Analytic Solutions of the Madelung Equation
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Abstract
We present analytic self-similar solutions for the one, two and three dimensional Madelung hydrodynamical equation for a free particle. There is a direct connection between the zeros of the Madelung fluid density and the magnitude of the quantum potential.

Keywords: Quantum mechanics; Partial differential equations; Madelung equation; Planck’s constant

Introduction
Finding classical physical basements of quantum mechanics is a great challenge since the advent of the theory. Madelung was one among the firsts who gave one explanation, this was the hydrodynamical foundation of the Schrödinger equation [1,2]. His exponential transformation simply indicates that one can model quantum statistics hydrodynamically. Later, it became clear that the Madelung Ansatz is just the complex Cole-Hopf transformation [3,4] which is sometimes used to linearize non-linear partial differential equations(PDEs).

The transformed equation has an attractive feature that the Planck’s constant appears only once, as the coefficient of the quantum potential or pressure. Thus, the fluid dynamicist can gather experience of its effects by translating some of the elementary situations of the quantum theory into their corresponding fluid mechanical statements and vice versa.

The quantum potential also appears in the de Broglie-Bohm pilot wave theory [5,6] (in other context) which is a non-mainstream attempt to interpret quantum mechanics as a deterministic non-local theory. In the case of $\hbar \to 0$ the Euler equation goes over the Hamilton-Jacobi equation.

As an interesting peculiarity Walldstrom showed with mathematical means that the initial-value problem of the Madelung equation is not well-defined and additional conditions are needed [7].

Nowadays, hydrodynamical description of quantum mechanical systems is a popular technical tool in numerical simulations. Review articles on quantum trajectories can be found in a booklet edited by Huges in 2011 [8].

From general concepts as the second law of thermodynamics a weakly non-local extension of ideal fluid dynamics can be derived which leads to the Schrödinger-Madelung equation as well [9].

In our following study we investigate the Madelung equation with the self-similar Ansatz and present analytic solutions with discussion.

This way of investigation is a powerful method to study the global properties of the solutions of various non-linear PDEs [10]. Self-similar Ansatz describes the intermediate asymptotic of a problem: it is held when the precise initial conditions are no longer important, but before the system has reached its final steady state. This is much simpler than the full solutions and so easier to understand and study in different regions of parameter space. A final reason for studying them is that they are solutions of a system of ordinary differential equations(ODEs) and hence do not suffer the extra inherent numerical problems of the full PDEs. In some cases self-similar solutions helps to understand diffusion-like properties or the existence of compact supports of the solution.

In the last years we successfully applied the multi-dimensional generalization of the self-similar Ansatz to numerous viscous fluid equations [11,12] ending up with a book chapter in ref. [13].

To our knowledge there are no direct analytic solutions available for the Madelung equation. Baumann and Nonnenmacher [14] exhaustively investigated the Madelung equation with Lie transformations and presented numerous ODEs, however non exact and explicit solutions are presented in a transparent way. Additional numerous studies exist where the non-linear Schrödinger equation is investigated with the Madelung Ansatz ending up with solitary wave solutions, [15] however that is not the field of our present interest.

Theory and Results
Following the original paper of Madelung [2] the time-dependent Schrödinger equation reads:

$$\Delta \Psi = \frac{8\pi^2 m}{\hbar} U \Psi - i \frac{4\pi m}{\hbar} \frac{\partial \Psi}{\partial t} = 0,$$

where $\Psi, U, m, \hbar$ are the wave function, potential, mass and Planck’s constant, respectively. Taking the following Ansatz $\Psi = \sqrt{\rho \exp(iS/\hbar)}$ where $\rho(x,t)$ and $S(x,t)$ are time and space dependent functions. Substituting this trial function into eqn. (1) going through the derivations the real and the complex part gives us the following continuity and Euler equations with the form of:

$$\rho \frac{\partial \Psi}{\partial t} = 0,$$

$$\nabla \cdot \rho \nabla \Psi = \frac{\hbar^2}{8\pi^2 m^2} \left( \frac{\Delta \rho}{\rho} + \frac{1}{m} \right) \nabla U,$$

(2)

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where the following substitution has to be made \( v = \frac{\hbar}{m} \nabla \mathcal{S} \). The \( \rho \) is the density of the investigated fluid and \( v \) is the velocity field. Madelung also showed that this is a rotation-free flow. The transformed equations has an attractive feature that the Planck’s constant appears only once, at the coefficient of the quantum potential or pressure, which is the first term of the right hand side of the second equation. Note, that these are most general vectorial equation for the velocity field \( v \) which means that one, two or three dimensional motions can be investigated as well. In the following we will consider the two dimensional flow motion \( v = (u, v) \) in Cartesian coordinates without any external field \( U = 0 \). The functional form of the three and one dimensional solutions will be mentioned briefly as well.

We are looking for the solution of eqn. (2) with self-similar Ansatz which is well-known from ref. [10]

\[
\rho(x,y,t)=\exp(i(x\cdot\mathbf{f}(\eta)+y\cdot\mathbf{g}(\eta)+\mathbf{h}(\eta)c)), \quad \mathbf{u}(x,y,t)=\exp(i(x\cdot\mathbf{f}(\eta)+y\cdot\mathbf{g}(\eta)+\mathbf{h}(\eta)c)), \quad \mathbf{v}(x,y,t)=\exp(i(x\cdot\mathbf{f}(\eta)+y\cdot\mathbf{g}(\eta)+\mathbf{h}(\eta)c)),
\]

where \( f, g, h \) are the shape functions of the density and the velocity field, respectively. The similarity exponents \( a, b, d, e \) are of primary physical importance since \( a, b, d, e \) represents the damping of the magnitude of the shape function while \( \beta \) represents the spreading. More about the general properties of the Ansatz can be found in our former papers [11,12]. Except some pathological cases all positive similarity exponents mean physically relevant dispersive solutions with regard to zero. (A non-zero \( \beta \) remains an additive constant in the final ODE as a conserved quantity and the parallel solution for the velocity fields \( v = (u, v) \) in the original Schrödinger equation would generate an extra fifth term in eqn. (5) like, \( f'(\eta)s_\eta^2 \). Unfortunately, no other analytic closed form solutions can be found for such terms (Figures 1 and 2).

The solution of (5) can be expressed with the help of the Bessel functions of the first and second kind [16] and has the following form of:

\[
f(\eta) = \frac{2}{\eta m^2} \left[ J_0 \left( \frac{\sqrt{m f(\eta)} - \frac{\sqrt{m g(\eta)}}{8}}{8} \right) + Y_0 \left( \frac{\sqrt{m f(\eta)} - \frac{\sqrt{m g(\eta)}}{8}}{8} \right) \right] \]

where \( c_1 \) and \( c_2 \) are the usual integration constants. The correctness of this solutions can be easily verified via back substitution into the original ODE.

To imagine the complexity of these solutions Figures 1 and 2 present \( f(\eta) \) for various \( m, c_1 \) and \( c_2 \) values. It has a strong decay with a stronger and stronger oscillation at large arguments. The function is positive for all values of the argument, (which is physical for a fluid density), but such oscillatory profiles are completely unknown in regular fluid mechanics [13]. The most interesting feature is the infinite number of zero values which cannot be interpreted physically for a classical real fluid.

The presented form of the shape function cannot be simplified

**Figure 1:** The solution of eqn. (6) \((c_1, c_2 = 1)\) the yellow curve is for \( m = 1 \) and blue curve is for \( m = 0.5 \).

**Figure 2:** The solution of eqn. (6) \((m = 1)\) the blue curve is for \( c_1, c_2 = 1 \), the yellow for \( c_1 = 1/4 \) and \( c_2 = 1/2 \) and the green for \( c_1 = -1/2 \) and \( c_2 = 1/2 \).
further, only $Y$s can be expressed with the help of $J$s [16]. Applying the recurrence formulas the orders of the Bessel functions can be shifted as well. With the parabolic cylinder functions the Bessel functions with 3/4 and 1/4 orders can be expressed, too. Unfortunately, all these formulas and manipulations are completely useless now. However, the denominator of (6) can be simplified to a power function, therefore $f(\eta)$ can be written in a much simpler form:

$$f(\eta) = \frac{\pi \eta c_2}{64} \left( \frac{\sum m^2}{8} - c_2 Y_{\frac{1}{4}} \frac{\sum m^2}{8} \right).$$  (7)

Both $I$ and $Y$ Bessel functions with linear argument form an orthonormal set, therefore integrable over the $L^1$ space. In our case, the integral $\int f(\eta)d\eta$ is logarithmically divergent, unfortunately it cannot be interpreted as a physical density function of the original Schrödinger equation. However, $\sqrt{f}$ could be interpreted as the fluid mechanical analogue of the real part of the wave function of the free quantum mechanical particle which can be described with a Gaussian wave packet. To obtain the complete original wave function, the imaginary part has to be evaluated as well. It is trivial from $\eta = \frac{x+y}{\sqrt{2}} = 2(x+\eta)$ that,

$$S = \frac{m}{h} \int_0^\infty V dx dy = \frac{m (x+y)^2}{4t}. \quad \text{(8)}$$

Now,

$$\Psi_{x,y,t} = \frac{\sqrt{x+y}}{8} \left( c_1 J_{\frac{1}{4}} \frac{\sum m(x+y)^2}{8t} - c_2 Y_{\frac{1}{4}} \frac{\sum m(x+y)^2}{8t} \right). \quad \text{(9)}$$

Figure 3 shows the projection of the real part wave function to the x,t sub-space. At small times the oscillations are clear to see, however at larger times the strong damping is evident.

For arbitrary quantum systems, the wave function can be evaluated according to the Schrödinger equation, however we never know directly how large is the quantum contribution to the classical one. Now, it is possible for a free particle to get this contribution. (The Schrödinger equation gives the Gaussian wave function for a freely propagating particle). With the Madelung Ansatz we got the classical fluid dynamical analogue of the motion with the physical parameters $\rho(x,y,t), u(x,y,t)$ which can be calculated analytically via the self-similar Ansatz thereafter original wave function $\Psi(x,y,t)$ of the quantum problem can be evaluated as well. The magnitude of the quantum potential $Q$ directly informs us where quantum effects are relevant. This can be evaluated from the classical density of the Madelung eqn. (2) via:

$$Q = \frac{\hbar^2}{8\pi^2 m^2} V \left( \Delta \sqrt{\rho} \right) = \frac{\hbar^2}{8\pi^2 m^2} \frac{\partial}{\partial \eta} \left( c_1 8 \frac{m^2}{4\sqrt{2}} - c_2 8 \frac{m^2}{4\sqrt{2}} \right). \quad \text{(10)}$$

Figure 4 shows the shape function of quantum potential $Q$ comparing to the shape function of the density $f$. Note, that where the density has zeros the quantum potential is singular. Such singular potentials might appear in quantum mechanics, however the corresponding wave function should compensate the effect. This question is analyzed in the book of Holland [17] for various other quantum systems.

**Conclusion**

After reviewing the historical development and interpretation of the Madelung equation we introduced the self-similar Ansatz which is a not-so-well-known but powerful tool to investigate nonlinear PDEs. The free particle Madelung equation was investigated in two dimensions with this method, (the one and three dimensional solutions were mentioned as well.) We found analytic solution for the fluid density, velocity field and the original wave function. All can be eliminated with the help of the Bessel functions. The classical fluid density has interesting properties, oscillates and has infinite number of zeros which is quite unusual and has not yet been seen in such analysis.

**References**