdot)$ with multiplication $(x,y)\mapsto x\cdot y$ is called a *quasigroup* if the equations $a\cdot y=b$ and $x\cdot a=b$ have precisely one solution in Q which we denote by $y=a\backslash b$ and $x=b/a$. The element $1_l(a)=a/a$ (resp., $1_r(a)=a\backslash a$) is the left (resp., the right) local unit element of the element a. If the left (right) local unit elements coincide for all elements of $(Q,\cdot)$, then the element $1_l=1_l(a)$ (resp., $1_r=1_r(a)$) is called the left (resp., right) unit element of $(Q,\cdot)$. If a quasigroup $(Q,\cdot)$ has both left and right unit elements, then they coincide $1=1_l=1_r$; in this case $(Q,\cdot)$ is called a *loop*.

In 1944, R. H. Bruck has described a very general construction method which he called the extension of a set by a quasigroup (cf. [3, 4]). Epimorphisms of quasigroups in general cannot be described by cosets of a normal subquasigroup, but only by congruence relations in the sense of universal algebra. Bruck's construction takes this into account giving a manageable description of quasigroup epimorphisms. In this note we discuss this method for endomorphisms of quasigroups.

A quasigroup $(Q,\cdot)$ is called an *LF-quasigroup* if the identity

$$x\cdot yz=xy\cdot(x\backslash x\cdot z)$$

holds in Q. In his book [1], Belousov initiated a systematic study of *LF*-quasigroups using isotopisms. Recently progress has been made in this topic (cf. [7, 8, 9]). It is known that in an *LF*-quasigroup the map $e(x)=x\backslash x$ is an endomorphism, which we call the left deviation. In this situation Bruck's theory is available. We use it to construct a class of examples for *LF*-quasigroups Q in which $e(Q)$ is a group.

More generally, we consider the variety $\mathfrak{D}_l$ of quasigroups which is defined by the property that the left deviation is an endomorphism and its subvariety $\mathfrak{a}\mathfrak{D}_l$ where the image of the left deviation is a group. We characterize quasigroups belonging to these varieties using their Bruck decomposition with respect to their left deviation.

2 The Bruck decomposition of a quasigroup

In Bruck's papers [3, Theorem 10 A, pp. 166–168] and [4, pp. 778–779], a principal construction for quasigroups is given. Let $(E, \cdot, \backslash, /)$ be a quasigroup, let T be a set, and let $\{\nabla_{a,b};\ a, b \in E\}$ be a family of multiplications on T. Define on the set $Q = T \times E$ a multiplication by

$$(\alpha, a) \circ (\beta, b) = (\alpha \nabla_{a,b} \beta, ab), \quad (\alpha, a), (\beta, b) \in T \times E \tag{2.1}$$

Then $(Q, \circ)$ is a quasigroup if and only if for any $a, b \in E$ the multiplication $\nabla_{a,b}$ on T defines a quasigroup $\mathcal{T}_{a,b} = (T, \nabla_{a,b})$. In this case we call $\mathcal{B} = (E, T, (\nabla_{a,b})_{a,b \in E})$ a *Bruck system* and put $Q(\mathcal{B}) = Q$. Obviously, the projection $((\alpha, a) \mapsto a) : Q(\mathcal{B}) \to E$ is an epimorphism of quasigroups (cf. [6, pp. 35–36]). We call this epimorphism the *canonical epimorphism* for $\mathcal{B}$.

Conversely, let $(Q, \circ)$ and $(E, \cdot)$ be quasigroups and let $\pi : Q \to E$ be an epimorphism. For the inverse images $T_a = \pi^{\leftarrow}(a),\ a \in E$, one has

$$T_a \circ T_b = T_{ab}$$

hence the set $\{T_a,\ a \in E\}$ forms a quasigroup $\mathcal{E}'$ isomorphic to $(E, \cdot)$. Using a transversal for the partition $\{T_a,\ a \in E\}$ of the set Q, one can identify the inverse images $T_a,\ a \in E$, with a subset $T \subset Q$ and the set Q with the cartesian product $T \times E$. The multiplication in Q can be written in the form (2.1), where $\alpha, \beta \in T,\ a, b \in E$. This means that we have obtained a Bruck system $\mathcal{B}_\pi = (E, T, (\nabla_{a,b})_{a,b \in E})$ for which $Q(\mathcal{B}_\pi)$ is isomorphic to Q and which has π as canonical epimorphism. We call this representation of $(Q, \circ)$ a *Bruck decomposition of* $(Q, \circ)$ *with respect to* $\pi : Q \to E$.

Let $\eta : Q \to Q$ be an endomorphism. We consider the set E of the congruence classes $T_x = \eta^{\leftarrow}(x),\ x \in \eta(Q)$. The multiplication $T_x \star T_y = T_{xy}$ defines a quasigroup $(E, \star)$ such that the mapping $\pi : (x \mapsto T_x) : Q \to E$ is an epimorhism. We put $\iota = \eta \circ \pi^{\leftarrow} = (T_x \mapsto \eta(x)) : E \to Q$. Then ι is an injective homomorphism (see [5, Theorem 6.12, p. 50]).

We consider the Bruck decomposition $Q = T \times E$ with respect to the epimorphism π. For $a \in E = \{T_x,\ x \in \eta(Q)\}$ one has $\iota(a) = (\gamma(a), g(a))$, where $\gamma(a) \in T,\ g(a) \in E$. The maps $\gamma : E \to T$ and $g : E \to E$ satisfy

$$\begin{aligned}\big(\gamma(ab), g(ab)\big) &= \iota(ab) = \iota(a)\iota(b) = \big(\gamma(a), g(a)\big)\big(\gamma(b), g(b)\big) \\ &= \big(\gamma(a) \nabla_{g(a),g(b)} \gamma(b), g(a)g(b)\big)\end{aligned}$$

It follows that $g : E \to E$ is an endomorphism and

$$\gamma(ab) = \gamma(a) \nabla_{g(a),g(b)} \gamma(b) \tag{2.2}$$

holds for all $a, b \in E$. We call the structure described here the *Bruck decomposition of the quasigroup* $(Q, \circ)$ *with respect to the endomorphism* $\eta : Q \to Q$.

If the quasigroup Q is a loop and $\eta : Q \to Q$ is an endomorphism, then $K = \eta^{\leftarrow}(1)$ is a normal subloop of Q. In this situation Q is a semidirect product of K and $\eta(Q)$ if and only if $\eta = \eta^2$ holds. We describe the Bruck decomposition with respect to an idempotent endomorphism for arbitrary quasigroups.

Proposition 2.1. *An endomorphism η of a quasigroup Q is idempotent if and only if in the Bruck decomposition with respect to η the maps $\gamma : E \to T$ and $g : E \to E$ satisfy*

$$g^2 = g \quad \textit{and} \quad \gamma \circ g = \gamma$$

i.e., if and only if the endomorphism $g : E \to E$ is idempotent and the map $\gamma : E \to T$ factors over the congruence relation defined by g on E.

Proof. Since $\eta(\alpha, a) = \iota(\pi(\alpha, a)) = \iota(a) = (\gamma(a), g(a))$ the assertion follows from

$$\eta\big(\eta(\alpha, a)\big) = \eta\big(\gamma(a), g(a)\big) = \gamma\big(g(a), g\big(g(a)\big)\big) \qquad \square$$

3 The left deviation of a quasigroup

For a quasigroup $(Q, \cdot, \backslash, /)$ we call the map $e = (x \mapsto x\backslash x) : Q \to Q$ the *left deviation.* As mentioned in the preliminaries, the left deviation of $x \in Q$ is the local right unit element of x. In a Bruck decomposition $Q = T \times E$ with respect to an epimorphism $Q \to E$, the deviation is $e(\alpha, a) = (\alpha\backslash\alpha, a\backslash a)$, where $a\backslash a$ is computed in the quasigroup E and $\alpha\backslash\alpha$ is computed in the quasigroup $\mathcal{T}_{a,a\backslash a} = (T, \triangledown_{a,a\backslash a})$. Obviously, the quasigroups in which *the left deviation is an endomorphism form a variety* $\mathfrak{D}_l$ of quasigroups. For $(Q, \cdot, \backslash, /) \in \mathfrak{D}_l$ we consider the Bruck decomposition $Q = T \times E$ with respect to the left deviation. In this case $e(\alpha, a) = (\gamma(a), g(a))$ and hence $g(a) = a\backslash a$ (computed in E) and

$$\alpha \triangledown_{a,g(a)} \gamma(a) = \alpha \tag{3.1}$$

Theorem 3.1. *A quasigroup Q belongs to the variety $\mathfrak{D}_l$ if and only if there exists a Bruck system $\mathcal{B} = (E, T, (\triangledown_{a,b})_{a,b\in E})$ satisfying*

(i) *$Q \cong Q(\mathcal{B})$,*
(ii) *for any $a \in E$ the quasigroup $\mathcal{T}_{a,a\backslash a} = (T, \triangledown_{a,a\backslash a})$ has a right unit element, denoted by $\epsilon(a)$,*
(iii) *the map $(a \mapsto (\epsilon(a), a\backslash a)) : E \to Q(\mathcal{B})$ is a homomorphism.*

In this case the left deviation of $Q(\mathcal{B})$ is the map

$$e = \big((\alpha, a) \longmapsto \big(\epsilon(a), a\backslash a\big)\big) : Q(\mathcal{B}) \longrightarrow Q(\mathcal{B})$$

Proof. Assume first that Q belongs to $\mathfrak{D}_l$ and consider the Bruck decomposition with respect to the left deviation $e(x) = x\backslash x$ of Q. Then E is isomorphic to the subquasigroup $e(Q)$ of Q. Putting $\epsilon = \gamma$ the assertion (ii) follows from equation (3.1) and the assertion (iii) follows from equation (2.2).

Conversely, if $\mathcal{B} = (E, T, (\triangledown_{a,b})_{a,b\in E})$ is a Bruck system satisfying (i), then

$$e(\alpha, a) = (\alpha, a)\backslash(\alpha, a) = (\alpha', a\backslash a)$$

where $\alpha = \alpha \triangledown_{a,a\backslash a} \alpha'$. From (ii) it follows that $\alpha' = \epsilon(a)$ and the deviation satisfies $e(\alpha, a) = (\epsilon(a), a\backslash a)$. Hence we obtain from (iii) that the quasigroup Q belongs to the variety $\mathfrak{D}_l$. $\square$

Corollary 3.2. *Let $\mathcal{B} = (E, T, (\triangledown_{a,b})_{a,b\in E})$ be a Bruck system satisfying the conditions* (ii) *and* (iii) *of the previous theorem. Then $\mathcal{B}$ is a Bruck decomposition of the quasigroup $Q(\mathcal{B})$ with respect to the left deviation of $Q(\mathcal{B})$ if and only if the homomorphism*

$$\iota = \big(a \longmapsto \big(\epsilon(a), a\backslash a\big)\big) : E \longrightarrow Q(\mathcal{B})$$

is injective.

Example 3.3. Let $(E, \cdot)$, $\mathcal{T}^{(1)} = (T, \circ)$, and $\mathcal{T}^{(2)} = (T, \star)$ be quasigroups such that the following properties are satisfied:

(a) E is a $\mathfrak{D}_l$-quasigroup,

(b) $\mathcal{T}^{(1)}$ has a right unit element ϵ,

(c) ϵ is idempotent in the quasigroup $\mathcal{T}^{(2)}$.

Put

(i) $\mathcal{T}_{a,a\backslash a} = (T, \triangledown_{a,a\backslash a}) = \mathcal{T}^{(1)} = (T, \circ)$,

(ii) $\mathcal{T}_{a\backslash a,b\backslash b} = (T, \triangledown_{a\backslash a,b\backslash b}) = \mathcal{T}^{(2)} = (T, \star)$ if $b\backslash b \neq (a\backslash a)\backslash(a\backslash a)$,

(iii) $\mathcal{T}_{c,d} = (T, \triangledown_{c,b\backslash d})$ arbitrary in all other cases.

According to Theorem 3.1 the multiplication $(\alpha, a) \circ (\beta, b) = (\alpha \triangledown_{a,b} \beta, ab)$ on the set $T \times E$ is a $\mathfrak{D}_l$-quasigroup Q with left deviation $e(\alpha, a) = (\epsilon, a\backslash a)$. For $(E, \cdot)$ one can take idempotent quasigroups or groups. Clearly, the construction of the quasigroup $Q = T \times E$ gives a Bruck decomposition with respect to the left deviation of Q if and only if the left deviation of $(E, \cdot)$ is an automorphism. If $(E, \cdot)$ is a group with unit element 1, then the left deviation of Q is the constant $(\epsilon, 1)$ which is the right unit element of Q.

4 Associative image of the left deviation map

The class of $\mathfrak{D}_l$-quasigroups for which *the image of the left deviation map is a group* forms a variety, too, as can be seen from the identities

$$xy\backslash xy = x\backslash x \cdot y\backslash y, \quad (x\backslash x \cdot y\backslash y) \cdot z\backslash z = x\backslash x \cdot (y\backslash y \cdot z\backslash z)$$

We denote this variety by $\mathfrak{a}\mathfrak{D}_l$ and investigate the Bruck decomposition in $\mathfrak{a}\mathfrak{D}_l$. As an immediate consequence of Theorem 3.1 one obtains the following.

Theorem 4.1. *A quasigroup Q belongs to the variety $\mathfrak{a}\mathfrak{D}_l$ if and only if there exists a Bruck system $\mathcal{B} = (E, T, (\triangledown_{a,b})_{a,b \in E})$ with $Q \cong Q(\mathcal{B})$ satisfying*

(i) *E is a group (with unit element 1),*

(ii) *the quasigroup $\mathcal{T}_{a,1} = (T, \triangledown_{a,1})$ has right unit element, denoted by $\epsilon(a)$, for any $a \in E$,*

(iii) *the map $\epsilon : E \to \pi^{\leftarrow}(1) = \mathcal{T}_{1,1} = (T, \triangledown_{1,1})$ is a homomorphism of the group E into the normal subquasigroup $\pi^{\leftarrow}(1) = \mathcal{T}_{1,1} = (T, \triangledown_{1,1})$ of $Q(\mathcal{B})$.*

In this case one has $e(\alpha, a) = (\epsilon(a), 1)$ for any $(\alpha, a) \in T \times E$.

Proof. E is a group since it is isomorphic to the image of the left deviation map. Hence $e(\alpha, a) = (\epsilon(a), a\backslash a) = (\epsilon(a), 1)$ for any $(\alpha, a) \in T \times E$ and $\epsilon : (a \mapsto (\epsilon(a), 1)) : E \to \pi^{\leftarrow}(1) = \mathcal{T}_{1,1}$. □

Example 4.2. Let $(E, \cdot)$, $(T, \circ)$ be quasigroups and let $\epsilon : E \to T$ be a homomorphism such that the following properties are satisfied:

(i) $(E, \cdot)$ is a group with unit element 1,

(ii) $\epsilon(1)$ is a right unit element of $(T, \circ)$.

Put

(a) $\alpha \triangledown_{a,1} \beta = (\alpha / \epsilon(a)) \circ \beta$ in the quasigroup $\mathcal{T}_{a,1} = (T, \triangledown_{a,1})$, where $/$ is the right division in $(T, \circ)$,

(b) $\mathcal{T}_{a,b} = (T, \triangledown_{a,b})$ arbitrary for $b \neq 1$.

According to Theorem 4.1 the multiplication $(\alpha, a) \circ (\beta, b) = (\alpha \triangledown_{a,b} \beta, ab)$ on the set $T \times E$ is an $\mathfrak{a}\mathfrak{D}_l$-quasigroup. For $(E, \cdot)$ and $(T, \circ)$ one can take groups having $\epsilon : E \to T$ as a group homomorphism. The decomposition $Q = T \times E$ is a Bruck decomposition with respect to the left deviation of Q if and only if the homomorphism $\epsilon : E \to T$ is injective.

5 *LF*-quasigroups

It is known that the LF-quasigroups form a subvariety of $\mathfrak{D}_l$ (cf. [2, p. 108] and [9, Lemma 2.1]). We will now give examples of LF-quasigroups even belonging to the variety $\mathfrak{a}\mathfrak{D}_l$. Let E, T be groups and let $\epsilon : E \to T$ be a homomorphism. Put

$$\alpha \nabla_{a,b} \beta = \alpha \cdot \epsilon(a)^{-1} \cdot \beta$$

for all $a, b \in E$. Then every $\mathcal{T}_{a,b} = (T, \nabla_{a,b})$ is a group (with unit element $\epsilon(a)$) which is isotopic and hence isomorphic to the group T.

As in the previous example the multiplication $(\alpha, a) \circ (\beta, b) = (\alpha \nabla_{a,b} \beta, ab)$ on the set $T \times E$ defines an $\mathfrak{a}\mathfrak{D}_l$-quasigroup (with left unit element $(1,1)$) in which the left deviation is given by $e(\alpha, a) = (\epsilon(a), 1)$ (Theorem 4.1).

Theorem 5.1. *Let E, T be groups and let $\epsilon : E \to T$ be a homomorphism. The set $Q = T \times E$ equipped with multiplication $(\alpha, a) \circ (\beta, b) = (\alpha \cdot \epsilon(a)^{-1} \cdot \beta, ab)$ is an LF-quasigroup with left unit element $(1,1)$ satisfying the left inverse property.*

Proof. An easy calculation shows

$$(\alpha, a) \circ \big((\beta, b) \circ (\gamma, c)\big) = \big(\alpha \cdot \epsilon(a)^{-1} \cdot \beta \cdot \epsilon(b)^{-1} \cdot \gamma, abc\big)$$

On the other hand,

$$\begin{aligned}\big((\alpha, a) \circ (\beta, b)\big) \circ \big((\epsilon(a), 1\big) \circ (\gamma, c)\big) &= \big(\alpha \cdot \epsilon(a)^{-1} \cdot \beta, ab\big) \circ (\epsilon(a) \cdot \gamma, c) \\ &= \big(\alpha \cdot \epsilon(a)^{-1} \cdot \beta \cdot \epsilon(ab)^{-1} \cdot \epsilon(a) \cdot \gamma, abc\big)\end{aligned}$$

Hence $(Q, \circ)$ is an LF-quasigroup in which $(1,1)$ is the left unit element. The left inverse of an element (α, a) is given by $(\epsilon(a) \cdot \alpha^{-1} \cdot \epsilon(a)^{-1}, a^{-1})$. Indeed one has $(\epsilon(a) \cdot \alpha^{-1} \cdot \epsilon(a)^{-1}, a^{-1}) \circ ((\alpha, a) \circ (\beta, b)) = (\beta, b)$. □

The decomposition $Q = T \times E$ is a Bruck decomposition with respect to the left deviation of Q if and only if the homomorphism $\epsilon : E \to T$ is injective.

We note that in accordance with [9, Theorem 4.1] the quasigroup $Q = T \times E$ is isotopic to the direct product of the groups T and E; the isotopism is given by the triple $(\phi, \mathrm{id}, \mathrm{id})$, where $\phi : (\alpha, a) \mapsto (\alpha \cdot \epsilon(a)^{-1}, a)$.

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