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Average Sentinel for a Heat Equation with Incomplete Data

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Abstract

In this paper, we analyse the problem of identification of the pollution term in a heat system when the dynamics of the state is governed by a parameterized operator. In this way, we introduce a notion of average sentinel. First, we prove the existence of such sentinels introduced by Lions by solving a problem of null average controllability given by Zuazua. Secondly, we identify the information for pollution terms by using the average sentinel.

Keywords: Average sentinel; Average controllability; Averaged observability; Pollution term

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Introduction

In the modelling of the evolution system type, the source terms as well as the initial or boundary conditions may be unknown [1]. Here we analyse the problem of identification of pollution term (the source) in a systems governed by a heat equation depending on unknown parameters in a deterministic manner [2]. We look for a control, independent of the values of these parameters that need to introduce the sentinel in an averaged sense to be made precise [3]. The notion of averaged control for parameter-dependent family of parabolic system is introduced by Lions [4] and the sentinel method introduced by Lions [5,6] is adapted to the estimation of this incomplete or unknown data in the problems governed by parabolic system in general, for example, pollution in river or a lake. So since the introduction of the sentinel method many authors developed several applications, such as in environment, in ecology [3].

The notion of average sentinel, as formulated here, has not been analysed until now, this notion is very interested in the identification of the missing data when the system is depending on unknown parameters [7].

By duality this leads to averaged observability problems, we prove the null average controllability of the adjoint system with constraint on the averaged control and we give information for the unknown source [8].

Let Ω be a bounded domain in R^d, $d \geq 1$, with smooth boundary $\Gamma$and w be an open non-empty subset of Ω [9]. Denote by $Q=\Omega \times (0;T)$ the space time cylinder where the equation holds and $\sum=\Gamma \times (0,T)$ for the lateral boundary, we will assume that a parameter $\theta \in (0,1)$, and $y_\theta(x,t)=y(x,t,\theta)$ the solution of the following system:

$$\begin{cases} \dfrac{\partial y_\theta(x,t)}{\partial t} - div(a(x,\theta)\nabla y_\theta(x,t)) = f(x,t) & in\ Q, \\ y_\theta = 0 & on\,\Sigma, \\ y_\theta(x,0) = y_0(x) & in\ \Omega. \end{cases} \tag{1}$$

where the diffusivity coefficients $a(x,\theta)$, taken to be scalar to simplify the study, are assumed to be bounded above and below by positive constants, and to depend on the uncertainty parameter $\theta \in (0,1)$ in a continuous manner. However, the dynamics of the state is governed by a parameterized operator $A(\theta)=\text{div}(a(\theta)\nabla y_\theta)$ [10].

We assume that $y_0 \in L^2(\Omega)$, $f \in L^2(Q)$ and so that eqn. (1) admits a unique solution

$$y_\theta \in C((0,T)), L^2(\Omega)) \cap L^2(0,T,H_0^1(\Omega)),\ for\ all\ \theta \in (0,1)$$

The motivation of the problem we consider is the following:

We address to the system in eqn. (1) whose initial datum and the source term are unknown and the effective value of the parameter being unknown [11],

$$\begin{aligned} f &= \zeta + \lambda\hat{\zeta}, \\ y_0 &= g + \tau\hat{g}, \end{aligned} \tag{2}$$

Where ζ and g are given. However, the terms $\lambda\hat{\zeta}$ and $\tau\hat{g}$ are unknown functions with λ, τ are a small reals parameters [12].

We aim at choosing a control that would perform optimally in an averaged sense, i.e., so that, rather than controlling specific realisations of the ad joint state, the average with respect to is controlled. This allows building a control independent of the parameter and dene the average sentinel to obtain a good estimation of the source term which called pollution term independly of the initial condition called missing data [13].

The notion of sentinel permits to distinguish and to analyse two types of incomplete data, the pollution term at which we look for information independently of the missing term that we do not want to identify [14].

In this paper, we study this system with incomplete initial data; we use the average sentinel concept; which relies on the following three objects: some state equation, some observation function and some control function to be determined.

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Setting of the Problem

Let $y(\lambda,\tau,\theta)=y(x,t,\lambda,\tau,\theta)$ be the unique solution of the problem in eqn.(1). We denote by an observation which is a measure of the concentration of the pollution taken on a non-empty open subset w at the interval time $(0,T)$.

$$\int_0^1 y(x,t,\lambda,\tau,\theta)d\theta = y_{obs}(x,t),\ for\ all(x,t)\in\omega\times(0,T), \tag{3}$$

Let h be some function in $L^2(w\times(0,T))$, for any control function $u\in L^2(w\times 0,T)$ we introduce the functional $S_m(\lambda,\tau)$ as follows:

$$S_m(\lambda,\tau)=\int_0^T\int_\Omega (h+u)\mathcal{X}_w\int_0^1 y(x,t,\lambda,\tau,\theta)d\theta dxdt. \tag{4}$$

where $\mathcal{X}_w$ is the characteristic functions for the open set w, such that:

$\mathcal{X}_w: L^2(\Omega)\to L^2(w)$.

Definition

Let S_m be a real function in equation in eqn. (4) depending only on the parameters λ and τ: S_m is said an average sentinel defined by h if the following conditions are satisfied:

$$\left.\frac{\partial S_m}{\partial_\tau}(\lambda,\tau)\right|_{\lambda=0,\tau=0}=0, \tag{5}$$

and there exists a average control $u\in L^2(w\times(0,T))$ such that:

$$\|u\|_{L^2(w\times(0,T))}=\min_{\upsilon\in U}\|\upsilon\|, \tag{6}$$

$$where\ U=\left\{\upsilon\in L^2(w\times(0,T)),\ such\ that\ \left.\frac{\partial S_m}{\partial_\tau}(\lambda,\tau)\right|_{\lambda=0,\tau=0}=0\right\}$$

Average Controllability Problem

We consider the function y_θ^0 which solve the problem in eqn. (1) for λ=τ=0:

$$\begin{cases}\dfrac{\partial y_\theta^0}{dt}(x,t)+div(a,(x,\theta)\nabla y_\theta^0(x,t)=\varsigma & in\ Q,\\ y_\theta^0=0 & on\ \Sigma,\\ y_\theta^0(x,0)=g(x) & in\ \Omega.\end{cases} \tag{7}$$

We denoted by

$$y_m(x,t,\lambda,\tau)=\int_0^1 y(x,t,\lambda,\tau,\theta)d\theta.$$

We consider the functions y_θ^τ and y_m^τ defined by

$$y_\theta^\tau=\left.\frac{\partial y_\theta}{\partial\tau}(\lambda,\tau)\right|_{\lambda=0,\tau=0},$$

and

$$y_m^\tau=\left.\frac{\partial y_m}{\partial\tau}(\lambda,\tau)\right|_{\lambda=0,\tau=0},$$

where y_θ^τ is the unique solution of the problem

$$\begin{cases}\dfrac{\partial y_\theta^\tau}{dt}(x,t)+div(a,(x,\theta)\nabla y_\theta^\tau(x,t)=\varsigma\ in\ Q,\\ y_\theta^\tau=0 & on\ \Sigma,\\ y_\theta^\tau(x,0)=\hat{g}(x) & in\ \Omega.\end{cases} \tag{8}$$

Remark

The condition in eqn. (5) holds if and only if:

$$\int_0^T\int_\Omega (h+u)\mathcal{X}_w y_\theta^\tau(x,t)dxdt=0 \tag{9}$$

In order to realize equation in eqn. (9), we introduce the classical adjoint system in eqn. (8):

$$\begin{cases}-\dfrac{\partial q_\theta(x,t)}{\partial t}+div(a(x,\theta)\nabla q_\theta(x,t))=(h+u)\mathcal{X}_w\ in\ Q,\\ q_\theta(x,t)=0 & on\ \Sigma,\\ q_\theta(x,T)=0 & in\ \Omega.\end{cases} \tag{10}$$

with $q_\theta(x,t)=q(x,t,\theta)$.

Theorem

Let q_θ the solution to the backward problem in eqn. (10), then the existence of an average sentinel insensitive to the missing data is equivalent to the average null-controllability problem

$$q_m(x,0)=\int_0^1 q_\theta(x,0)d\theta=0 \tag{11}$$

Proof

Multiplying the first equation in eqn. (10) by y_θ^τ the solution in the eqn. (8), and integrating by parts over Q × (0,1) we find

$$-\int_\Omega\int_0^1 q_\theta(x,0)y_\theta^\tau(0)d\theta dx+\int_\Omega\int_0^1 q_\theta(x,T)y_\theta^\tau(T)d\theta dx$$
$$=\int_0^1\int_0^T\int_\Omega (h+u)\mathcal{X}_w\ y_\theta^\tau(x,t)dxdtd\theta,$$

Then

$$\int_\Omega\int_0^1 q(x,0,\theta)\hat{g}(x)dx-\int_0^T\int_\Omega (h+u)\mathcal{X}_w\int_0^1 y_\theta^\tau(x,t)d\theta dxdt=0$$

Since $\hat{g}(x)$ is independent of θ, then if (9) is verified we will have

$$\int_0^1 q(x,0,\theta)d\theta=0$$

Thus, condition in eqn. (4) holds if and only if we have in eqn. (11) which is a average null controllability problem.

Characterization of Optimal Control

Theorem

Solving a problem of the average null-controllability is equivalent to finding a control u so that the solution of eqn. (10) satisfies eqn. (11) such that

$$u(x,t)=-\int_0^1\rho_\theta(x,t)\mathcal{X}_w d\theta \tag{12}$$

where ρ_θ is the solution of in eqn. (16).

Proof

To satisfy in eqn. (10), we separate the two component of in eqn. (10)

$q(x,t,\theta)=q_1(x,t,\theta)+q_2(x,t,\theta)$

we denote

$$\begin{cases}q_{1,\theta}(x,t)=q_1(x,t,\theta),\\ q_{2,\theta}(x,t)=q_2(x,t,\theta),\end{cases}$$

which are the solutions of the following system

$$\begin{cases} -\dfrac{\partial q_1,\theta(x,t)}{\partial t} + div\big(a(x,\theta)\nabla q_{1,\theta}(x,t)\big) = h_{\mathcal{X}_w} \ in\ Q \\ q_{1,\theta} = 0 \qquad on\ \Sigma \\ q_{1,\theta}(x,T) = 0 \qquad in\ \Omega \end{cases} \tag{13}$$

And

$$\begin{cases} -\dfrac{\partial q_2,\theta(x,t)}{\partial t} + div\big(a(x,\theta)\nabla q_{2,\theta}(x,t)\big) = u_{\mathcal{X}_w} \ in\ \ Q \\ q_{2,\theta} = 0 \qquad on\ \ \Sigma \\ q_{2,\theta}(x,T) = 0 \qquad in\ \ \Omega \end{cases} \tag{14}$$

so, we have a control u, such that

$$\int_0^1 q_{2,\theta}(x,t)\,d\theta = -\int_0^1 q_{1,\theta}(x,t)\,d\theta \tag{15}$$

Of course, in the present situation, the solution of the adjoint system in eqn. (16) of eqn.(14) depends also on the parameter θ :

$$\begin{cases} \dfrac{\partial \rho_\theta(x,t)}{\partial t} + div\big(a(x,\theta)\nabla \rho_\theta(x,t)\big) = 0 \ \ in\ \ Q \\ \rho_\theta = 0 \qquad on\ \ \Sigma \\ \rho_\theta(x,0) = \rho_0(x) \qquad in\ \ \Omega \end{cases} \tag{16}$$

with $\rho_0(x)$ is a unknown term independent of θ.

We want find ρ_0 such that the averaged control u is given by:

$$u\big|_w(x,t) = -\int_0^1 \rho_\theta(x,t)\mathcal{X}_w\,d\theta \tag{17}$$

We assume that $q_{1,\theta}$ is independant of θ at time zero.

Then using eqn.(16) in (13), we find:

$$\begin{cases} -\dfrac{\partial q_{2,\theta}(x,t)}{\partial t} + div\big(a(x,\theta)\nabla q_{2,\theta}(x,t)\big) = -\int_0^1 \rho_\theta(x,t)\mathcal{X}_w\,d\theta \ \ in\ Q\ on\,\Sigma\, in\ \Omega \\ q_{2,\theta} = 0 \\ q_{2,\theta}(x,T) = 0 \end{cases} \tag{18}$$

Then, for given $\rho_0(x)$, the first equation in (16) have unique solution. To find ρ_0, such that the solution in eqn. (16) satisfied in eqn. (19)

$$\int_0^1 q_{2,\theta}(x,0)\,d\theta + \int_0^1 q_{1,\theta}(x,0)\,d\theta = 0 \tag{19}$$

we define an linear operator Λ by

$$\Lambda\big(\rho_0(x)\big) = -\int_0^1 q_{2,\theta}(x,0)\,d\theta$$

We must therefore solve in a suitable functional space F, the equation

$$\Lambda\big(\rho_0(x)\big) = -\int_0^1 q_{1,\theta}(x,0)\,d\theta \tag{20}$$

For realize this , we multiply in eqn. (18) by $\hat{\rho}$ where $\hat{\rho}_0$ is the solution in eqn. (16) correspondent to $\hat{\rho}_0$ who is independent of θ and if we integrated by part over $(0,T)$, we obtain

$$-\big\langle \hat{\rho}_\theta(T), q_{2,\theta}(T)\big\rangle_{L^2(\Omega)} + \big\langle \hat{\rho}_\theta(0), q_{2,\theta}(0)\big\rangle_{L^2(\Omega)} = \Big\langle \hat{\rho}_0, -\int_0^1 \rho_\theta \mathcal{X}_w\,d\theta\Big\rangle_{L^2(\Omega)}$$

if we integrate for θ over (0,1), we will have

$$\int_0^1 \big\langle \hat{\rho}_0, q_{2,\theta}(0)\big\rangle\,d\theta = \int_0^T \int_w \int_0^1 \Big\langle \hat{\rho}_\theta \int_0^1 \rho_\theta \mathcal{X}_w\,d\theta\Big\rangle\,d\theta\,dx\,dt,$$

we denote $\Lambda\hat{\rho}_0 = -\int_0^1 q_{2,\theta}(0)\,d\theta$ then

$$\big\langle \hat{\rho}_0, \Lambda\hat{\rho}_0\big\rangle = \int_0^T \int_w \int_0^1 \Big\langle \hat{\rho}_0, \int_0^1 \rho_\theta \mathcal{X}_w\,d\theta\Big\rangle\,d\theta$$

If $\hat{\rho}_0$ ρ_0 then

$$\big\langle \rho_0, \Lambda\rho_0\big\rangle = \int_0^T \int_w \Big|\int_0^1 \rho_\theta\,d\theta\Big|^2\,dx\,dt,$$

We then introduce, in a natural way, the norm defined by

$$\|\rho_0\|_F = \int_0^T \int_w \Big(\int_0^1 \rho(x,t,\theta)\,d\theta\Big)^2\,dx\,dt, \tag{21}$$

by the result of Mizohata [9]

$$\|\rho_0\|_F = 0 \Rightarrow \rho = 0 \ on\ \ w \times (0,T)$$

The space $L^2(\Omega)$ is not complete for the norm in eqn.(21), then we introduce F as a Hilbert space completed of $L^2(\Omega)$, we note F' the dual space of F, then the linear operator Λ

$$\Lambda : F \mapsto F' \tag{22}$$

is an isomorphism, hence the result.

Identification of the Pollution Term

To show how the average sentinel permits to estimate the pollution term, we consider y_{obs} be the measured state of the system on the observatory w during the interval $[0,T]$, then the measured sentinel associate to y_{obs} is given by:

$$S_{obs}(\lambda,\tau) = \int_0^1 \int\int_{\Omega\times]0,T[} \big(h_{\mathcal{X}_w} + u_{\mathcal{X}_w}\big) y_{obs}(x,t,\lambda,\tau)\,dx\,dt\,d\theta$$

Theorem

The pollution term is identified as follows

$$\int_0^1 \int_\Omega q(h)\hat{\zeta}\,d\Omega = S_{obs}(\lambda,\tau) - S(0,0)$$

Proof

We have

$$S_{obs}(\lambda,\tau) = S(0,0) + \lambda \frac{\partial S}{\partial \lambda}\Big|_{\lambda=0,\tau=0} + O(\lambda,\tau)$$

With

$$\frac{\partial S}{\partial \lambda}(\lambda,\tau) = \int_0^1 \int_\Omega (h+w)\mathcal{X}_w y_\lambda\,dx\,dt$$

And

$$\lambda \frac{\partial S}{\partial \lambda}(\lambda,\tau)\Big|_{\lambda=0,\tau=0} = S_{obs}(\lambda,\tau) - S(0,0),$$

Hence

$$\int_0^1 \int_\Omega q(h)\hat{\zeta}\,d\Omega = S_{obs}(\lambda,\tau) - S(0,0)$$

where y_λ is the solution of the following system

$$\begin{cases} \dfrac{\partial y_\theta^\lambda(x,t)}{\partial t} - div\big(a(x,\theta)\nabla y_\theta^\lambda(x,t)\big) = \hat{\zeta}, \ in\ \ Q \\ y_\theta^\lambda = 0 \qquad on\ \ \Sigma \\ y_\theta^\lambda(x,0) = 0 \qquad in\ \ \Omega \end{cases}$$

Conclusion

In this work, we have introduced the problem of identification of the pollution term in a heat system when the dynamics of the state is governed by a parameterized operator; we have introduced a notion of average sentinel to obtain information about the pollution term.

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