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Bicompletable Standard Fuzzy Quasi-Metric Space

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Abstract

In this paper we introduce the definition of standard fuzzy quasi-metric space then we discuss several properties after we give an example to illustrate this notion. Then we showed the existence of a standard fuzzy quasi-metric space which is not bicompletable. Here we prove that every bicompletable standard qussi-metric space admits a unique [up to F-isometic] bicompletion.

Keywords: Standard fuzzy metric space, Standard fuzzy quasi-metric space, Bicompletable standard qussi-metric space

Introduction

In [1] Kider started the study of a notion of standard fuzzy metric space that constitutes an interesting modification of the notion of metric fuzziness due to George and Veeramani [2]. In this paper we extend the notion standard fuzzy metric space to a standard fuzzy quasi-metric space [3]. On the other hand, it was presented in [4] an example of a standard fuzzy metric space that is not completable, also it has been obtained an internal characterization of completable standard fuzzy metric spaces. taking these results into account and the fact that the concept of bicompletion provides a theory of completion to quasi-metric spaces in the classical sense [5]. It seems natural and interesting to discuss the problem of characterizing standard fuzzy quasi-metric spaces that are bicompletable. The main purpose of this paper is to solve this problem. Following the modern terminology of [5] by a quasi-metric on a set X we mean a function $d: XX \to [0,\infty)$ such that for all x, y, z $\in$ X.

(i) d(x,y)=d(y,x)=0 if and only if x=y

(ii) $d(x,y) \le d(x,z) + (z,y)$

Each quasi-metric d on X generates a T0-topology which has a base the family of Υ_d Open balls $\{B_\varepsilon(x): X, \varepsilon > 0\}$ where $B_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$.

Standard Fuzzy Metric Space

Definition 1.1: A binary operation : [0,1] × [0,1] → [0,1] is a continuous t-norm if * satisfies the following conditions [1]:

1- * is associative and commutative.

2- * is continuous.

3- a * 1=a for all a $\in$ [0,1].

4- a * b ≤ c * d whenever a ≤ c and b ≤ d where a,b, c,d $\in$ [0,1].

Remark 1.2: For any r1 > r2 we can find r3 such that r1* r3 ≥ r2 and for any r4 we can find an r5 such that r5* r5 ≥ r4 where r1,r2,r3,r4,r5 $\in$ (0,1) [2].

We introduce the following definition.

Definition 1.3: A triple (X,M,*) is said to be standard fuzzy metric space if X is an arbitrary set, is a continuous t- norm and M is a fuzzy set * on X^2 satisfying the following conditions [1]:

(FM1) M(x,y) > 0 for all x, y $\in$ X

(FM2) M(x,y)=1 if and only if x=y

(FM3) M(x,y)=M(y,x) for all x, y $\in$ X

(FM4) M(x,z) ≥ M(x,y) M(y,z) for all x, y and z $\in$ X

(FM5) M(x,y) is a continuous fuzzy set

Example 1.4: Let X= N, and let a * b=a.b for all a, b $\in$ [0,1] [1].

$$\text{Define } M(x,y) = \begin{cases} \frac{x}{y} \text{ if } x \le y \\ \frac{y}{x} \text{ if } y \le x \end{cases}$$

for all x, y N .

Then (N, M,) is a standard fuzzy metric space.

Example 1.5: Let X= R and let a b=a.b for all a, b $\in$ [0,1] [1].

Define $M(x,y) = \frac{1}{e|x-y|}$ for all x, y $\in$ R

Then (R, M,) is a standard fuzzy metric space.

Definition 1.6: Let (X,M) be a standard fuzzy metric space then M is continuous if whenever xn→x and yn→y in X then M(xn,yn) →M(x,y) that is $\lim_{n\to\infty} M(x_n, y_n) = M(x,y)$ [1].

Definition 1.7: Let (X,M) be a standard fuzzy metric space .Then B(x,r) ={y$\in$X: M(x,y)>1-r} is an open ball with center x$\in$X and radius r, 0<r<1 [1].

Proposition 1.8: Let B(x,r1) and B(x,r2) be two open balls with same center x in a standard fuzzy metric space (X,M,).Then either [1]

B(x,r1) $\subseteq$ B(x,r2) or B(x,r2)$\subseteq$ B(x,r1) where r1,r2$\in$(0,1).

Definition 1.9: A subset A of a standard fuzzy metric space (X,M,) is said to be open if given any point a in A there exists r, 0<r<1 such that B(a,r) $\subseteq$ A. A subset B is said to be closed if B^c is open [1].

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Definition 1.10: Let (X,M,*) be a standard fuzzy metric space and let A $\subset$ X then the closure of A is denoted by $\overline{A}$ A or CL(A) and is defined to be the smallest closed set contains A [1].

Definition 1.11: A subset A of a standard fuzzy metric space (X,M,*) is said to be dense in X if $\overline{A}$ = X [1].

Theorem 1.12: Every open ball in a standard fuzzy metric space (X,M,*) is an open set [1].

Theorem 1.13: Let (X,M,) is a standard fuzzy metric space. Define $\Gamma_M = \{A \subset X : x \in A$ if and only if there exists 0<r<1 such that $B(x,r) \subset A\}$ then Γ_M is a topology on X.

Theorem 1.14: Every standard fuzzy metric space is a Hausdorff space.

Definition 1.15: A sequence (xn) in a standard fuzzy metric space (X,M,*) is said to be converge to a point x in X if for each r ,0<r<1 there exists a positive number N such that M(xn,x)>(1-r) , for each n ≥ N .

Theorem 1.16: Let (X,M,) be a standard fuzzy metric space then for a sequence (xn) in X converge to x if and only if $\lim_{n\to\infty} M(x_n, x) = 1$

Definition 1.17: A sequence (xn) in a standard fuzzy metric space (X,M,) is Cauchy if for each r, 0 < r < 1, there exists a positive number N such that M(xn,xm) > (1-r), for each m, n ≥ N.

Proposition 1.18: Let (X,d) be an ordinary metric space and let a b=a.b for all $a, b \in [0,1]$ Define $M_d(x, y) = \dfrac{1}{1+d(x, y)}$ then (X,M_d,*) is a standard fuzzy metric space and it is called the standard fuzzy metric induced by the metric d [1].

Proposition 1.19: Let (X,d) be a metric space and let (X,M_d,*) be the standard fuzzy metric space induced by d. Let (X_n) be a sequence in X. Then (X_n) converges to x$\in$X in (X,d) if and only if (X_n) converges to x in (X,M_d,*).

Proposition 1.20: Let (X,d) be a metric space and let M_d(X,Y)= $\dfrac{1}{1+d(x, y)}$. Then (X_n) is a Cauchy sequence in (X,d) if and only if (X_n) is a Cauchy sequence in (X, M_d,*) [1].

Definition 1.21: Let (X,M,*) be a standard fuzzy metric space. A subset A of X is said to be F-bounded if there exists 0<r<1 such that, M(x,y)> 1- r, for all x, y$\in$A [1].

Proposition 1.22: Let (X,d) be a metric space and let M_d(X,Y) $= \dfrac{1}{1+d(x, y)}$ then a subset A of X is F-bounded if and only if it is bounded [1].

Definition 1.23: A standard fuzzy metric space (X,M,*) is complete if every Cauchy sequence in X converges to a point in X [1].

Definition 1.24: Let (X,M_x,*) and (Y,M_Y,*) be standard fuzzy metric spaces and A $\subseteq$ X. A function f:A→Y is said to be continuous at a$\in$A, if for every 0< ε <1, there exist some 0< δ <1, such that M_Y (f(x),f(a))> (1- ε) whenever x$\in$A and M_X (x,a)> (1- δ). If f is continuous at every point of A, then it is said to be continuous on A.

Theorem 1.25: Let (X, M_X,*) and (Y,M_Y,*) be standard fuzzy metric spaces and A$\subseteq$X. A function f:A→Y is continuous at a$\in$A if and only if whenever a sequence (X_n) in A converge to a, the sequence (f(X_n)) converges to f(a).

Theorem 1.26: A function f:X→Y is continuous on X if and only if f^{-1}(G) is open in X for all open subset G of Y.

Theorem 1.27: A mapping f:X→Y is continuous on X if and only if f^{-1} (F) is closed in X for all closed subset F of Y [1].

Lemma 1.28: Let A be a subset of a standard fuzzy metric space (X,M,*) then a$\in\overline{A}$ if and only if there is a sequence (a_n) in A such that a_n → a [2].

Theorem 1.29: Let A be a subset of a standard fuzzy metric (X,M,) then A is dense in X if and only if for every x$\in$X there is a$\in$A such that M(x,a) 1- ε for some 0< ε <1 [3].

Definition 1.30: Let (X,M_x,*) and (Y,M_Y ,*) be any two standard fuzzy metric spaces. A mapping f:X→Y which is both one-to-one and onto is said to be a homeomorphism if and only if the mapping f and f^{-1} are continuous on X and Y, respectively. Two standard fuzzy metric spaces X and Y are said to be homeomorphic if and only if there exists a homeomorphism of X onto Y, and in this case, Y is called a homeomorphic image of X [4].

Remark 1.31: If X and Y are homeomorphic, the homeomorphism puts their points in one-to-one correspondence in such a way that their open sets also correspond to one another. For standard fuzzy metric space X and Y, let $X \sim Y$ means that X and Y are homeomorphic. It is easily verified that the relation is reflexive, symmetric and transitive.

Definition 1.32: A mapping f from a standard fuzzy metric space (X, M_X,*) into a standard fuzzy metric space (Y,M_Y ,*) is an F-isometry if

M_Y(f(x),f(y))=M_X(x,y) for all x, y$\in$X.

It is obvious that an F-isometry is one-to-one and uniformly continuous. X and Y are said to be F-isometric if there exists an F-isometry between them that is onto. An F-isometry is necessarily a homeomorphism but the converse is not true [4].

Proposition1.33: Let (X,d) be a metric space and let (X,M_d ,*) be the induced standard fuzzy metric space . Let (Y) be another metric space and let (Y,N_d ,*) be the induced standard fuzzy metric space. Let f:X→Y be a mapping then f is isometry if and only if f is F-isometry [4].

Theorem 1.34: Let A be a dense subset of a standard fuzzy metric space (X,M,*). If every Cauchy sequence of point of A converges in X then (X,M,*) is complete.

Proof: Let (X_n) be a Cauchy sequence in X, since A is dense then for every Xn$\in$X

there is $a_n \in$ A such that M(X_n,a_n)>(1- s) for some 0<s<1 by

Theorem 1.29 Then by Remark 1.2 there is (1- ε) $\in$(0,1) such that

(1- s) * (1- s)> (1- ε).

Since (X_n) is Cauchy so (a_n) is Cauchy so a_n→x by assumption

Now M(x_n,x)$\geq$ M(x_n ,x) * M(a_n ,x)$\geq$ (1- s) * (1- s) > (1- ε)

Hence x_n→x

Definition 1.35: Let (X,M,*) be a standard fuzzy metric space. A completion of (X,M,*) is a complete standard fuzzy metric space (Y,N,*) such that (X,M,*) is F-isometric to a dense subset of Y.

In [4] it was presented the following example of a standard fuzzy metric space

That is not completable.

Example 1.36:

Let a b=max {0, a + b -1} for all a, b $\in$ [0,1] . Now let { X_n: n=3, 4, 5,… , ∞} and {y_n : n=3,4,…, ∞} be two sequences of distinct points such that $A\cap B=\phi$ where A= {X_n : n $\geq$3} and B={ y_n: n $\geq$3}. Put X=A B, define M : X x X→ [0,1] as follows :

$M(X_n,X_m)=M(y_n, y_m)=1-[\frac{1}{n\wedge m} - \frac{1}{n\vee m}]$

Where $n\wedge m$ =min{n,m} and $n\vee m$ =max{n,m}.

$M(X_n, Y_m)=M(y_m, X_n)=\frac{1}{n}+\frac{1}{m}$

It was shown that (X,M,*) is a standard fuzzy metric space and (X,M,*) is not completable.

Definition 1.37: A standard fuzzy metric space (X,M,*) is called completable if it admits a completion.

Theorem 1.38: Every completable standard fuzzy metric space admits a completion.

Standard Fuzzy Quas-Metric Space

Definition 2.1: The triple (X,M,*) is called a standard fuzzy quasi-metric space where X is a nonempty set, * is a continuous t-norm and M is a fuzzy set on X x X satisfying the following conditions:

(1) For all x, y$\in$X, M(x,y) > 0

(2) M(x,y)=M(y,x)=1 if and only if x=y

(3) M(x,y) * M(y,z) $\leq$ M(x,z) for all x, y, z$\in$X

(4) M is a continuous fuzzy set

Propostion 2.2:

If (X,M, *) is a standard fuzzy quasi-metric space then define

M^{-1}: XxX→[0,1] by : M^{-1} (x,y)=M(y,x) for all x, y$\in$X. Then

(X,M^{-1},*) is a standard fuzzy quasi-metric space.

Proof:

(1) M^{-1} (x,y)> 0 since M(y,x)> 0 for all x, y$\in$X

(2) M^{-1} (x,y)=1 if and only if M(y,x)=1=M(x,y) ⇔ y=x

(3) M^{-1} (x,y)* M^{-1} (y,z)=M(y,x) * M(z,y)

= M(z,y) * M(y,x)

$\leq$M(z,x)=M^{-1} (x,z)

(4) M^{-1} is continuous since M is continuous.

Therefore (X, M^{-1}, *) is a standard fuzzy quasi-metric space

Proposition 2.3:

Let (X,M,*) be a standard fuzzy quasi-metric space. Define G:XxX→[0,1] by: G(x,y)=min{M(x,y), M^{-1} (x,y)}. Then (X,G,*) is a standard fuzzy space. We shall refer to (X,G, *) as the standard fuzzy metric induced by (X,M,*).

Proof:

It is sufficient to show that G(x,y)=G(y,x) for each x, y$\in$X.

If G(x,y)=M(x,y) then G(y,x) must equal to M^{-1} (y,x) but

M^{-1} (y,x)=M(x,y) that is G(x,y)=M(x,y).

Hence G(x,y)=G(y,x)

Similarly if G(x,y)=M^{-1} (x,y) then G(x,y)=G(y,x)

Therefore (X,G, *) is a standard fuzzy metric space

Proposition 2.4:

Let (X,M,*) be a standard fuzzy quasi-metric space. Then Γ_m = {A $\subset$ X : a$\in$A ⇔ $\exists$ r, 0< r <1, such that B(a,r) $\subset$ A} is a topology on X.

Proof: The proof is similar to the proof of Theorem 1.13, hence is omitted [5].

Example 2.5:

Let (X,d) be an ordinary quasi-metric space and let M_d be the function defined on X X to [0,1] by: $M_d(x,y)=\frac{1}{1+d(x,y)}$

Then for each continuous t-norm *, (X, M_d,*) is a standard fuzzy quasi-metric space,which is called the standard fuzzy induced by the quasi-metric d. Furthermore, it is easy to check that $(M_d)^{-1}=M_{d-1}$ and $G_d=M_ds$ where d^{-1} (x,y)=d(y,x), d^s (x,y)=max {d(x,y), d^{-1} (x,y)} G_d (x,y)=min{M_d (x,y),M_d^{-1} (x,y)}

Definition 2.6: A standard fuzzy quasi-metric space (X,M,*) is called bicomplete if (X,G,*) is a complete standard fuzzy metric space.

Definition 2.7: Let (X,M,*) be a standard fuzzy quasi-metric space. A bicompletion of (X,M,*) is a bicomplete standard fuzzy quasi-metric space (Y,N,*) such that (X,M,*) is F-isometric to a dense subset of Y.

Lemma 2.8: Let (X,M,*) be a standard fuzzy quasi-metric space. Denote by S the collection of all Cauchy sequence in (X,G,*). Define a relation $\sim$ on S by $(x_n) \sim (x'_n)$ if and only if lim G(x_n, x'_n)=1, where by lim G(x_n, x'_n) we denote the lower limit of the sequence (G(x_n, x'_n)) i.e G(x_n, x'_n)=$\sup_k \inf_{n\geq k}$ G(x_n, x'_n) Then $\sim$ is an equivalence relation on S.

Proof:

1- $\sim$ is reflexive because G(X_n,X_n)=1 for all n$\in$N so (X_n) $\sim$ (X_n)

2- If (X_n)   (y_n), it immediately follows that (y_n) (X_n) because

G(X_n , y_n)=G(y_n, X_n) for all n$\in$N , So that

Lim G(y_n, X_n) Lim=G(X_n , y_n)=1

3- is transitive, suppose that (X_n) $\sim$ (y_n) and (y_n) $\sim$ (Z_n) . We shall prove Lim G(X_n ,Z_n)=1. Since (X_n) $\sim$ (Y_n) then G(X_n , y_n)=1.

Also (Y_n) $\sim$ (Z_n) so Lim G(y_n, X_n)=1 for all n$\in$N.

Now G(X_n ,Z_n) $\geq$ G(X_n ,Y_n) *G(y_n, Z_n)

Hence Lim G(X_n ,Z_n)=1

Lemma 2.9: Define M_s ((X_n),(y_n))=lim M(X_n,y_n) for all (X_n), (y_n)$\in$S where : SxS→[0,1]. Then M_s satisfies 1, 3 and 4 of Definition 2.1.

Proof:

1- M_s ((X_n), (y_n)) > 0 because M(X_n, y_n) >0 so, lim M(X_n, y_n) >0.

3- Let (X_n), (y_n),(Z_n)$\in$S and put α = M_s ((X_n),(y_n)), β =M_s ((y_n),(Z_n))

and $\gamma = M_s((X_n),(Z_n))$. We shall show that $\alpha * \beta \le \gamma$

If=0 or=0 the conclusion is obvious. So we assume that 0

and β =0. Choose an arbitrary $\varepsilon \in (0, \min \frac{\{\alpha\beta\}}{2})$. Then

$\alpha - \varepsilon < M_s((X_n),(y_n))$ and $\beta - \varepsilon < M_s((y_n),(z_n))$

Furthermore, there exists such that for all k $k \ge N_\varepsilon$

$M_s((x_n),(y_n)) - \varepsilon < M(x_k, y_k)$ And $M_s((y_n),(z_n)) - \varepsilon < M(y_k, z_k)$

Then
$$\begin{aligned}(\alpha - 2\varepsilon) * (\beta - 2\varepsilon) &\le [M_s((x_n),(y_n)) - \varepsilon] * [M_s((y_n),(z_n)) - \varepsilon] \\ &\le M(x_k, y_k) * M(y_k, z_k) \\ &\le M(x_k, z_k) \text{ for all } K \ge N_\varepsilon\end{aligned}$$

Therefore
$$\begin{aligned}(\alpha - 2\varepsilon) * (\beta - 2\varepsilon) &\le \inf_{k \ge N_\varepsilon} M(X_k, z_k) \\ &\le \lim M(X_n, Z_n) = \gamma\end{aligned}$$

By continuity of *, it follows that $\alpha * \beta \le \gamma$

4- M_s is continuous because M is continuous

Notation 2.10:

We denote the quotient $s/\sim$ by $\tilde{X}$ and $[(X_n)]$ the class of the element (X_n) of S.

Lemma 2.11:

If $(X_n) \sim Y$ **and** $(y_n) \sim (b_n)$ Then $M_s((X_n),(y_n)) = M_s((a_n),(b_n))$

Proof:

$$\begin{aligned}M_s((X_n),(y_n)) &\ge M_s((x_n),(a_n)) * M_s((a_n),(b_n) * M_s((b_n),(y_n)) \\ &= M_s((a_n),(b_n))\end{aligned}$$

Thus $M_s((X_n),(y_n)) \ge M_s((a_n),(b_n))$ **Now**

$$\begin{aligned}M_s((a_n),(b_n)) &\ge M_s((a_n),(b_n)) * M_s((x_n),(y_n) * M_s((y_n),(b_n)) \\ &= M_s((x_n),(y_n))\end{aligned}$$

So, $M_s((a_n),(b_n)) \ge M_s((x_n),(y_n))$

Therefore $M_s((x_n),(y_n)) \ge M_s((a_n),(b_n))$

Definition 2.12:

For each $[(x_n)], [(y_n)] \in \tilde{X}$ define $\tilde{M}([(x_n)],[(y_n)]) = M_s(x_n),(y_n))$. Then $\tilde{M}$ is a function from $\tilde{X}x\tilde{X}$ to [0,1] and it is well defined by Lemma 2.11. Also we define T:X→ $\tilde{X}$ such that for each x ∈ X, T(x) is the class of constant sequence x, x, …………… .

Now, from the above construction we obtain the main result in this section.

Theorem 2.13:

Let (X,M,*) be a standard fuzzy quasi-metric space.

(a) (X,M,*) is a standard fuzzy quasi-metric space

(b) T(X) is dense in (X,M,*)

(c) (X,M,) is F-isometry to (T(X), M,*)

(d) (X,M,*) is bicomplete

Proof (a):

M satisfies conditions 1, 3 and 4 of Definition 2.1 as an immediate consequence of Lemma 2.9. Now, let (Xn), (Yn) ∈ S such that

$\tilde{M}([x_n],[(y_n)])$ if $(z_n) \in [(y_n)]$ it follows that from Lemma 2.11 that $M_s((z_n),(y_n))$ 1 The same argument shows that $(z_n) \in [(x_n)]$ implies that $M_s((z_n),(x_n)) = 1$ We conclude that $\tilde{M}([x_n],[(y_n)]) = 1$ if and only if $[(x_n)] = [(y_n)]$ Hence $(\tilde{X}, \tilde{M}, *)$ is a standard fuzzy quasi-metric space.

Proof (b):

Let $(x_n) \in S$ and 0< < 1. Since (x_n) is Cauchy sequence in (X,M,*) then there is

N_ε such that $M(x_k, x_{N_\varepsilon}) > (1 - \frac{\varepsilon}{2})$ for all $k \ge N_\varepsilon$

Thus $\tilde{M}([X_n], T(x_{n_\delta})) = M_s((x_n), T()x_{N_\delta})$

$= \sup_n \inf_{k > n} M(x_k, x_{N_\varepsilon})$

$\ge 1 - \frac{\varepsilon}{2} > 1 - \varepsilon$

We have shown that T(X) is dense in $(\tilde{X}, \tilde{M}, *)$

Proof (c):

This is almost obvious because for each x, y ∈ X, we have $\tilde{M}(Tx, Ty) = M(x, y)$

Proof (d):

Let $\tilde{G}([x_n].[x_n]) = \min\{\tilde{M}([(x_n),[(x_n)]), \tilde{M}^{-1}([(x_n),[(x_n)]])\}$

Let $(\tilde{X}_n)$ be a Cauchy sequence in $(\tilde{X}, \tilde{G}, *)$ then there is an increasing sequence (n_k) in N such that $\tilde{G}(\tilde{X}_n, \tilde{X}_m) > 1 - 2^{-k}$ for all n, m $\ge n_k$ Since T(X) is dense in $(\tilde{X}, \tilde{G}, *)$ then for each $k \in N$ there is $y_k \in X$ such that $\tilde{G}(\tilde{X}_{nk}, T(y_k)) > 1 - 2^{-k}$ for all $k \in N$ We show that (y_k) is a Cauchy sequence in (X,G, *). To this end, choose $0 < \varepsilon < 1$ Take $j \in N$ such that $(1\text{-}2^{-j})*(1\text{-}2^{-j})*(1\text{-}2^{-j})>(1\text{-}\varepsilon)$ Then for each k, m $\ge j$ we have

$$\begin{aligned}M(y_k y_m) &= \tilde{M}(T(y_k), T(y_m)) \\ &\ge \tilde{M}(T(y_k), \tilde{X}_{nk}) * \tilde{M}(\tilde{x}_{nm}, T(y_m)) \\ &\ge (1 - 2^{-k}) * (1 - 2^{-(k \wedge m)}) * (1 - 2^{-m}) \\ &\ge (1 - 2^{-j}) * (1 - 2^{-j}) * (1 - 2^{-j}) > (1 - \varepsilon)\end{aligned}$$

And consequently (y_k) is a Cauchy sequence in (X,G,*). Therefore $\tilde{y} \in \tilde{X}$ where $\tilde{y} = [(y_k)]$ Finally, we prove that $(\tilde{X}_n)$ converges to $\tilde{y}$ in $(\tilde{X}, \tilde{G}, *)$

Indeed, as in part (c) choose $0 < \varepsilon < 1$ Take $j \in N$

$(1 - 2^{-j}) * (1 - 2^{-j}) * (1 - 2^{-j}) > (1 - \varepsilon)$

Since (y_k) is a Cauchy sequence in $(\tilde{X}, \tilde{G}, *)$ the proof of part (b) shows that there is $k \ge j$ such that $\tilde{G}(\tilde{y}, T(y_k)) > 1 - 2^{-j}$

Then for $n \ge n_k$ we obtain

$\tilde{G}(\tilde{y}, \tilde{x}_n) \ge \tilde{G}(\tilde{y}, T(y_k)) * \tilde{G}(T(y_k), \tilde{X}_{nk}) * \tilde{G}(\tilde{X}_{nk}, \tilde{X}_n)$

$\ge (1 - 2^{-j}) * (1 - 2^{-k}) * (1 - 2^{-k}) *$

$\ge (1 - 2^{-j}) * (1 - 2^{-j}) * (1 - 2^{-j}) > (1 - \varepsilon)$

We conclude that $(\tilde{X}, \tilde{M}, *)$ is bicomplete

Definition 2.14: A standard fuzzy quasi-metric space (X,M,*) is called bicompletable if it admits a bicompletion

Theorem 2.15: Let (X,M,*) be a standard fuzzy quasi-metric space and let (Y,N,*) be a bicomplete standard fuzzy quasi-metric space. If there is an F-isometry mapping f from a dense subset A of X to Y then f has a unique extension f^*: X→Y.

Proof: We consider any $x \in X$ but $X=\bar{A}$ so $x \in \bar{A}$ then there is a sequence (X_n) in A such that (X_n) converges to x by Lemma 1.28. Then (X_n) is Cauchy.

Since f is F-isometry ($f(X_n)$) is Cauchy in Y but Y is complete hence there is $y \in Y$ such that ($f(X_n)$) converges to y. Now we define f^* (x)=y.

We now show that this definition is independent of the particular choice of the sequence in A converging to x .Suppose that (X_n) in A converges to x and (z_n) in A converges to x. Then (v_m) converges to x where (v_m)=(x_1 ,z_1 x_2,z_2 , ,…). Hence ($f(v_m)$) converges and the two subsequence ($f(x_n)$) and ($f(z_n)$) of ($f(v_m)$) must have the same limit. This prove is uniquely defined at every $x \in X$. Clearly (x)=f(x) for every $x \in A$ so that is an extension of f.

Theorem 2.16: Let (X,M,*) be a standard fuzzy quasi-metric space and let (Y,N,*) be a bicomplete standard fuzzy quasi-metric space .If f is an F-isometry mapping from a dense subset A of X to Y then the unique extension f^*:X→Y is an F-isometry.

Proof:

Let $x, y \in X$ then there exists two sequences (x_n) and (y_n) in A such that $x_n \to x$ and $y_n \to y$. Choose an arbitrary $0< \varepsilon < 1$. Now:

$\varepsilon + M(x,y) > M(x,y)$. Furthermore, it follows that (x_n) and (y_n) are Cauchy sequences in A so ($f^*(x_n)$) and ($f^*(y_n)$) are Cauchy sequences in Y. But Y is complete hence ($f^*(y_n)$) converges to (y) and ($f^*(x_n)$) converges to $f^*(x)$. Then there is a positive integer N such that

$M(x,x_n) > (1-\varepsilon), M(y_n,y) > (1-\varepsilon)$

$N(f^*(x_n), f^*(x))>(1-\varepsilon)$ and $N(f^*(y_n),(y))>(1-\varepsilon)$ for all $n \geq N$

Thus we have

$$\begin{aligned}\varepsilon + M(x,y) &> M(x,y)\\ &\geq M(x,x_n) * M(x_n,y_n) * M(y_n,y)\\ &\geq (1-\varepsilon) * N(f^*(x_n), f^*(y_n)) * (1-\varepsilon)\end{aligned}$$

But

$$\begin{aligned}N(f^*(x_n), f^*(y_n)) &\geq N(f^*(x_n), f^*(x)) * N(f^*(x), f^*(y)) * N(f^*(y_n), f^*(y))\\ &\geq (1-\varepsilon) * N(f^*(x), f^*(y)) * (1-\varepsilon) \text{ for all } n \geq N\end{aligned}$$

Therefore $\varepsilon + M(x,y) > (1-\varepsilon) * [(1-\varepsilon) * N(f^*(x), f^*(y)) * (1-\varepsilon)] * (1-\varepsilon)$

By continuity of * and * it follows that $M(x,y) \geq N((x), (y))$

A similar argument shows that $N((x), (y)) \geq M(x,y)$ For all $x, y \in X$

We conclude that f* is an F-isometry from (X,M,*) to (Y,N,*)

Theorem 2.17: Every bicompletable standard fuzzy quasi-metric space admits a unique [up to F-isometry] bicompletion.

Proof: Let (Y,M_1,*) and (Z,M_2,0) be two bicompletions of (X,M,*) then we will prove that (Y,M_1,*) and (Z,M_2,0) are F- isometric. Since (Y, M_1,*) is a bicompletion of (X,M,*) then there is an F-isometry f from (X,M,*) to a dense subset of (Y, M_1,*). By Theorem 2.15 and Theorem 2.16 f admits a unique extension f* onto (Y, M_1,*) which is also an F-isometry. Similarly is an isometry extension (X,M,*) onto (Z, M_2,0). To prove that and are F-isometric it remains to see that and are onto we will show that is onto. Indeed given $y \in Y$ there is a sequence (x_n) in X such that (x_n) →y. Since is an F-isometry (x_n) is a Cauchy sequence, so it converges to some point $x \in X$. Consequently f* (x)=y. Similarly we can prove that is onto. Hence f* and f are F-isometric.

Now (Y, M_1,*) is F-isometric to (X,M,*) and (X,M,*) is F-isometric to (Z, M_2,0). Hence (Y, M_1,*) is F-isometric to (Z, M_2,0).

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