Journal of Generalized Lie Theory and Applications
Barna et al., J Generalized Lie Theory Appl 2017, 11:2
DOI: 10.4172/1736-4337.1000271
Research Article
Open Access

Analytic Solutions of the Madelung Equation

Barna IF[1,2]*, Pocsai MA[1,3] and Mátyás L[4]

[1]*Wigner Research Center of the Hungarian Academy of Sciences Konkoly-Thege út 29-33, 1121 Budapest, Hungary*
[2]*ELI-HU Nonprofit Kft, Dugonics Tér 13, H-6720 Szeged, Hungary*
[3]*University of Pécs, Institute of Physics, Ifjúság útja 6 H-7624 Pécs, Hungary*
[4]*Sapientia University, Department of Bioengineering, Libertatii sq. 1, 530104 Miercurea Ciuc, Romania*

Abstract

We present analytic self-similar solutions for the one, two and three dimensional Madelung hydrodynamical equation for a free particle. There is a direct connection between the zeros of the Madelung fluid density and the magnitude of the quantum potential.

Keywords: Quantum mechanics; Partial differential equations; Madelung equation; Planck's constant

Introduction

Finding classical physical basements of quantum mechanics is a great challenge since the advent of the theory. Madelung was one among the firsts who gave one explanation, this was the hydrodynamical foundation of the Schrödinger equation [1,2]. His exponential transformation simply indicates that one can model quantum statistics hydrodynamically. Later, it became clear that the Madelung Anzatz is just the complex Cole-Hopf transformation [3,4] which is sometimes used to linearize non-linear partial differential equations(PDEs).

The transformed equation has an attractive feature that the Planck's constant appears only once, as the coefficient of the quantum potential or pressure. Thus, the fluid dynamicist can gather experience of its effects by translating some of the elementary situations of the quantum theory into their corresponding fluid mechanical statements and vice versa.

The quantum potential also appears in the de Broglie-Bohm pilot wave theory [5,6] (in other context) which is a non-mainstream attempt to interpret quantum mechanics as a deterministic non-local theory. In the case of $\hbar \to 0$ the Euler equation goes over the Hamilton-Jacobi equation.

As an interesting peculiarity Wallstrom showed with mathematical means that the initial-value problem of the Madelung equation is not well-defined and additional conditions are needed [7].

Nowadays, hydrodynamical description of quantum mechanical systems is a popular technical tool in numerical simulations. Review articles on quantum trajectories can be found in a booklet edited by Huges in 2011 [8].

From general concepts as the second law of thermodynamics a weakly non-local extension of ideal fluid dynamics can be derived which leads to the Schrödinger-Madelung equation as well [9].

In our following study we investigate the Madelung equation with the self-similar Ansatz and present analytic solutions with discussion.

This way of investigation is a powerful method to study the global properties of the solutions of various non-linear PDEs [10]. Self-similar Ansatz describes the intermediate asymptotic of a problem: it is hold when the precise initial conditions are no longer important, but before the system has reached its final steady state. This is much simpler than the full solutions and so easier to understand and study in different regions of parameter space. A final reason for studying them is that they are solutions of a system of ordinary differential equations(ODEs) and hence do not suffer the extra inherent numerical problems of the full PDEs. In some cases self-similar solutions helps to understand diffusion-like properties or the existence of compact supports of the solution.

In the last years we successfully applied the multi-dimensional generalization of the self-similar Ansatz to numerous viscous fluid equations [11,12] ending up with a book chapter in ref. [13].

To our knowledge there are no direct analytic solutions available for the Madelung equation. Baumann and Nonnenmacher [14] exhaustively investigated the Madelung equation with Lie transformations and presented numerous ODEs, however non exact and explicit solutions are presented in a transparent way. Additional numerous studies exist where the non-linear Schrödinger equation is investigated with the Madelung Ansatz ending up with solitary wave solutions, [15] however that is not the field of our present interest.

Theory and Results

Following the original paper of Madelung [2] the time-dependent Schrödinger equation reads:

$$\Delta\Psi - \frac{8\pi^2 m}{h^2} U\Psi - i\frac{4\pi m}{h}\frac{\partial \Psi}{\partial t} = 0, \tag{1}$$

where Ψ, U, m, h are the wave function, potential, mass and Planck's constant, respectively. Taking the following Ansatz $\Psi = \sqrt{\rho} e^{iS}$ where $\rho(x,t)$ and $S(x,t)$ are time and space dependent functions. Substituting this trial function into eqn. (1) going through the derivations the real and the complex part gives us the following continuity and Euler equations with the form of:

$$\rho_t + \nabla\cdot(\rho v) = 0,$$

$$v_t + v\cdot\nabla v = \frac{h^2}{8\pi^2 m^2}\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \frac{1}{m}\nabla U, \tag{2}$$

***Corresponding author:** Barna IF, Wigner Research Center of the Hungarian Academy of Sciences Konkoly,Thege út 29-33, 1121 Budapest, Hungary, Tel: +36-1-392-2222/3504; Fax: +36-1-392-2598; E-mail: barna.imre@wigner.mta.hu

Received June 16, 2017; **Accepted** July 05, 2017; **Published** July 10, 2017

Citation: Barna IF, Pocsai MA, Mátyás L (2017) Analytic Solutions of the Madelung Equation. J Generalized Lie Theory Appl 11: 271. doi: 10.4172/1736-4337.1000271

where the following substitution has to be made $v = \frac{\hbar}{m}\nabla S$. The ρ is the density of the investigated fluid and v is the velocity field. Madelung also showed that this is a rotation-free flow. The transformed equations has an attractive feature that the Planck's constant appears only once, at the coefficient of the quantum potential or pressure, which is the first term of the right hand side of the second equation. Note, that these are most general vectorial equation for the velocity field v which means that one, two or three dimensional motions can be investigated as well. In the following we will consider the two dimensional flow motion $v=(u,v)$ in Cartesian coordinates without any external field $U=0$. The functional form of the three and one dimensional solutions will be mentioned briefly as well.

We are looking for the solution of eqn. (2) with self-similar Ansatz which is well-known from ref. [10]

$$\rho(x,y,t) = t^{-\alpha} f\left(\frac{x+y}{t^{\beta}}\right) := t^{-\alpha} f(\eta), \quad u(x,y,t) = t^{-\delta} g(\eta), \quad v(x,y,t) = t^{-\varepsilon} h(\eta), \quad (3)$$

where f,g and h are the shape functions of the density and the velocity field, respectively. The similarity exponents $\alpha,\beta,\delta,\varepsilon$ are of primary physical importance since $\alpha,\delta,\varepsilon$ represents the damping of the magnitude of the shape function while β represents the spreading. More about the general properties of the Ansatz can be found in our former papers [11,12]. Except some pathological cases all positive similarity exponents mean physically relevant dispersive solutions with decaying features at $x,y,t \to \infty$. Substituting the Ansatz (3) into (2) and going through some algebra manipulation the next ODE system can be expressed for the shape functions.

$$-\frac{1}{2}f - \frac{1}{2}f'\eta + f'g + fg' + f'h + fh' = 0,$$
$$-\frac{1}{2}g - \frac{1}{2}g'\eta + gg' + hg' = \frac{\hbar^2}{2m^2}\left(\frac{f'^3}{2f^3} - \frac{f'f''}{f^2} + \frac{f'''}{2f}\right),$$
$$-\frac{1}{2}h - \frac{1}{2}h'\eta + gh' + hh' = \frac{\hbar^2}{2m^2}\left(\frac{f'^3}{2f^3} - \frac{f'f''}{f^2} + \frac{f'''}{2f}\right). \quad (4)$$

The first continuity equation can be integrated giving us the mass as a conserved quantity and the parallel solution for the velocity fields $\eta=2(g+h)+c_0$ where c_0 is the usual integration constant, which we set to zero. (A non-zero c_0 remains an additive constant in the final ODE (5) as well.) It is interesting, and unusual (in our practice) that even the Euler equation can be integrated once giving us the conservation of momenta. For classical fluids this is not the case. After some additional algebraic steps a decoupled ODE can be derived for the shape function of the density.

$$2f''f - (f')^2 + \frac{m^2\eta^2 f^2}{2\hbar^2} = 0. \quad (5)$$

All the similarity exponents have the fixed value of $+\frac{1}{2}$ which is usual for regular heat conduction, diffusion or for Navier-Stokes equations [13]. Note, that the two remaining free parameters are the mass of the particle m and $\hbar$ which is the Planck's constant divided by 2π. For a better transparency we fix $\hbar = 1$. This is consistent with experience of regular quantum mechanics that quantum features are relevant at small particle masses.

At this point it is worth to mention, that the obtained ODE for the density shape function is very similar to eqn. (5) for different space dimensions, the only difference is a constant in the last term. For one, two or three dimensions the denominator has a factor of 1,2 or 3, respectively.

An additional space dependent potential U (like a dipole, or harmonic oscillator interaction) in the original Schrödinger equation would generate an extra fifth term in eqn. (5) like, $f(\eta),\eta^2$. Unfortunately, no other analytic closed form solutions can be found for such terms (Figures 1 and 2).

The solution of (5) can be expressed with the help of the Bessel functions of the first and second kind [16] and has the following form of:

$$f(\eta) = \frac{2\left(-J_{\frac{1}{4}}\left[\frac{\sqrt{2}m\eta^2}{8}\right]\cdot c_1 + Y_{\frac{1}{4}}\left[\frac{\sqrt{2}m\eta^2}{8}\right]\cdot c_2\right)^2}{\eta^3 m^2\left(J_{-\frac{3}{4}}\left[\frac{\sqrt{2}m\eta^2}{8}\right]\cdot Y_{\frac{1}{4}}\left[\frac{\sqrt{2}m\eta^2}{8}\right] - J_{\frac{1}{4}}\left[\frac{\sqrt{2}m\eta^2}{8}\right]\cdot Y_{-\frac{3}{4}}\left[\frac{\sqrt{2}m\eta^2}{8}\right]\right)^2} \quad (6)$$

where c_1 and c_2 are the usual integration constants. The correctness of this solutions can be easily verified via back substitution into the original ODE.

To imagine the complexity of these solutions Figures 1 and 2 present $f(\eta)$ for various m, c_1 and c_2 values. It has a strong decay with a stronger and stronger oscillation at large arguments. The function is positive for all values of the argument, (which is physical for a fluid density), but such oscillatory profiles are completely unknown in regular fluid mechanics [13]. The most interesting feature is the infinite number of zero values which cannot be interpreted physically for a classical real fluid.

The presented form of the shape function cannot be simplified

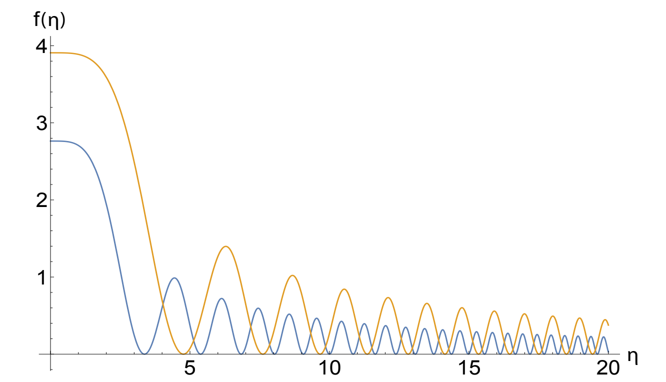

Figure 1: The solution of eqn. (6) ($c_1=c_2=1$) the yellow curve is for $m=1$ and blue curve is for $m=0.5$.

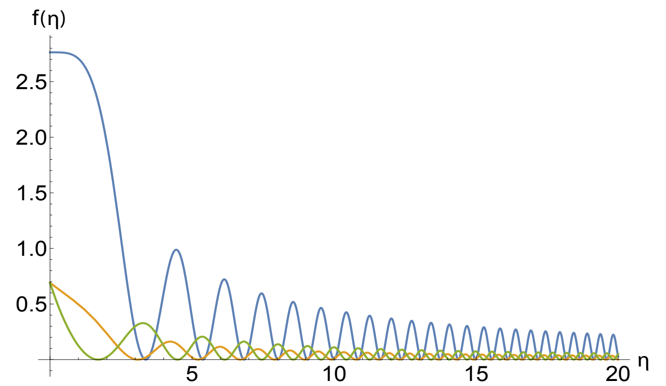

Figure 2: The solution of eqn. (6) ($m=1$) the blue curve is for $c_1=c_2=1$, the yellow for $c_1=1/4$ and $c_2=1/2$ and the green for $c_1=-1/2$ and $c_2=1/2$.

Citation: Barna IF, Pocsai MA, Mátyás L (2017) Analytic Solutions of the Madelung Equation. J Generalized Lie Theory Appl 11: 271. doi: 10.4172/1736-4337.1000271

further, only Y_νs can be expresses with the help of J_νs [16]. Applying the recurrence formulas the orders of the Bessel functions can be shifted as well. With the parabolic cylinder functions the Bessel functions with 3/4 and 1/4 orders can be expressed, too. Unfortunately, all these formulas and manipulations are completely useless now. However, the denominator of (6) can be simplified to a power function, therefore $f(\eta)$ can be written in a much simpler form:

$$f\eta = \frac{\pi\eta}{64} c_1 J_{1/4} \frac{\sqrt{2}m\eta^2}{8} - c_2 Y_{1/4} \frac{\sqrt{2}m\eta^2}{8}^{2}. \quad (7)$$

Both J_ν and Y_ν Bessel functions with linear argument form an orthonormal set, therefore integrable over the L^2 space. In our case, the integral $\int_0^\infty f(\eta)d\eta$ is logarithmically divergent, unfortunately it cannot be interpreted as a physical density function of the original Schrödinger equation. However, $\sqrt{f}$ could be interpreted as the fluid mechanical analogue of the real part of the wave function of the free quantum mechanical particle which can be descried with a Gaussian wave packet. To obtain the complete original wave function, the imaginary part has to be evaluated as well. It is trivial from $\eta = \frac{x+y}{t^{1/2}} = 2(g+h)$ that,

$$S = \frac{m}{\hbar}\int_{r_0}^{r_1} v dr = \frac{m}{\hbar}\frac{(x+y)^2}{4t}. \quad (8)$$

Now,

$$\Psi x, y, t = \frac{\sqrt{\pi}}{8}\frac{(x+y)^{1/2}}{t^{1/4}} c_1 J_{1/4} \frac{\sqrt{2}m(x+y)^2}{8t} - c_2 Y_{1/4} \frac{\sqrt{2}m(x+y)^2}{8t} e^{\frac{im}{\hbar}\frac{(x+y)^2}{4t^2}} \quad (9)$$

Figure 3 shows the projection of the real part wave function to the x,t sub-space. At small times the oscillations are clear to see, however at larger times the strong damping is evident.

For arbitrary quantum systems, the wave function can be evaluated according to the Schrödinger equation, however we never know directly how large is the quantum contribution to the classical one. Now, it is possible for a free particle to get this contribution. (The Schrödinger equation gives the Gaussian wave function for a freely propagating particle). With the Madelung Ansatz we got the classical fluid dynamical analogue of the motion with the physical parameters $\rho(x,y,t)$,$v(x,y.t)$ which can be calculated analytically via the self-similar Ansatz thereafter original wave function $\Psi(x,y,t)$ of the quantum problem can be evaluated as well. The magnitude of the quantum potential Q directly informs us where quantum effects are relevant. This can be evaluated from the classical density of the Madelung eqn. (2) via:

$$Q = \frac{h^2}{8\pi^2 m^2}\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \frac{h^2}{8\pi^2 m^2}\frac{\partial}{\partial\eta}\left(\frac{-\eta^2 m^2}{c_1 8 J_{\frac{1}{4}}[\frac{m\eta^2}{4\sqrt{2}}] - c_2 8 Y_{\frac{1}{4}}[\frac{m\eta^2}{4\sqrt{2}}]}\right). \quad (10)$$

Figure 4 shows the shape function of quantum potential Q() comparing to the shape function of the density f(). Note, that where the density has zeros the quantum potential is singular. Such singular potentials might appear in quantum mechanics, however the corresponding wave function should compensate the effect. This question is analyzed in the book of Holland [17] for various other quantum systems.

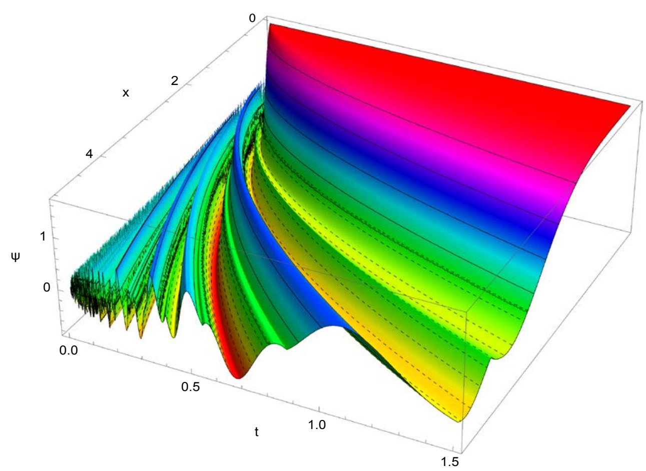

Figure 3: The projection of real part of the wave function $\Psi(x,t)$ from eqn. (9) for $m=1=c_1=c_2$.

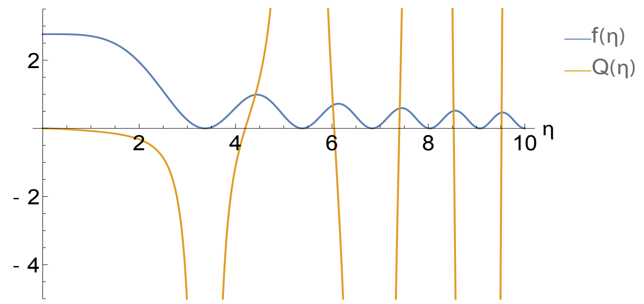

Figure 4: The shape function of the density $f(\eta)$ is the blue curve and the shape function of the quantum potential Q() is the yellow solid line. All the corresponding parameters are $c_1=c_2=m=1$.

Conclusion

After reviewing the historical development and interpretation of the Madelung equation we introduced the self-similar Ansatz which is a not-so-well-known but powerful tool to investigate non-linear PDEs. The free particle Madelung equation was investigated in two dimensions with this method, (the one and three dimensional solutions were mentioned as well.) We found analytic solution for the fluid density, velocity field and the original wave function. All can be eliminated with the help of the Bessel functions. The classical fluid density has interesting properties, oscillates and has infinite number of zeros which is quite unusual and has not yet been seen in such analysis.

References

1. Madelung E (1926) Eine anschauliche Deutung der Gleichung von Schrödinger. Naturwissenschaften 14: 1004.
2. Madelung E (1927) Zur Frage der Elektronenaffinität des Wasserstoffs. Z Phys 40: 331-332.
3. Hopf E (1950) Commun Pure Appl Math 3: 201.
4. Cole JD (1951) On a quasi-linear parabolic equation occurring in aerodynamics. Quart Appl Math 9: 225-336.
5. de Broglie L (1927) L'univers à cinq dimensions et la mécanique ondulatoire. Journ de Phys et la Rad 8: 65-73.
6. Bohm D (1952) Phys Rev 15: 166.
7. Wallstrom TC (1994) Phys Lett A 184: 229.
8. Hughes KH, Parlant G (2011) Quantum Trajectories (CCP6, Daresbury Laboratory, 2011).
9. Ván P, Fülöp T (2006) Weakly non-local fluid mechanics: the Schrödinger equation. Proc R Soc A, pp: 462-541.

10. Sedov LI (1993) Similarity and Dimensional Methods in Mechanics CRC Press.
11. Barna IF, Mátyás L (2014) Analytic solutions for the three-dimensional compressible Navier–Stokes equation. Fluid Dyn Res 46: 055508.
12. Barna IF, Mátyás L (2015) Chaos Solitons and Fractals 78: 249.
13. Campos D (2016) Handbook on Navier-Stokes Equations, Theory and Applied Analysis, Nova Publishers, New York.
14. Baumann G, Nonnenmacher TF (1987) Lie transformations, similarity reduction, and solutions for the nonlinear Madelung fluid equations with external potential. Journ Math Phys 28: 1250.
15. Grecu D, Alexandru T, Grecu AT, De Nicola S (2008) Solitary Waves in a Madelung Fluid Description of Derivative NLS Equations. Journal of Nonlinear Mathematical Physics 15: 209-219.
16. Olver FWJ, Lozier DW, Boisvert RF, Clark CW (2010 NIST Handbook of Mathematical Functions Hardback and CD-ROM. Cambridge University Press.
17. Holland PR (1995) The quantum theory of motion. Cambridge University Press.