Boundary Integral Equations Method for the Time-Harmonic Electromagnetic Scattering

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Abstract

In contrast to the classical boundary layer approach which is not uniquely solvable at the irregular wave numbers, the modified layer method always has a unique solution to the exterior boundary problem for the vector Helmholtz equation. We compare the stability of the two methods numerically.

Keywords: Boundary integral equation; Electromagnetic scattering; Helmholtz equation

Introduction

Consider the exterior boundary value problem for the scalar Helmholtz equation in $\mathbb{R}^n, n = 2, 3$

$$\Delta u + k^2 u = 0, \quad k \in \mathbb{R}, \quad k \neq 0, \quad \text{in } \mathbb{R}^n \setminus D$$

with prescribed Dirichlet or Neumann boundary condition. For the uniqueness of this problem, the asymptotic behavior of the solution at infinity will also be given. Using the classical boundary layer approach (see [1,2]), this kind of problem can be solved uniquely using the double layer potential for the Dirichlet boundary condition, the single layer potential for the Neumann boundary condition, respectively. We provide $k$ is not an interior eigenvalue of the $-\Delta$ operator (see [3]). Since physically the uniqueness of the exterior boundary value problem is not in question, the cause of the non-uniqueness must come from our mathematical method. Remaining in the framework of boundary integral equation method, various modifications have been developed to overcome this non-uniqueness for all $k \neq 0$ ([4] by adding a volume potential, [5-7] by using the combined boundary layers).

In [8,9], the method of combined layers has been modified slightly and extended to the case of a vector Helmholtz equation. In this paper, we compare the numerical results of the classical boundary integral equation methods with those of the modified boundary integral equation methods with combined boundary layers for the vector Helmholtz equation. The plan of this paper is as follows. In the second section, we deduce the exterior boundary value problem from the time harmonic electromagnetic scattering problem from an ideal conductor. In the third section, we use the modified boundary layers approach to win a system of Fredholm integral equation of the second kind which is uniquely solvable. A section on numerical method is then followed by another giving some numerical examples. At the end of the paper, we give some properties of the system of the boundary integral equations from section 3 in the appendix.

Boundary Value Problem

Consider the electromagnetic scattering from a homogeneous, isotropic obstacle in $\mathbb{R}^3$. The electric Field $\mathbf{E}$ and the magnetic Field $\mathbf{H}$ satisfy the Maxwell equations

$$\nabla \times \mathbf{E} - \mu \frac{\partial \mathbf{H}}{\partial t} = 0, \quad \nabla \times \mathbf{H} - \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \sigma \mathbf{J}.$$  

Where $\varepsilon$ is the dielectricity, $\mu$ is the permeability and $\sigma$ is the conductivity of the medium. For the time-harmonic electromagnetic wave of the form

$$\mathbf{E}(x, t) = \text{Re} \left\{ \left( \varepsilon + \frac{i \sigma}{\omega} \right)^{-1/2} \mathbf{E}(x) e^{i\omega t} \right\},$$

$$\mathbf{H}(x, t) = \text{Re} \left\{ \left( \varepsilon + \frac{i \sigma}{\omega} \right)^{-1/2} \mathbf{H}(x) e^{i\omega t} \right\},$$

with a frequency $\omega > 0$, the space-dependent complex-valued functions $\mathbf{E}$ and $\mathbf{H}$ solve the reduced Maxwell equations $\nabla \times \mathbf{E} = 0$, $\nabla \times \mathbf{H} = 0$, where the wave number $k$ is given by $k^2 = \varepsilon^2 \mu^2$.

According to Stratton and Chu [10], every solution $(\mathbf{E}, \mathbf{H})$ of the reduced Maxwell equations is divergence free and satisfies the vector Helmholtz equation.

$$\Delta \mathbf{E} + k^2 \mathbf{E} = 0,$$

$$\Delta \mathbf{H} + k^2 \mathbf{H} = 0.$$  

On the other hand, if $\mathbf{E}$ solves the vector Helmholtz equation $\Delta \mathbf{E} + k^2 \mathbf{E} = 0$ and is divergence free, then the electromagnetic Field $(\mathbf{E}, \mathbf{H})$ with $\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{E}$ solves the reduced Maxwell equations. This motivates the following boundary value problem for the vector Helmholtz equation.

**Problem 1:** Let $D \subset \mathbb{R}^3$ be a simple connected bounded domain with $C^2$ boundary $\partial D$. Let $v$ be the unit outward normal vector to $\partial D$. Given the tangent field $\mathbf{c}$ and a function $\mathbf{z}$, both defined on the boundary $\partial D$, find a solution $\mathbf{E}$ to the vector Helmholtz equation

$$\Delta \mathbf{E} + k^2 \mathbf{E} = 0,$$

$$\mathbf{E} = 0, \quad \mathbf{H} = 0,$$

on $D$.

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which satisfies the boundary conditions
\[ \nabla\times\mathbf{E} = \mathbf{c} \text{ on } \partial D, \]
\[ \nabla\cdot\mathbf{E} = 0 \text{ on } \partial D, \]
and the radiation condition
\[ \text{rot} \mathbf{E}(x) \times \hat{x} + \hat{\mathbf{E}}(x) = \mathbf{0} \text{ as } |x| \to \infty, \]

Uniformly for all directions \( \hat{x} := \frac{x}{|x|} \).

Now consider the case of an infinitely long cylinder which runs in the \( x_3 \) direction and has a cross section \( D \) parallel to the \( x_1, x_2 \) plane. We have thus the following two dimensional boundary value problems for the vector Helmholtz equation.

**Problem 2: (H)** Let \( D \) be a bounded simply connected domain in \( \mathbb{R}^2 \) with \( C^2 \) boundary \( \Gamma := \partial D \). Denote \( \partial (x) \) the unit tangent vector at \( x \in \partial D \) which is given by \( \partial (x) := e \times \mathbf{v}(x) \). Given \( \xi, \rho \in C_{0,\alpha}(\Gamma, C), 0 < \alpha < 1 \), find a vector field \( \mathbf{E} \in C^2(\mathbb{D}^+) \) which satisfies the boundary conditions

1. \( \mathbf{E} \in C^2(\mathbb{D}^+), 0 < \alpha < 1 \), find a vector field \( \mathbf{E} \in C(\mathbb{D}^+) \) satisfies
2. \( \Delta \mathbf{E} + 2k^2 \mathbf{E} = 0 \), \( \mathbf{E} \in C^2(\mathbb{D}^+) \)
3. \( \partial \mathbf{E} + 2k^2 \mathbf{E} = 0 \) in \( \mathbb{D}^+, k > 0 \), \( \mathbf{E} \in C^2(\mathbb{D}^+) \)
4. \( \text{rot} \mathbf{E}(x) \times \hat{x} + \hat{\mathbf{E}}(x) = 0 \) for \( |x| \to \infty \) uniformly in all directions \( \hat{x} \).

The last condition in the boundary value problem (H) is the Sommerfeld radiation condition. It describes an outgoing wave and is essential for the uniqueness of the problem. Knauff and Kress [8] proved the unique solvability of the 3-dimensional problem 1. With slight modifications, their proof can also be used here to show the well-posedness of the problem (H) (See sec. 3.5 in [3] for details).

**Boundary Integral Equations**

Motivated by the representation theorem from stratton and Chu [10] (see also [8]), we define the solution ansatz for our problem H.

**Definition (solution ansatz)**

Given \( \varphi, \varphi_2 \in C^{0,\alpha}(\Gamma, C), 0 < \alpha < 1 \), and a constant \( \eta > 0 \). We call \( \mathbf{E}(x) := e_3 \times \text{grad} \int_{\Gamma'} \varphi(y) \Phi(x, y) ds(y) + \eta \int_{\Gamma'} \varphi(y) \partial_3 \Phi(x, y) ds(y) - \int_{\Gamma'} \partial_3 \varphi_2(y) \Phi(x, y) ds(y) + \eta \text{grad} \int_{\Gamma'} \varphi_2(y) \Phi(x, y) ds(y), \)

for all \( x \in \mathbb{D} \),

the function \( \Phi(x, y) := \frac{1}{4i} H_0^{(1)}(k|x - y|), x, y \in \mathbb{R}^2, x \neq y, \)

is the fundamental solution of the scalar Helmholtz equation.

At this place, let’s define the Banach space \( C_{0,\alpha} \) by

\[ C_{0,\alpha} := C_{0,\alpha}(\Gamma, C) \times C_{0,\alpha}(\Gamma, C), 0 < \alpha < 1, \text{ and the functions } \phi, \psi \in C_{0,\alpha} \]

\[ \varphi := (\phi, \varphi_2) \]

From the solution theory of the problem H (see [6]), we see that the solution ansatz \( \mathbf{E} \) is the solution of \( H \) in and only if \( \varphi \) is the solution of the following boundary integral equation (in operator form)

\[ (L + A) \varphi = f, \]

where

\[ A_{10} \varphi(x) := 2 \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y), \]

\[ A_{12} \varphi(x) := 2 \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y), \]

\[ A_{21} \varphi(x) := -2 \int_{\Gamma} \varphi(y) \Phi(x, y) ds(y), \]

\[ A_{22} \varphi(x) := -2 \mathbf{k}^2 \int_{\Gamma} \varphi(y) \Phi(x, y) ds(y), \]

The operator matrices \( L, A : C^{0,\alpha} \to C^{0,\alpha} \) are defined by

\[ L := \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \]

For the numerical treatment of this problem, we need to parametrize this boundary integral equation. Write

\[ \Gamma = \{ x(t) = (x_1(t), x_2(t)) | t \in [0, 2\pi] \} \]

for a 2\pi-periodic, two times continuously differentiable function \( x : \mathbb{R} \to \mathbb{R}^2 \).

After a lengthy mathematical computation, which will be given in some details in the appendix, the boundary integral equation (4) will be brought into the following form

\[ (L + A) \varphi = f, \]

which is an integral equation of the second kind with a bounded invertible \( L \) and a compact \( A \).
Numerical Method

We will apply the Nyström method to solve the boundary integral equation (5) numerically. This means that we need some quadrature rules for our integrals appear in the operators \( L, A \). To begin with, we need to approximate the integrals. Because of the singular behavior of the Hankel function, we have 3 different kinds of integration (see appendix). Therefore it is wise to use different quadrature rules. We use the following well-known convergent quadrature rules:

\[
\int_0^{2\pi} \phi(t) dt \approx \frac{1}{2\pi} \sum_{i=0}^{2n-1} \phi(t_i) + \frac{1}{4\pi} \phi(0) + \frac{1}{4\pi} \phi(2\pi), \quad t_i = \frac{2\pi i}{2n-1}, \quad i = 0, 1, \ldots, 2n-1, \]

\[
\int_0^{2\pi} \phi(t) dt \approx \frac{1}{2\pi} \sum_{i=0}^{2n-1} \phi(t_i) + \frac{1}{4\pi} \phi(0) + \frac{1}{4\pi} \phi(2\pi), \quad t_i = \frac{2\pi i}{2n-1}, \quad i = 0, 1, \ldots, 2n-1, \]

\[
\int_0^{2\pi} \phi(t) dt \approx \frac{1}{2\pi} \sum_{i=0}^{2n-1} \phi(t_i) + \frac{1}{4\pi} \phi(0) + \frac{1}{4\pi} \phi(2\pi), \quad t_i = \frac{2\pi i}{2n-1}, \quad i = 0, 1, \ldots, 2n-1. \]

The quadrature (6) is used by Garrick [11], (7) is used in Martensen [12] and in Kussmaul [13]. The quadrature rule (8) is just the composite trapezoidal rule for continuous functions. After approximating the integrals in (5) by interpolatory quadratures (6)-(8), the boundary integral equation is brought into a semidiscrete form

\[
(L + A)n \phi_n = f, \quad \phi_n := (\phi_n^k, \phi_n^\ell) \in C^{2n}, \quad k, \ell = 0, 1, \ldots, 2n-1. \]

We call (9) semi-discrete because it is still a functional equation. To find a full discrete system, we solve this equation at the collocation points which are the same as the interpolation points used for the trigonometric interpolation.

Mathematically, we define a projection operator \( P_n: C^{2n} \rightarrow C^n \times C^n \). Then the full discrete system reads

\[
P_n (L + A)n \phi_n = f. \]

We note here that the convergence \( \phi_n \rightarrow \Phi \) can be showed as a consequence of the convergent quadratures (6)-(8) and the interpolation. We omit the proofs which are not essential in this paper.

Numerical Results

In this section we will demonstrate the efficiency of our method through some examples. To test the advantage of the method, the parameter \( \eta \) is taken to be 1. The boundary value problem will be solved by the Nyström method with trigonometric polynomials as the underlying interpolatory space. From the asymptotic behavior of the Hankel function, we have the following representation for the far field value problem

\[
E_n(\hat{x}) = \sum_{k=1}^{\infty} \left[ \left( k \cdot \hat{x} \cdot \hat{\Phi}(\psi) \right) e^{-i(\psi \cdot k \cdot \hat{x})} - i \phi(\psi) e^{-i(\psi \cdot k \cdot \hat{x})} \right] ds(\psi), \quad \hat{x} \in \Omega. \]

Example 1

In the first example, we choose the ellipse with the parametrization \( x(t) = (\cos(t), 0.5 \sin(t)), t \in [0, 2\pi] \).

For the incident field \( E_i \), we use the plane wave multiplied by a constant vector

\[
E(\psi) = e^{i(\psi \cdot d)} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \]

with the incident direction \( d = (1/\sqrt{2}, 1/\sqrt{2}) \). The far field pattern \( E_\infty = (E_\infty^1, E_\infty^2) \) is calculated for the two directions \( d \) and \( -d \). We deal with the case of a smaller wave number (\( k=1 \), Table 1) and the case of a larger wave number (\( k=10 \), Table 2). We see that the convergence is very fast in the case of an ellipse.

Example 2

In the second example, we choose a domain which is non-convex and non-symmetric. The bean-shaped domain is parametrized by \( x(t) = (\rho(t) \cos(t), \rho(t) \sin(t)), t \in [0, 2\pi] \).

Again, we compute the two cases \( k = 1 \) and \( k = 10 \) (Table 3-4). We
The solution to the boundary value problem is
\[ z_1 = 2.4048255576957727686 \ldots, \quad z_5 = 14.9309177084877859477 \ldots \]

Note that these values are close to the zeros of the Bessel functions \( J_n \) and \( J_{n+1} \), respectively. The results for \( n = 1 \) and \( n = 0 \) will then be compared. The results are listed in the Tables 5-10.

At this point, we want to draw some conclusions from our numerical results. The first thing to note is that in the case where \( k \) is not very close to the zero of \( J_n \) but close to the zero of \( J_{n+1} \), both the modified method \( \eta = 1 \) and the classical method \( \eta = 0 \) converge fast (Table 5, Table 6, Table 8). In the case where \( k \) is close to the zero of \( J_n \), while the modified method converges very fast regardless of the value of \( k \), the classical method does not converge (Table 7, Table 9, Table 10). Secondly, the modified approach is very robust in the sense that it is very stable even when the wave number \( k \) is numerical the same as the eigenvalue of the interior problem (Table 10).

From this conclusion we make the following statement is that the modified approach is stable for wave numbers.

**Appendix**

Here we will explain the mathematical transition from (4) to (5). It is known that the method of boundary integral equations has the advantage that it reduces the dimension by one. On the other hand, it

\[ n \]

\[ \eta = 1 \]

\[ \eta = 0 \]

\[ n \]

\[ \eta = 0 \]

\[ n \]

\[ \eta = 1 \]

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\[ n \]

\[ \eta = 1 \]

\[ \eta = 0 \]
is also known that one often has to struggle with stronger singularities. Therefore, one main point here is to split the singularities from the equations and to treat them in a unified profitable way. The singularities come from the Hankel functions $H_0^{(1)}, H_1^{(1)}$ in the fundamental solution $\Phi$ and its derivative which are buried in the solution ansatz (3). The Hankel functions have the following asymptotic behavior for $t \to 0$

$$H_0^{(1)}(t) = \frac{2i}{\pi} \ln \frac{2}{t} + 1 + \frac{2i}{\pi} C + O(t^2 \ln t), \quad (12)$$

$$H_1^{(1)}(t) = -\frac{2i}{\pi} \ln \frac{1}{t} + O(t \ln t). \quad (13)$$

After some cumbersome calculations, which we'd like to omit, the functional equation (4) becomes (5) with

$$L := \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \quad A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} := \begin{pmatrix} M_{11} + R_{11} & M_{12} + R_{12} \\ M_{21} + R_{21} & M_{22} + R_{22} \end{pmatrix},$$

where

$$L_{ij} := \int_0^{2\pi} \cos \frac{s-t}{2} \psi(s,t) \, ds dt,$$

$$A_{ij} := \int_0^{2\pi} \cos \frac{s-t}{2} \psi''(s,t) \, ds dt,$$

and

$$M_{ij} := \frac{1}{2\pi} \int_0^{2\pi} \ln \left( \frac{s-t}{2} \right) m_{ij}(s,t) \psi(s,t) \, ds dt,$$

$$R_{ij} := \frac{1}{2\pi} \int_0^{2\pi} \psi''(s,t) \, ds dt.$$