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Balanced Folding Over a Polygon and Euler Numbers

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Abstract

In this paper we introduced a new folding over a polygon we called it balanced folding, then we proved that for a balanced folding of a simply connected surface *M* there is a subgroup of the group of all homeomorphisms of *M* that acts *1*-transitively on the *2*-cells of *M*. Also we explored the relationship between balanced folding and covering spaces. Finally we obtained a general relation of the Euler number of surfaces which may balance folded over a polygon and we also listed all the possibilities if *M* is a sphere balanced folded over a triangle and we gave the subgroup mentioned above in each case.

Keywords: Surface; Cellular folding; Singularities; Cayley color graph; 1-Transitive; Universal covering; Euler number

Introduction

Let *K* and *L* be cellular complexes and $f:|K|\to|L|$ a continuous map. Then $f: K\to L$ is a cellular map if

(i) for each cell $\sigma\in K, f(\sigma)$ is a cell in *L*,

(ii) $\dim(f(\sigma))\le\dim(\sigma)$, [1].

A cellular map $f: K\to L$ is a cellular folding iff

(i) for each i-cell $\sigma^i\in K$, $f(\sigma^i)$ is an i-cell in *L* , i.e., *f* maps *i*-cells to *i*-cells,

(ii) if $\overline{\sigma}$ contains *n* vertices, then $\overline{f(\sigma)}$ must contains *n* distinct vertices, [2].

A cellular folding $f: K\to L$ is neat if $L^n - L^{n-1}$ consists of a single *n*-cell, interior *L*. The set of all cellular folding of *K* into *L* is denoted by *C(K, L)* and the set of all neat foldings of *K* into *L* by N(*K*, *L*).

If $f\in C(K, L)$, then $x\in K$ is said to be a singularity of *f* iff *f* is not a local homeomorphism at *x*. The set of all singularities of *f* corresponds to the "folds" of the map. This set associates a cell decomposition C_f of *M*. If *M* is a surface, then the edges and vertices of C_f form a graph Γ_f embedded in *M* [3].

Now there is a graph K_f associated to C_f in a natural way. In fact the vertices of K_f are just the *n*-cells of C_f, and its edges are the *(n-1)*-cells. If $e\in C_f$ is *(n-1)*-cell, then e lies in the frontiers of exactly two *n*-cells σ, σ/. We then say that *e* is an edge in K_f with end points σ, σ/. The graph K_f can be realized as a graph $\tilde{K}_f$ embedded in *M* as follows. For each *n*-cell σ, choose any point $\tilde{\sigma}\in\sigma$. If the *n*-cells σ, σ/ are end points of *e*, then we can join $\tilde{\sigma}$ to $\tilde{\sigma}'$ by an arc $\tilde{e}$ in *M* that runs from $\tilde{\sigma}$ through σ and σ′ to $\tilde{\sigma}'$, crossing *e* transversely at a single point. Trivially, the correspondence $\sigma\to\tilde{\sigma}$, $e\to\tilde{e}$ is a graph isomorphism. Figure 1 illustrates this relationship in case n=2.

It should be noted that the graph K_f may have more than one edge joining a given pair of vertices. For instance, consider the cellular folding *f* of the torus into itself with the cellular subdivision shown in Figure 2. The graph K_f has just two vertices but two edges, see Figure 2.

Balanced Folding

Definition 1: Let *M* be a compact connected surface, and P_n a cell complex having *n* 0-cells, *n* 1-cells and just one 2-cell. Again a continuous map $f: M\to P_n$ is called neat folding if there is a cell decomposition C_f of *M* such that:

(i) *f* is a cellular map of C_f onto $C(P_n)$.

(ii) for each closed cell $\overline{\sigma}$ of C_f, $f|\overline{\sigma}$ is a homeomorphism of $\overline{\sigma}$ onto a closed cell of $C(P_n)$.

To avoid trivial cases, we require that that each 0-cell of *M* is an end point of more than two 1-cells. Thus the 0-cells and 1-cells of this decomposition form a finite graph Γ_f without loops (but possibly with multiple edges) and *f* folds *M* along the edges or 1-cells of Γ_f [4].

Let $f: M\to P_n$ be a neat folding. Then for any *n*-cells *A* and *B* there is a homeomorphism $f_{AB}: A\to B$ given by $f_{AB}(a)=b$ iff $f(a)=f(b)$, where $a\in A$ and $b\in B$. We can always extend f_{AB} to a homeomorphism, $\overline{f}_{AB}:\overline{A}\to\overline{B}$, but there need not exist an extensions to any open neighbourhoods of *A* and *B*. The following two examples explore this fact.

Example 1: Let *M* be a desk in the plane R^2 with the cellular subdivisions shown in Figure 3. Let P_4 be a desk with four 0-cells, four 1-cells and one 2-cell.

Define a map $f: M\to P_4$ by $f(\sigma_i)=\sigma$, $i=1, 2, \ldots,9$,

$$f(e^1_{17})=f(e^1_3)=f(e^1_{16})=f(e^1_2)=f(e^1_{15})=f(e^1_1)=\overline{e}^1_1$$

$$f(e^1_{20})=f(e^1_{13})=f(e^1_6)=f(e^1_{18})=f(e^1_{11})=f(e^1_4)=\overline{e}^1_2$$

$$f(e^1_{24})=f(e^1_{23})=f(e^1_{22})=f(e^1_{10})=f(e^1_9)=f(e^1_8)=\overline{e}^1_3$$

$$f(e^1_{21})=f(e^1_{14})=f(e^1_7)=f(e^1_{19})=f(e^1_{12})=f(e^1_5)=\overline{e}^1_4$$

$$f(e^0_{11})=f(e^0_3)=f(e^0_9)=f(e^0_1)=\overline{e}^0_1 ,$$

$$f(e^0_{12})=f(e^0_4)=f(e^0_{10})=f(e^0_2)=\overline{e}^0_2$$

$$f(e^0_{15})=f(e^0_7)=f(e^0_{13})=f(e^0_5)=\overline{e}^0_4 \text{ and}$$

$$f(e^0_{16})=f(e^0_8)=f(e^0_{14})=f(e^0_6)=\overline{e}^0_3 .$$

Let $A=\sigma_4$, $B=\sigma_7$ be the 2-cells shaded in Figure 3. Then there is a homeomorphism $f_{AB}: A\to B$ given by $f_{AB}(x,y)=f(x',y')$ iff $f(x,y)=f(x',y')$

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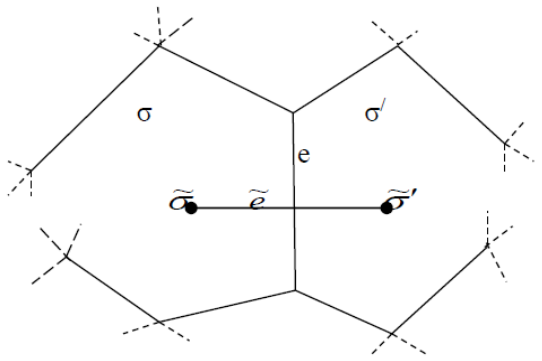

Figure 1: Relationship in case n=2.

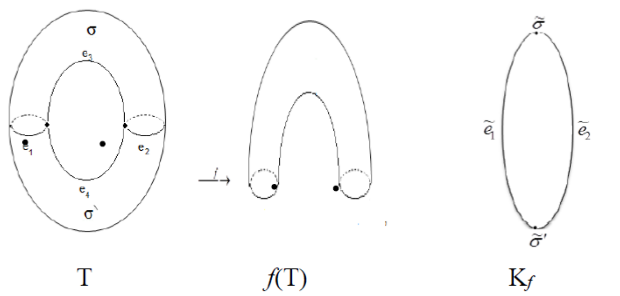

Figure 2: Cellular subdivision.

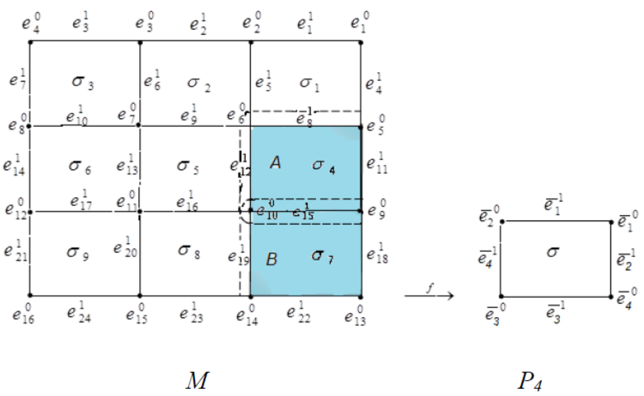

Figure 3: Cellular subdivisions 2.

where $(x,y)\in A$ and $(x,y)\in B$. This homeomorphism has an extension to a homeomorphism $\bar{f}_{AB}:\bar{A}\to\bar{B}$ given by $\bar{f}_{AB}(x,y)=(x',y')$ iff $f(x,y)=f(x',y')$, where $(x,y)\in\bar{A}$ and $(x',y')\in\bar{B}$. Now consider any open neighbourhoods $\tilde{A},\tilde{B}$ of $\bar{A},\bar{B}$ respectively. We see that there is no extension of $\bar{f}_{AB}$ to a homeomorphism $\tilde{f}_{AB}:\tilde{A}\to\tilde{B}$. This is because three 1-cells of A are interior to M, while two 1-cells of B have this property.

Example 2: Let M be a sphere partitioned by the cells shown in Figure 4.

A cellular folding f may be defined from M to a polygon P_3. The vertices are labelled in such a way that vertices with the same image under f are labelled alike.

Now, it can be checked that a homeomorphism f_{AB}: $A\to B$, (where A and B are the 2-cells shaded in Figure 4) cannot be extended to a homeomorphism of any neighborhoods $\tilde{A},\tilde{B}$ of $\bar{A},\bar{B}$ respectively. This is because the valencies of the vertices of the 2-cell A are 12, 4, 4 while those of B are 12, 8, 4.

Definition 2: We will call a neat folding: f: $M\to P_n$ a balanced folding if for all 2-cells A, B and each homeomorphism f_{AB}: $A\to B$ given by $f_{AB}(a)=b$ iff $f(a)=f(b)$, we can extend f_{AB} to a homeomorphism for any neighbourhoods $\tilde{A}$, $\tilde{B}$ of $\bar{A},\bar{B}$ respectively.

We denote the set of all balanced foldings of M into P_n by $\mathfrak{B}(M, P_n)$.

Example 3: Let M be a sphere partitioned by the cells shown in Figure 5. The valencies of the vertices of each 2-cells are 4, 6 and 8.

A neat folding f may be defined from M to a polygon P_3. The vertices are labeled in such a way that vertices with the same image under f are labeled alike.

If we considered any 2-cells A and B of M (e.g. the shaded 2- cells in Figure 5) then, it can be checked that a homeomorphism f_{AB}: $A\to B$, (where A and B are the 2-cells shaded in Figure 5) can be extended to a homeomorphism of any neighborhoods $\tilde{A},\tilde{B}$ of $\bar{A},\bar{B}$ respectively. This is because the vertices of the 2-cells A and B have the same valencies. It follows that f is balanced.

The Properties of the Graph K_f of Neat Folding

Let $f\in \mathfrak{N}(M, P_n)$, then K_f has the following special features.

(a) Edge coloring: Let $e_1, e_2,\ldots, e_n$ be the 1-cells of P_n, we can regard the indices i, $i=1, 2,\ldots, n$ "colors". Each edge of K_f is mapped by f to one

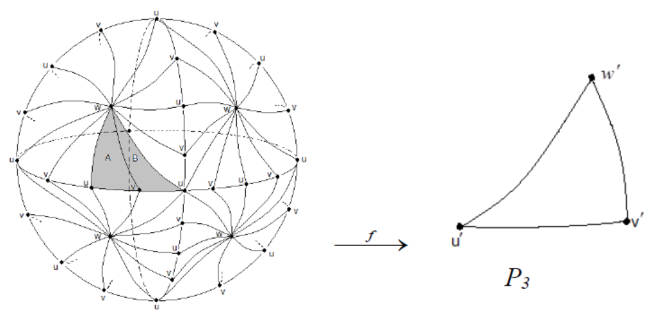

Figure 4: Sphere partitioned by the cells.

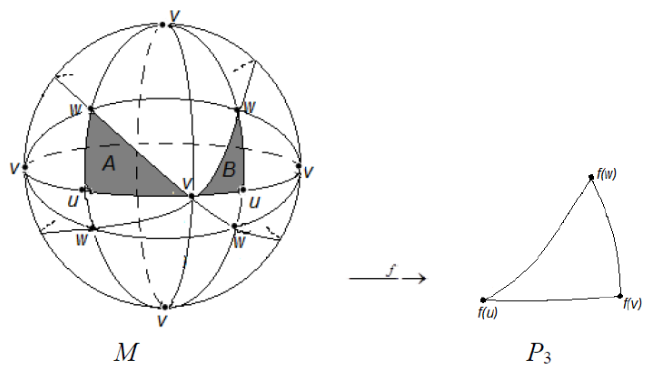

Figure 5: Sphere partitioned by the cells 2.

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of these. We may then give K_f an edge-coloring by assigning to each edge e of K_f the color i of its image $f(e)=e_i$.

(b) Sources and sinks: If A and B are 2-cells of M with a common 1-cell in their frontiers, then A and B are given opposite orientations by f. It follows that each edge of the graph K_f may be oriented in such a way that every vertex is either a source or a sink (where a vertex u is a source if all the oriented edges with u as a vertex begin at u, and is a sink if all the edges end at w), see Figure 6. For such a graph, every circuit has an even number of edges (and hence of vertices).

(c) Regularity: If $f\in \mathcal{N}(M, P_n)$, so that every 2-cell of M is mapped homeomorphically by f onto interior P_n, then the graph K_f is regular. This follows immediately from the fact that the 1-cells in the frontier of each 2-cell is 1-1 correspondence under f with those of P_n. It is also worth observing that every color i occurs once in the set of colored edges at each vertex of K_f. Consequently, the valency of each vertex of K_f is the cardinality of the set of 1-cells of P_n, that is to say, of the set of colors.

The properties of the graph K_f we have already discussed suggest that in certain cases the graph K_f may be a Cayley color graph. In the following we can show that this is indeed the case for a large class of balanced foldings.

Note first that, for any map f: $M \to N$, we can associate a group $G(f)$ namely the group of all homeomorphisms h: $M \to M$ such that $foh=f$. In case $f\in \mathcal{N}(M, P_n)$, we may ask whether the induced action of $G(f)$ on the 2-cells of M is transitive. In particular, we ask whether there is a subgroup $H(f)$ of $G(f)$ that acts 1-transitively on the set of 2-cells.

The Action of the Group of Homeorphisms on the 2-Cells

Theorem 1: Let M be a simply connected surface, f: $M \to P_n$ be a balanced folding then there is a subgroup $H(f)$ of $G(f)$ that acts 1-transitively on the set of 2-cells of M. Moreover K_f is a Cayley color graph of the group $H(f)$.

Proof: Let $f\in \mathcal{B}(M, P_n)$. Let A, B be 2-cells of the cell decomposition of M associated by f. Then f_{AB}: $A \to B$ extends to a homeomorphism $\tilde{f}_{AB}:\tilde{A}\to\tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are open neighborhoods of A and B respectively.

Now let C be a 2-cell such that $C\neq A$ and $C\cap\tilde{A}\neq\varphi$. Let $\tilde{f}_{AB}(C)\subset D$. Then there are open neighborhoods $\tilde{C}$ and $\tilde{D}$ of C and D such that f_{CD} extends to a homeomorphism $\tilde{f}_{CD}:\tilde{C}\to\tilde{D}$, where $\tilde{f}_{CD}$ and $\tilde{f}_{AB}$ agree on $\tilde{A}\cap\tilde{C}$. Iterate this procedure to extend f_{AB} to a map F_{AB}: $M \to M$.

The existence and uniqueness of this extension are guaranteed by the fact that M is 1-connected.

Now, to prove that F_{AB} is onto, let $y \in M$ a non-singular point.

Then y belongs to a 2-cell σ. Let $B_1, B_2,\ldots, B_{k+1}=\sigma$, be a sequence of 2-cells such that B_i, B_{i+1} are contiguous, $i=1, 2, \ldots, k$. The sequence B_1, $B_2,\ldots, B_{k+1}$ of 2-cells is the image under F_{AB} of a unique sequence A_1, $A_2,\ldots, A_{k+1}=\sigma'$ of 2-cells such that A_i, A_{i+1} are contiguous, $i=1, 2,\ldots, k$ and each $F_{A_iB_i}:A_i\to B_i$ extends to a homeomorphism $\tilde{F}_{A_iB_i}:\tilde{A}_i\to\tilde{B}_i$ where $\tilde{F}_{A_iB_i}$ and $\tilde{F}_{A_{i+1}B_{i+1}}$ agree on $\tilde{A}_i\cap\tilde{A}_{i+1}$. Hence F_{AB} is onto.

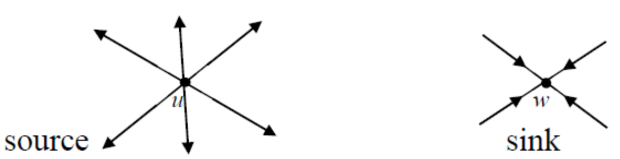

Figure 6: The oriented edges.

We have now shown that F_{AB} is a local homeomorphism of the simply connected manifold M onto itself. In fact, F_{AB} is a covering map. Thus F_{AB} is a homeomorphism.

The set of all such homeomorphisms is the required group $H(f)$, which by its construction acts 1- transitively on the set of 2-cells.

The relationship of $H(f)$ to the graph K_f is as follows:

Choose some 2-cells A. Thus A is a vertex of K_f. Identify any other vertex (2-cell) B of K_f with the unique element F_{AB} of $H(f)$ such that $F_{AB}(A)=B$.

It follows trivially that the graph K_f is a Cayley color graph of $H(f)$, with generators $f_B=f_{AB}$, where B runs through the set of 2-cells $B\neq A$ having a 1-cell in its common frontier with A.

Note that in general any neat folding f of a surface M to a surface N, the set of singularity forms the edges and vertices of a graph Γ_f. If f is balanced, then the valencies of the vertices are invariant under any of the extended homeomorphisms $\tilde{F}_{AB}$. In particular, if $f\in\mathcal{N}(M, P_n)$ be such that Γ_f is a regular graph embedded in M, then $f\in\mathcal{B}(M, P_n)$. Moreover, if M is simply connected, then $H(f)$ will acts 1-transitively on the set of 2-cells of M and K_f will be a Cayley color graph of the group $H(f)$.

Example 4: Let $M=S^2=\{x\in R^3:\|x\|=1\}$, be the unit sphere in the Euclidean 3-space. Let f: $M \to M$, be given by $f(x,y,z)=(|x|,|y|,|z|)$. Then f is a neat folding and the graph Γ_f is a regular graph of valency 4, with 6 vertices, twelve 1-cells and eight 2-cells. The image is the positive octant P_3 where $x\geq0$, $y\geq0$, $z\geq0$ see Figure 7a. Since Γ_f is a regular graph, it follows that f is a balanced folding and the graph , which is a Cayley color graph, has the form given in Figure 7b. Hence $H(f)$ is isomorphic to $Z_2\times Z_2\times Z_2$ and it acts 1-transitively on the set of eight 2-cells $A_1, A_2,\ldots, A_8$.

We now explore the relationship between balanced foldings and covering maps.

Theorem 2: Let $f\in \mathcal{N}(M,P_n)$, and let $p:\tilde{M}\to M$ be the universal covering. Suppose that

$\tilde{f}=fop\in\mathcal{B}(\tilde{M},P_n)$ and that $G(p)\lhd H(\tilde{f})$. Then there is a subgroup $H(f)$ of $G(f)$, isomorphic to $H(\tilde{f})/G(p)$, acting 1-transitively on the set of 2-cell of M by f.

Proof: We first construct the group $H(f)$. Let $\tilde{h}\in H(\tilde{f})$, we now show that $\tilde{h}$ covers a (unique) homeomorphism h: $M \to M$, i.e. $hop=po\tilde{h}$. Let $a\in M$, and let $\tilde{a}\in p^{-1}(a)$. Put $b=p(\tilde{b})$, where $\tilde{b}=\tilde{h}(\tilde{a})$. The point b is independent of the choice of $\tilde{a}\in p^{-1}(a)$. For if $p(\tilde{c})=a$, and $p(\tilde{d})=d$ where $\tilde{d}=\tilde{h}(\tilde{c})$, then there is an element $g\in G(p)$ such that $g(\tilde{a})=\tilde{c}$. Consider $g'=\tilde{h}og\tilde{h}^{-1}$. Then $g'(\tilde{b})=\tilde{d}$. Since $G(p)\lhd H(\tilde{f})$, $g'\in G(p)$. Thus $b=p(\tilde{b})=p(\tilde{d})=d$.

Now, define h: $M \to M$ by $h(a)=b$. Then h is a homeomorphism of M, and, trivially, the set $H(f)=\{h:\tilde{h}\in H(\tilde{f})\}$ is a subgroup of $G(f)$ isomorphic to $H(\tilde{f})/G(p)$. Thus there is an epimorphism $\theta:H(\tilde{f})\to H(f)$ given by $\theta(\tilde{h})=h$.

Secondly, we show that $H(f)$ acts 1-transitively on the set of 2-cells of M by f. Let A, B be 2-cells of M by f. Then there are 2-cells $\tilde{A}$ and $\tilde{B}$ of $\tilde{M}$ by $\tilde{f}$ such that $p(\tilde{A})=A$ and $p(\tilde{B})=B$. Let $\tilde{h}$ be the unique

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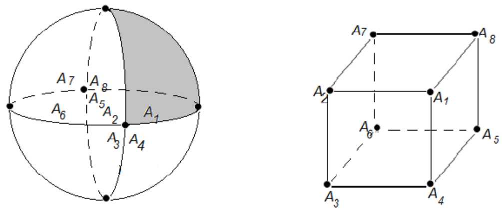

(a) S^2 with the graph Γ_f (b) The graph K_f is a Cayley color graph

Figure 7: Cayley color graph.

element of $H(\tilde{f})$ such that $\tilde{h}(\tilde{A})=\tilde{B}$ and let $h=\theta(\tilde{h})$. Then $h(A)=B$, and there is only one such element of $H(f)$.

It should be noted that if $p:\tilde{M}\to M$ is a covering map, and $\tilde{f}=fop$, where $f\in\aleph(M,P_n)$, then $\tilde{f}\in\beta(\tilde{M},P_n)$ implies that $f\in\beta(M,P_n)$.

Example 5: Let $M=P_n(R)$ be the real projective, n-space, and let P_n be the n-polygon $\{t\in R^{n+1}:\sum_{i=1}^{n+1}t_i=1, 0\le t_i\le 1\}$. Define f: $M\to P_n$ by $f(\{x\})=(|x_1|,\ldots,|x_{n+1}|)/\|x\|$. Then $\tilde{M}$ may be identified with S^n, and $p:\tilde{M}\longrightarrow M$ is given by $p(x)=\{x\}$. In this case $G(p)\cong Z_2$ is generated by the map g: $S^n\to S^n$, $g(x)=-x$ and $H(\tilde{f})\cong(Z_2)^{n+1}$ is generated by the reflexions g_i: $R^{n+1}\to R^{n+1}$, $g_i(x_1,\ldots,x_{n+1})=(x_1,\ldots,x_{i-1},-x_i,x_{i+1},\ldots,x_{n+1})$ and $\tilde{f}(x)=(fop)(x)=(|x_1|,\ldots,|x_{n+1}|)/\|x\|$ as above.

Theorem 3: Let $\tilde{f}$ and f be as in Theorem 1 such that $G(p)\triangleleft H(\tilde{f})$. Let γ: $L\to M$ be a regular covering. Then $H(g)$, where $g=fo\gamma$, acts 1-transitively on the set of 2-cells of L by g.

Proof: Since M is simply connected, for any other covering map γ: $L\to M$ there exists a universal covering map $h:\tilde{M}\to L$ such that $\gamma oh=p$ (Figure 8).

Now $G(p)\cong\Pi_1(M)$ and $G(h)\cong\Pi_1(L)$. Since γ: $L\to M$ is regular $\gamma_*\Pi_1(L,y)\triangleleft\Pi_1(M,x)$, where $\gamma(y)=x$. There is isomorphism Φ: $G(p)\to\Pi_1(M)$ and Ψ: $G(h)\to\Pi_1(L)$ such that following diagram is commutative (Figure 9).

It follows from elementary group theory that, since $\Pi_1(L)$ is embedded in $\Pi_1(M)$ as a normal subgroup, then $G(h)$ is embedded by α in $G(p)$ as a normal subgroup. But $G(p)\triangleleft H(\tilde{f})$ by assumption. Hence $G(h)\triangleleft H(\tilde{f})$ and Theorem 2 can be applied for g, yielding that $G(g)=H(\tilde{f})/G(h)$ acts 1-transitively on the set of 2-cells of L by g.

Euler Numbers of Balanced Folding onto a Polygon

General considerations

Let $f\in\aleph(M,N)$, where M and N are surfaces. To avoid too many complications, let us suppose that M is compact, connected and without boundary, and let N be connected.

Since M is compact the graph Γ_f is a finite graph. Let Γ_f divides M into k 2-cells, or faces, A_1, A_2,…, A_k. In this case $f\,|A_i$, $i=1$, …, k is a homeomorphism onto the interior of N.

We can triangulate N by a simplicial complex T_N such that every vertex of the cell decomposition C_f of ∂N is a vertex of T_N. Let T_M be the triangulation of M induced by f.

Consider the faces A_1, …, A_k and their closures B_1, …, B_k. Thus $e(B_i)=e(N)$, $i=1$, …, k, where $e(X)$ is the Euler number of X. If we now calculate the Euler number $e(M)$ of M using the triangulation T_N, then we can compare $e(M)$ with $\sum_{i=1}^{k}e(B_i)=ke(N)$. We note that for each vertex of Γ_f with valency υ exactly υ vertices have been counted in the calculation of the Euler number $ke(N)$ of the disjoint union of B_1, …, B_n. Likewise, every edge of Γ_f appears twice in these calculations. Figure 10 which shows the neighborhood of a vertex with valency 4, may help to clarify these remarks.

Thus to obtain $e(M)$ from $\sum_{i=1}^{k}e(B_i)$ we must subtract $(\upsilon-1)$ for each vertex of Γ_f (of valency υ) and add the number of edges of Γ_f. The first of these is V-nk, where V is the number of vertices of Γ_f, and n is the number of vertices of ∂N. The second is equal to $\frac{nk}{2}$. We conclude that:

$$e(M)=ke(N)+V-\frac{nk}{2} \quad (1)$$

The case in which N is the disc D^2, $e(N)=1$ and each 2-cell A of M is homeomorphic to D^2. Thus equation (1) now reduces to

$$2e(M)=k(2-n)+2V \quad (2)$$

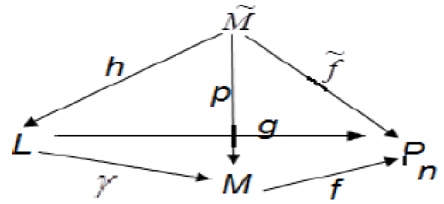

Figure 8: Universal covering map.

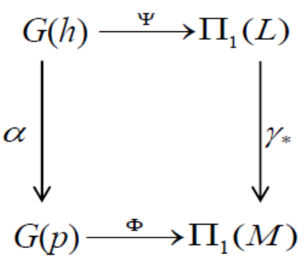

Figure 9: Universal covering map 2.

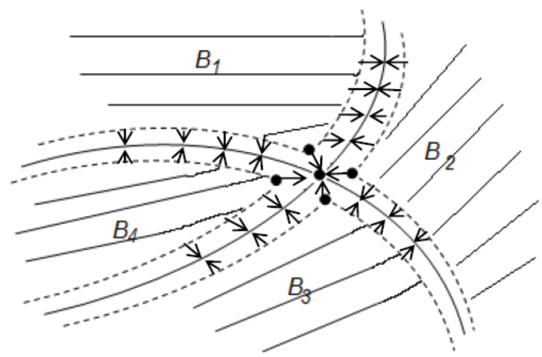

Figure 10: Neighborhood of a vertex with valency 4.

q_1	q_2	q_3	k	$H(f)$
2	2	p, p>1	4p	$\mathcal{D}_{2n}$
2	3	3	24	O
2	3	4	48	Ō
2	3	5	120	Ī

Table 1: Possibilities list.

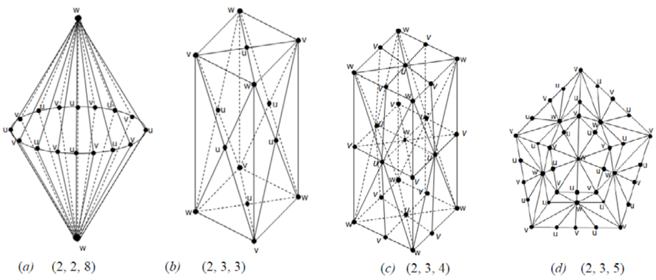

Figure 11: Triangulation.

Balanced folding over a polygon

Equations (1) and (2) can be refined slightly when f is balanced. In this case, if we label the vertices of the polygon Pn as $V_1, \ldots, Vn$, then each vertex in the set $f^{-1}(Vj)$ has the same valency $2q_j$, $j=1, \ldots, n$.

It follows that $f^{-1}(Vj)$ contains $\frac{k}{2q_j}$ elements. Thus the number of vertices of Γ_f is:

$$V = \frac{k}{2}\sum_{j=1}^{n}\frac{1}{q_j} \tag{3}$$

Hence for a balanced folding over a disc, equation (3) may be reduced to

$$2e(M) = k\ (2-n) + k\sum_{j=1}^{n}\frac{1}{q_j} = k\left\{(2-n) + \sum_{j=1}^{n}\frac{1}{q_j}\right\} \tag{4}$$

Certain cases of equation (4) are of special interest. For instance, let n=3, so that M is triangulated by Γ_f, and equation (4) becomes

$$2e(M) = k\left\{\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} - 1\right\} \tag{5}$$

Thus if M is a sphere, then $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} > 1$ and $k \geq 4$. The only possibilities are listed in the following Table 1:

The group $H(f)$ associated with f according to Theorem 1 is shown in column5, and the corresponding triangulation of S^2 are shown in Figures 11a-11d. Note that in Figure 11d we have drawn only one side. The vertices are labeled in such a way that vertices with the same image under f are labelled alike.

References

1. Kinsey LC (1993) Topology of surfaces. Springer, New York, USA.
2. El-Kholy E, Shahin RM (1998) Cellular folding. J Inst Math and Comp Sci 11: 177-181.
3. Robertson SA, El-Kholy E (1986) Topological foldings. Commun Fac Sci Univ Ank Ser 35: 101-107.
4. Farran HR, El-Kholy E, Robertson SA (1996) Folding a surface to a polygon. Geometriae Dedicata 63: 255-266.