

# Analysis of the Fractional Integrodifferentiability of Power Functions and Hypergeometric Representation

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## Abstract

In this work we show that it is possible to calculate the fractional integrals and derivatives of order (using the Riemann-Liouville formulation) of power functions  $(t^*)^\beta$  with  $\beta$  being any real value, so long as one pays attention to the proper choice of the lower and upper limits according to the original functions domain. We, therefore, obtain valid expressions that are described in terms of function series of the type  $(t^*)^{\pm \alpha+k}$  and we also show that they are related to the famous hypergeometric functions of the Mathematical-Physics.

**Keywords:** Integrodifferentiability; Mathematical-Physics; Riemann-Liouville

## Introduction

The non-integer order calculus, popularly known as fractional calculus (FC) was born in 1695. Only after nearly 250 years did the first event dedicated exclusively to the theme [1-3]. Today, after more than 40 years, FC has gained enormous visibility both from a theoretical point of view and in applications.

In the theoretical point of view we mention, in addition to the classical formulations of the derivative (Riemann-Liouville, Caputo and Grnwald-Letnikov, just to cite a few) [4] a recent one due to Caputo and Fabrizio [5] where the kernel is non-singular and whose properties were studied by Losada and Nieto [6] and as an application by Atangana [7] in the study of Fischers reaction-diffusion equation. As a generalization of these derivatives we cite the -Hilfer derivative [8] and the general Hilfer-Hadamard derivative [9]. On the other hand, also recent, there is a wide class of new derivatives that recover the classical (definition) derivative in the Newton/Leibniz sense and, even if fractional call, they are local derivatives [10].

As applications we can cite a wide range where the FC acts. Some of them are: Yang et al. [11] discuss anomalous discussion models with general fractional derivatives within the kernels of the extended Mittag-Leffler functions; Yang [12] propose a fractional derivative of constant and variable orders applied to anomalous relaxation models in heat-transfer problems; Atangana and Baleanu [13] introduce a new fractional derivative with non-local and non-singular kernel to discuss a particular heat transfer model; Yang et al. [14] present a new family of the local fractional partial differential equations; Yang et al. [15] discuss a LC-electric circuit modeled by a local fractional calculus; Gao and Yang [16] using local fractional Eulers method, discuss the steady heat-conduction problem; Gmez and Capelas de Oliveira [17] propose and discuss a nonlinear partial differential equation variational interaction method; Costa et al. [18] present a nonlinear fractional Harry Dym equation whose solution is written in terms of the Foxs H-function; Garrappa et al. [19] discuss models of dielectric relaxation based on completely monotone functions; Rosa and Oliveira [20] discuss some particular fractional differential equation involving dielectrics and discuss the complete monotonicity and Oliveira et al. [21] discuss analytic components for the hadronic total cross-section using Mellin transform.

When it comes to “Calculus and Analysis” concepts such as limits,

infinitesimals and continuity of functions are among the first that, at one step after another, leads to the definitions of the operations of differentiation and integration. Since these operations acts on a certain suitable class of functions, it is due to their simplicity and well understood behavior that in any introductory course to (Standard) Calculus the first class of functions to be “concretely” computed are the power functions, that is, functions of the type  $f(t)=(t-d)^\beta$ , with  $d, \beta \in \mathbb{R}$ .

It is therefore logical that in the theory of the so-called Fractional Calculus (FC) the rules for operating these type of functions must also be carefully analyzed and established, but there is more to it than this simple argument. Power functions defines a very important class of functions and there are many natural (and artificial) phenomena that behaves according to power law distributions (e.g., Newtonian laws of gravity and electrostatics, Keplers third law about the orbital period of planets, the square-cube law relating the ratio between surface area and volume of an object, the Stefan-Boltzman law describing the power radiated from a black body in terms of its temperature, etc...), particularly, there are an increasing number of works showing the interesting link between FC models and physical phenomena described in terms of power laws. Just to mention a few examples, it is presented a new fractional Laplacian time-space model to describe the frequency-dependent attenuation obeying empirical power law distribution [22]. It is studied the duality of Hamiltonian dynamics of a system of particles with power-like interactions with the solution of certain fractional differential equations [23]. It is presented a fractional generalization of the Kelvin-Voigt rheology in order to better simulate the power law stress-strain relation of some biological media [24]. The authors study the link between multiple relaxation models, power law attenuation and fractional wave equations, providing some physically based evidence for the use of FC in the modelling processes [25]. It is proposed a fractional model to describe a power

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law relation between pain transmission processes in the human body and analgesia measurements, and the lists may go on [26]. So all these works suggest that having a clear picture of the theoretical behavior and modus operandis of FC operators acting on power functions (and more generally to functions described in terms of power series such as analytic functions) may provide us with insights to several distinct power law related phenomena or, at the very least, a better and concise mathematical tool available to be used in the same way as already happens with the standard calculus.

In fact, in this work we are concerned exactly to this matter, because although the main literature provide some basic rules for fractionally operating power functions, we believe that there are still some fundamental aspects that are not fully cleared and understood and the matter still require more attention. To clarify this argument, we recall the following FC result: Let  $f(t)=(t-d)^\beta$ , with  $d, \beta \in \mathbb{R}$ . It is known that whenever  $\beta > -1$ , we can compute expressions for the Riemann-Liouville fractional integral (RLFI) and Riemann-Liouville fractional derivative (RLFD) of these power functions [27]:

$$[{}_d D_t^\alpha (x-d)^\beta](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-d)^{\beta-\alpha} \quad (1)$$

$$[{}_d D_t^\alpha (x-d)^\beta](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-d)^{\beta-\alpha} \quad (2)$$

for  $\text{Re}(\alpha) \geq 0$ .

The proofs for the expressions in eqn. (1) and eqn. (2) are very straightforward and can be found, e.g., they make use of the integral representation for the Euler's beta function [28].

$$B(\eta, \xi) = \int_0^1 x^{\eta-1} (1-x)^{\xi-1} dx \quad (3)$$

$$= \frac{\Gamma(\eta)\Gamma(\xi)}{\Gamma(\eta+\xi)}, \text{Re}[\eta], \text{Re}[\xi] > 0 \quad (4)$$

But here, we are particularly interested in evaluating possible expressions for the RLFI and the RLFD of order  $\alpha \in \mathbb{R}$  for the power functions stated above without restricting the value of the index  $\beta \in \mathbb{R}$ , just as it happens when were dealing with the integer order calculus. Obviously, as one might expect, we wont be able to use the same direct kind of Eulers beta function approach, because of the restriction of the integral representation of the Eulers beta function to the positive half complex-plane. We also point out that while the standard literature provide expressions for the RLFI and RLFD with the lower limit  $a$  of the operators coinciding with the shifting factor  $d$  in the argument of the power function (i.e., when  $a=d$  we call this situation, "centered"), we are also concerned with a more general setting when  $a$  and  $d$  are not necessarily equal (in such cases, we call this situation "displaced"). This is particularly important, because such "displaced" expressions frequently occurs when trying to solve *fractional delay differential equations* [29].

So in this work, we show that when one takes careful consideration on the choices for the lower and upper limits of these operations, it is possible to compute expressions for the RLFI and RLFD of any power function (regardless of the index of the power) in terms of series that can be related to the famous hypergeometric functions [30] so important and commonly found in many problems of the Mathematical-Physics. Its worth mentioning that the authors have provided two alternative definitions for fractional derivatives of power functions of any order, but they approached the problem in a very distinct way as they have not used the Riemann-Liouville formulation [31]. Also, while there

are many distinct formulations for a fractional differential operator [32,33], it is our hope that with this work we can, not only provide helpful expressions for the aforementioned calculations of RLFI and RLFD of power functions to be used on analytical or numerical related problems, but also set some ground for a future discussion on the theoretical aspects of computing a *fractional definite integral* versus knowing its *fractional primitives* whenever such definitions are meaningful.

We recall some basic concepts in the preliminary section, in the second and third ones we present the main results stated as theorems [34,35], we then provide a summary of the results for convenience compiling the main formulas obtained. Finally, we draw our conclusions and expose some further topics for researches. This work also contain an appendix where it is shown some calculations where we point out that our expressions can be identified with some hypergeometric functions [36-38].

## Preliminaries

We recall that the RLFD is defined in terms of the RLFI which is, by the way, defined in terms of "ordinary" *definite integral*, (Definitions 1 and 2). So by construction, not only this makes the operator non-local, but also address the matter of fractional integro-differentiability of a function to be dependant on the (upper and lower) limits of integration and the domain in which we want to operate, just as it happens in the classical integer order theory and we know that we can calculate the (classic integer order) *indefinite integral* of power functions  $f(t)=(t-d)^\beta$  regardless the values of  $\beta \in \mathbb{R}$ . This means that any power function has a "classical" primitive and, as a consequence, one can calculate the  $n$ -fold (indefinite) integral (or  $n$ -fold primitive) of any order  $n \in \mathbb{N}$ . The situation is similar when calculating the  $n$ th-order derivatives. Since the domain of the resulting power functions after the operations does eventually change and depend on the index  $\beta$  as well as the order  $n$  of the operators, then being (or not) able to calculate the *definite integrals* of these functions actually depends on where we perform the operations. After all in order to concretely calculate any definite results we must use the expressions for the functions obtained by the action of the formal operators:  $n$ -fold indefinite integrals and the  $n$ th-order derivatives. So, even though we can perform these integer order operators on power functions of any order, it is not necessarily true that one can calculate any "definite" *integrals* or *derivative* at a point for these functions at any arbitrary interval of  $\mathbb{R}$ . And we point out that the situation for the fractional case should be similar.

### Definition 1

Let  $\Omega=[a,b] \subset \mathbb{R}$ ,  $f \in L_1(\Omega)$  and  $\alpha \in \mathbb{R}_+$ . The expression for  $[{}_a J_t^\alpha f(x)](t)$

$$[{}_a J_t^\alpha f(x)](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x) dx \quad (5)$$

Defines (leftwise) the Riemann-Liouville fractional integral of order of the function  $f$ .

### Definition 2

Let  $\Omega=[a;b] \subset \mathbb{R}$ ,  $\mathbf{n}=[\alpha]+1$  where  $[\alpha]$  is the integer part of  $\alpha \in \mathbb{R}_+$  and  $f \in AC^n(\Omega)$ .

The expression for  $[{}_a D_t^\alpha f(x)](t)$

$$[{}_a D_t^\alpha f(x)](t) = \frac{d^n}{dt^n} [{}_a J_t^{n-\alpha} f(x)](t)$$

$$= \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} f(x) dx \quad (6)$$

defines (leftwise) the Riemann-Liouville fractional derivative of order of the function  $f$ .

One can similarly define the (right wise) versions of the RLFI and RLFD [34,35]. In the definitions for the RLFI in eqn. (5) and the RLFD in eqn. (6), if  $\alpha=0$ , we define both operators to be the identity operator  $I$ , while if one chooses  $\alpha=n \in \mathbb{N}$  both operators reduces, respectively, to their integer order counterparts, that is, as  $\alpha \rightarrow n$ ,  $[_a J_t^\alpha f(x)](t) \rightarrow [_a J_t^n f(x)](t)$  the  $n$ -fold integrals and  $[_a D_t^\alpha f(x)](t) \rightarrow [_a D_t^n f(x)](t) = \frac{d^n}{dt^n} f(t)$  the  $n$ th -order derivatives [28,36].

In both definitions, the  $\Gamma(\cdot)$  symbol refers to the gamma function and we will be using some of its properties related to the ascending and descending Pochhammer symbols defined, respectively, as:

$$(z)_k = \begin{cases} 1, & \text{if } k = 0 \\ z(z+1)\dots(z+k-1), & \text{if } k \in \mathbb{N} \end{cases} \quad (7)$$

$$(z)_{.k} = \begin{cases} 1, & \text{if } k = 0 \\ z(z-1)\dots(z-k+1), & \text{if } k \in \mathbb{N} \end{cases} \quad (8)$$

which can be rewritten in the following form

$$(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)} \quad (9)$$

$$(z)_{.k} = \frac{\Gamma(z+1)}{\Gamma(z-k+1)} \quad (10)$$

and are valid the relations below

$$(-z)_k = (-1)^k (z)_{-k} \quad (11)$$

$$(-z)_{.k} = (-1)^k (z)_k \quad (12)$$

We also recall that the gamma function is uniquely determined as the function satisfying the functional relation

$$\Gamma(1) = 1 \quad (13)$$

$$z\Gamma(z) = \Gamma(z+1), \text{Re}(z) > 0 \quad (14)$$

but the relation in eqn. (14) can be used to extend it analytically to all complex values, except on  $\mathbb{N}_0^+ = \{0, -1, -2, \dots\}$ , where  $\Gamma(-n) \rightarrow \pm\infty$ ,  $n \in \mathbb{N}$ . Yet, the relation in eqn. (14) is valid for all complex values and when dealing with the elements of  $\mathbb{N}_0$  one should consider

$$z^\pm \Gamma(z^\pm) = \Gamma(z^\pm + 1) \quad (15)$$

Although the gamma function is not defined for negative integers, the ratio of gamma functions of negative integers are defined [36]

$$\frac{\Gamma(-n)}{\Gamma(-m)} = (-1)^{m-n} \frac{m!}{n!}, m, n \in \mathbb{N} \quad (16)$$

and we point out that in eqn.(16) is also valid when choosing  $m$  or  $n$  to be zero, with

$$\frac{\Gamma(-0)}{\Gamma(-m)} = (-1)^m \frac{m!}{0!} = (-1)^m m! \quad (17)$$

$$\frac{\Gamma(-n)}{\Gamma(-0)} = (-1)^n \frac{0!}{n!} = \frac{(-1)^n}{n!} \quad (18)$$

$$\frac{\Gamma(0)}{\Gamma(0)} = 1 \quad (19)$$

Finally, due to the definition and properties of the (analytically extended) gamma function and its relation with the Pochhammer symbols above, we are allowed to generalize the binomial coefficients to non-integer values

$$[_a J_t^\alpha (x-d)^\beta](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} (x-d)^\beta dx \quad (20)$$

where we denote  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

In this work, we will restrict ourselves to  $0 < \alpha < 1$ . This will simplify the main analysis without losing great generality, since for any arbitrary order  $\alpha \in \mathbb{R}$ , we have  $\alpha = [\alpha] + \{\alpha\}$ , where  $[\alpha]$  is the integer part of  $\alpha$  and  $0 < \{\alpha\} < 1$  its fractional part and the main feature of FC is related exactly to the non-integer part. Hence we are considering the evaluation of these expressions:

$$[_a J_t^\alpha (x-d)^\beta](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} (x-d)^\beta dx \quad (21)$$

$$[_a D_t^\alpha (x-d)^\beta](t) = \frac{d}{dt} [_a J_t^{1-\alpha} (x-d)^\beta](t) = \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-x)^{-\alpha} (x-d)^\beta dx. \quad (22)$$

But before we begin with the explicit calculations, let's call attention to the domain in  $\mathbb{R}$  of  $f(t) = (t-d)^\beta$ , which essentially depend on the values of  $d$  and  $\beta$  as it plays an important role on (definite) integrability and differentiability (at a point). We have the following possibilities (at a point):

We have the following possibilities:

#### Case 1

$$\text{Dom}(f) = \begin{cases} \mathbb{R} & \text{if } \beta \in \mathbb{N}_0 \\ \mathbb{R} \setminus \{d\} & \text{if } \beta \in \mathbb{Z} \setminus \mathbb{N}_0. \end{cases} \quad (23)$$

**Case 2:** If  $\beta \in \mathbb{Q} \setminus \mathbb{Z}$ , that is, if  $\beta$  is a proper rational fraction, then we can assume without loss of generality that

$$\beta = \frac{p}{q} \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ and under this hypothesis, we know that}$$

$$f(t) = (t-d)^\beta = \sqrt[q]{(t-d)^p} \quad (24)$$

and since the operation  $\sqrt[q]{(*)}$ ,  $q \in \mathbb{N}$  is well defined in  $\mathbb{R}_+^0 = [0, +\infty)$ , we conclude that

$$\text{Dom}(f) = \begin{cases} \mathbb{R}, & \text{if } p \text{ is an even positive integer;} \\ [d, +\infty), & \text{if } p \text{ is an odd positive integer;} \\ (d, +\infty), & \text{if } p \text{ is a negative integer (even or odd)} \end{cases} \quad (25)$$

#### Case 3

Now if  $\beta \in \mathbb{R} \setminus \mathbb{Q}$ , that is, if  $\beta$  is irrational, then we can use the following identity:

$$(t-d)^\beta = e^{\ln(t-d)\beta} = e^{\beta \ln(t-d)}, \quad (26)$$

and since the domain in  $\mathbb{R}$  of the logarithm is  $\mathbb{R}_+ = (0, +\infty)$  while the domain in  $\mathbb{R}$  of the exponential is  $\mathbb{R}$  itself, then we conclude that in such case  $\text{Dom}(f) = (d, +\infty)$ .

### Riemann-Liouville Integration of Power Functions

We start with the following theorem.

#### Theorem

Let  $f(t) = (t-d)^\beta$ ,  $d, \beta \in \mathbb{R}$  and suppose  $t \in \Omega \subset \mathbb{R}$ , where  $\Omega$  is an interval

where  $f$  is properly defined as real valued function. Then

$$[{}_a J_t^\alpha (x-d)^\beta] (t) = \sum_{k=0}^{\infty} \frac{\Gamma(\beta+1)(a-d)^{\beta-k} (t-a)^{a+k}}{\Gamma(\beta-k+1)\Gamma(a+k+1)}, t \in [a, a + \frac{\epsilon}{2}] \quad (27)$$

$$[{}_a J_t^\alpha (x-d)^\beta] (t) = \sum_{k=0}^{\infty} \frac{\Gamma(\beta+1)(a-d)^{\beta-k} (t-a)^{a+k}}{\Gamma(\beta-k+1)\Gamma(a+k+1)}, t \in [d^+, d^+ + \epsilon]. \quad (28)$$

Particularly, if  $\beta = m \in \mathbb{N}_0$ , then

$$[{}_a J_t^\alpha (x-d)^m] (t) = \sum_{k=0}^m \frac{\Gamma(m+1)(a-d)^{m-k} (t-a)^{a+k}}{\Gamma(m-k+1)\Gamma(a+k+1)}, a, t \in \mathbb{R}, \quad (29)$$

while if  $\beta = -m$  with  $m \in \mathbb{N}$ , then we can use alternatively the following expressions as well

$$[{}_a J_t^\alpha (x-d)^{-m}] (t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(m+k)(a-d)^{-(m+k)} (t-a)^{a+k}}{\Gamma(m)\Gamma(a+k+1)}, \quad (30)$$

$$[{}_{a^+} J_t^\alpha (x-d)^{-m}] (t) = \sum_{k=0}^{\infty} \frac{(-1)^k (m+k) \epsilon^{-(m+k)} (t-d^+)^{a+k}}{\Gamma(m)\Gamma(a+k+1)}, \quad (31)$$

with  $t \in [a, a + \frac{\epsilon}{2}]$  and  $t \in [d^+, d^+ + \epsilon]$  respectively and where in all cases,  $\epsilon = |d-a| > 0$ .

**Proof**

Initially, let  $\beta \in \mathbb{R} \setminus \mathbb{Z}$ . Using the definition of the RLFI in eqn. (5) and integrating by parts a total of  $\mathbf{p}$  times, we get

$$[{}_a J_t^\alpha (x-d)^\beta] (t) = \sum_{k=0}^{\mathbf{p}-1} \frac{(\beta)_{-k} (a-d)^{\beta-k} (t-a)^{a+k}}{\Gamma(a+k+1)} + \mathcal{R}_p, \quad (32)$$

and

$$\mathcal{R}_p = \frac{(\beta)_{-p}}{\Gamma(a+p)} \int_a^t (x-d)^{\beta-p} (t-x)^{a+p-1} dx. \quad (33)$$

Now, using the identity in eqn.(10), we can write

$$[{}_a J_t^\alpha (x-d)^\beta] (t) = \sum_{k=0}^{\mathbf{p}-1} \frac{\Gamma(\beta+1)(a-d)^{\beta-k} (t-a)^{a+k}}{\Gamma(m-k+1)\Gamma(a+k+1)} + \mathcal{R}_p, \quad (34)$$

With

$$\mathcal{R}_p = \frac{\Gamma(\beta+1)_{-p}}{\Gamma(\beta-p+1)\Gamma(a+p)} \int_a^t (x-d)^{\beta-p} (t-x)^{a+p-1} dx. \quad (35)$$

We now estimate the remainder  $\mathcal{R}_p$

$$\begin{aligned} \mathcal{R}_p &\leq \left| \frac{\Gamma(\beta+1)}{\Gamma(\beta-p+1)\Gamma(a+p)} \right| \int_a^t |x-d|^{\beta-p} |t-x|^{a+p-1} dx \\ &\leq \left| \frac{\Gamma(\beta+1)}{\Gamma(\beta-p+1)\Gamma(a+p)} \right| |t-a|^{a+p-1} \int_a^t |x-d|^{\beta-p} dx \\ &= \left| \frac{\Gamma(\beta+1)}{\Gamma(\beta-p+1)\Gamma(a+p)} \right| |t-a|^{a+p-1} \left\{ \frac{|t-d|^{\beta-p+1} - |a-d|^{\beta-p+1}}{\beta-p+1} \right\} \\ &= \left| \frac{\Gamma(\beta+1)}{\Gamma(\beta-p+1)\Gamma(a+p)} \right| \left\{ \frac{|t-a|^a}{|t-d|^{-\beta}} \left| \frac{t-a}{t-d} \right|^{p-1} - \frac{|t-a|^a}{|a-d|^{-\beta}} \left| \frac{t-a}{a-d} \right|^{p-1} \right\}. \end{aligned} \quad (36)$$

While for the second factor in eqn. (36) it is straightforward that

$$\lim_{p \rightarrow \infty} \left\{ \frac{|t-a|^a}{|t-d|^{-\beta}} \left| \frac{t-a}{t-d} \right|^{p-1} - \frac{|t-a|^a}{|a-d|^{-\beta}} \left| \frac{t-a}{a-d} \right|^{p-1} \right\} = 0 \quad (37)$$

whenever  $t \in [a, a + \frac{|a-d|}{2}]$  if  $a \leq t < d$  or  $t \in [a, a + |a-d|]$  if  $d < a \leq t$ , for the first factor, we need some further analysis. Using the well known identity [37].

$$\Gamma(z-n) = \frac{(-1)^n \pi}{\sin(\pi z) \Gamma(n+1-z)}, \quad (38)$$

we can write

$$\begin{aligned} \left| \frac{\Gamma(\beta+1)}{\Gamma(\beta-p+1)\Gamma(a+p)} \right| &= \left| \frac{\Gamma(\beta+1) \sin[\pi(\beta+2)\Gamma(\mathbf{p}-\beta-1)]}{(-1)^p \pi \Gamma(a+p)} \right| |t-a|^{a+p-1} \int_a^t |x-d|^{\beta-p} dx \\ &\leq \left| \frac{\Gamma(\beta+1) \sin[\pi(\beta+2)]}{(-1)^p \pi} \right| \left| \frac{\Gamma(\mathbf{p}-\beta-1)}{\Gamma(a+p)} \right| \\ &= \left| \frac{\Gamma(\beta+1) \sin \pi \beta}{\pi} \right| \left| \frac{\Gamma(\mathbf{p}-\beta-1)}{\Gamma(a+p)} \right| \end{aligned} \quad (39)$$

$$\text{For our choice of } \beta, \text{ it is secured that } \left| \frac{\Gamma(\beta+1) \sin(\pi \beta)}{\pi} \right| = M$$

is always finite regardless of  $\mathbf{p}$  while from in eqn. (38) we have the asymptotic behaviour of a ratio of gamma functions.

$$\lim_{p \rightarrow \infty} \left\{ \frac{\Gamma(\mathbf{p}-\beta-1)}{\Gamma(a+p)} \right\} = p^{-a-\beta-1}, \quad (40)$$

with convergence depending on  $-a-1 \leq \beta$ . However, we are actually interested on the behaviour of the product of the terms in eqn. (37) and in eqn. (40). Since

$$\left| \frac{t-a}{t-d} \right|^{p-1} = e^{-\gamma p + \gamma}, \quad (41)$$

$$\left| \frac{t-a}{a-d} \right|^{p-1} = e^{-\eta p + \eta}, \quad (42)$$

With  $0 < \gamma = -\ln \left| \frac{t-a}{a-d} \right|$  and  $0 < \eta = -\ln \left| \frac{t-a}{a-d} \right|$  (for the proper neighbourhood as describe above) and the exponential decay (or growth) rate is always stronger (in the limit) than any power like rate of growth (decay), that is

$$\lim_{p \rightarrow \infty} \frac{\mathbf{p}^a}{e^p} = 0, \forall a \in \mathbb{R} \quad (43)$$

we can conclude that

$$\lim_{p \rightarrow \infty} |\mathcal{R}_p| \leq M \lim_{p \rightarrow \infty} \mathbf{p}^{-\alpha-\beta-1} \{ e^{-\gamma p + \gamma} - e^{-\eta p + \eta} \} = 0 \quad (44)$$

proving the results in eqn. (27) and in eqn. (28) for  $\beta \in \mathbb{R} \setminus \mathbb{Z}$ .

We now investigate the case  $\beta = -m, m \in \mathbb{N}$ . Initially, we will exclude the case  $m=1$ , since they lead to logarithms. Recall that for  $\beta = -2, -3, \dots$  the domain of  $f$  is

$$\text{Dom}(f) = \mathbb{R} \setminus \{d\} = (-\infty, d) \cup (d, \infty)$$

Therefore, when calculating  $[{}_a J_t^\alpha (x-d)^{-m}] (t)$  we need to take care of choosing the lower limit of integration  $a$  in one of the following intervals: (I)  $(-\infty; d)$  or (II)  $(d, \infty)$ .

(I) So let  $a \leq t < d$ . Then,

$$[{}_a J_t^\alpha (x-d)^{-m}] (t) = \frac{1}{\Gamma(\alpha)} \int_a^t (x-d)^{-m} (t-x)^{\alpha-1} dx$$

If we integrate by parts the expression in the right side in eqn. (45) a total of  $p$  times we get the following result

$$[{}_a J_t^\alpha (x-d)^{-m}] (t) = \frac{1}{\Gamma(\alpha)} \int_a^t (x-d)^{-m} (t-x)^{\alpha-1} dx \quad (45)$$

If we integrate by parts the expression in the right side of eqn. (45) a total of  $p$  times we get the following result

$$[{}_a J_t^\alpha (x-d)^{-m}] (t) = \sum_{k=0}^{p-1} \frac{(-1)^k \binom{m}{k} (a-d)^{-(m+k)} (t-a)^{\alpha+k}}{\Gamma(\alpha+k+1)} + \mathcal{R}_p \quad (46)$$

Where

$$\mathcal{R}_p = \frac{(-1)^p \binom{m}{p}}{\Gamma(\alpha+p)} \int_a^t (x-d)^{-(m+p)} (t-x)^{\alpha+p-1} dx \quad (47)$$

Using the identity in eqn. (9), the expressions in eqn. (46) and eqn. (47) can be rewritten as

$$[{}_a J_t^\alpha (x-d)^{-m}] (t) = \sum_{k=0}^{p-1} \frac{(-1)^k \Gamma(m+k) (a-d)^{-(m+k)} (t-a)^{\alpha+k}}{\Gamma(m)\Gamma(\alpha+k+1)} + \mathcal{R}_p \quad (48)$$

Where

$$\mathcal{R}_p = \frac{(-1)^p \Gamma(m+p)}{\Gamma(m)\Gamma(\alpha+p)} \int_a^t (x-d)^{-(m+p)} (t-x)^{\alpha+p-1} dx. \quad (49)$$

Observe that the remainder  $\mathcal{R}_p$  can be estimated by the inequalities

$$\begin{aligned} [{}_a J_t^\alpha (x-d)^{-m}] (t) &= \sum_{k=0}^{p-1} \frac{(-1)^k \Gamma(m+k) (a-d)^{-(m+k)} (t-a)^{\alpha+k}}{\Gamma(m)\Gamma(\alpha+k+1)} + \mathcal{R}_p \\ \mathcal{R}_p &= \frac{(-1)^p \Gamma(m+p)}{\Gamma(m)\Gamma(\alpha+p)} \int_a^t (x-d)^{-(m+p)} (t-x)^{\alpha+p-1} dx. \\ |\mathcal{R}_p| &\leq \left| \frac{(-1)^p \Gamma(m+p)}{\Gamma(m)\Gamma(\alpha+p)} \int_a^t (x-d)^{-(m+p)} |t-x|^{\alpha+p-1} dx \right. \\ &\leq \left| \frac{\Gamma(m+p)}{\Gamma(m)\Gamma(\alpha+p)} \right| |t-a|^{\alpha+p-1} \int_a^t (x-d)^{-(m+p)} dx \\ &\leq \left| \frac{\Gamma(m+p)}{\Gamma(m)\Gamma(\alpha+p)} \right| |t-a|^{\alpha+p-1} \int_a^t |t-d|^{-(m+p)} dx \\ &= \left| \frac{\Gamma(m+p)}{\Gamma(m)\Gamma(\alpha+p)} \right| \frac{|t-a|^{\alpha+p}}{|t-d|^{m+p}} \\ &= \left| \frac{\Gamma(m+p)}{\Gamma(m)\Gamma(\alpha+p)} \right| \frac{|t-a|^\alpha}{|t-d|^m} \frac{|t-a|^p}{|t-d|^p} \end{aligned} \quad (50)$$

Now, in the last equality in eqn. (50), we have that for each fixed value of  $t$  satisfying  $a \leq t < d$  the fraction  $\frac{|t-a|^\alpha}{|t-d|^m}$  has a fixed finite value. While

$$\lim_{p \rightarrow \infty} \frac{|t-a|^p}{|t-d|^p} = 0 \quad (51)$$

as long as  $\left| \frac{t-a}{t-d} \right| < 1$  which is guaranteed for  $\lim_{p \rightarrow \infty} \left| \frac{\Gamma(m+p)}{\Gamma(m)\Gamma(\alpha+p)} \right| = \frac{p^{m-\alpha}}{\Gamma(m)}$ .

On the other hand, we have that [38]

$$\lim_{p \rightarrow \infty} \left| \frac{\Gamma(m+p)}{\Gamma(m)\Gamma(\alpha+p)} \right| = \frac{p^{m-\alpha}}{\Gamma(m)} \quad (52)$$

with convergence depending on  $m-\alpha \leq 0$ . However, again we are

basically concerned with the limit of the product between in eqn. (51) and in eqn. (52) and since we can identify

$$\left| \frac{t-a}{t-d} \right|^p = e^{-\gamma p} \quad (53)$$

where  $0 < \gamma = -\ln \left| \frac{t-a}{t-d} \right|$  (for  $t \in [a, a + \frac{\epsilon}{2}]$ ),  $\epsilon = |d-a|$  and the exponential decay rate is faster (in the limit) than any power like rate, then we can conclude that

$$\lim_{p \rightarrow \infty} |\mathcal{R}_p| \leq \frac{|t-a|^\alpha}{\Gamma(m) |t-d|^m} \lim_{p \rightarrow \infty} \frac{p^{m-\alpha}}{e^{\gamma p}} = 0 \quad (54)$$

Therefore, as  $p \rightarrow \infty$ , we conclude that in eqn. (48) reduces to

$$[{}_a J_t^\alpha (x-d)^{-m}] (t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(m+k) (a-d)^{-(m+k)} (t-a)^{\alpha+k}}{\Gamma(m)\Gamma(\alpha+k+1)} \quad (55)$$

for  $t \in [a, a + \frac{\epsilon}{2}]$ , giving the expression claimed in eqn. (30). A very similar analysis but with some minor adjustment on the neighbourhood would prove in eqn. (31) as well.

It remains to explore the case where  $m=1$ , that is when  $\beta=-1$ . For that, recall from integer order calculus that integration of functions of the type  $(x-d)^{-1}$  lead to logarithms. Specially we have

$$\int_a^t (x-d)^{-1} f(x) dx = \ln \left( \frac{t-d}{a-d} \right), a > d \quad (56)$$

But we can also calculate the above integral in the following way: We consider the power series representation of  $f(t)=(t-d)^{-1}$  centered at  $t=a$  (for  $a > d$ ),

$$(t-d)^{-1} = -\sum_{k=0}^{\infty} \frac{(t-a)^k}{(d-a)^{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k (t-a)^k}{(a-d)^{k+1}} \quad (57)$$

and then integrate it (inside its radius of convergence) to get

$$\begin{aligned} \int_a^t (x-d)^{-1} dx &= \int_a^t -\sum_{k=0}^{\infty} \frac{(t-a)^k}{(d-a)^{k+1}} dx \\ &= -\sum_{k=0}^{\infty} \frac{1}{(d-a)^{k+1}} \frac{(t-a)^{k+1}}{k+1} \end{aligned} \quad (58)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \left( \frac{t-a}{a-d} \right)^{k+1} \quad (59)$$

where either of the last two series are valid representations for  $\ln \left( \frac{t-d}{a-d} \right)$ .

Now lets consider the explicit fractional case  $0 < \alpha < 1$ . By definition, we have

$$[{}_a J_t^\alpha (x-d)^{-1}] (t) = \frac{1}{\Gamma(\alpha)} \int_a^t (x-d)^{-1} (t-x)^{\alpha-1} dx \quad (60)$$

proceeding with integration by parts in the same way as the previous cases, one get the result

$$[{}_a J_t^\alpha (x-d)^{-1}] (t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1+k) \epsilon^{-(1+k)} (t-d)^{\alpha+k}}{\Gamma(\alpha+k+1)} \quad (61)$$

for  $t \in [d^-, d^+ + \epsilon]$ , which is nothing else but the formula in eqn. (31) when we set  $m=1$ . To recover the original expressions in eqn. (27) and eqn. (28) we can simply verify that since  $=-m$ , then

$$\frac{(-1)^k \Gamma(m+k)}{\Gamma(m)} = \frac{(-1)^k \Gamma(-\beta+k)}{\Gamma(-\beta)}$$

and by eqn. (11) together with the identities in eqn. (9) and eqn. (10), we have that

$$\frac{(-1)^k \Gamma(-\beta+k)}{\Gamma(-\beta)} = (-1)^k (-\beta)_k = \frac{\Gamma(\beta+1)}{\Gamma(\beta-k+1)}$$

Finally, for the simplest case  $\beta=m \in \mathbb{N}$ ,  $Dom(f)=\mathbb{R}$ , so it really doesn't matter where we take the lower limit  $t=a$ , so

$$\begin{aligned} [{}_a J_t^\alpha (x-d)^m](t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (x-d)^m (t-x)^{\alpha-1} dx \\ &= \text{after } m \text{ integration by parts} \\ &= \sum_{k=0}^m \binom{m}{k} (a-d)^{m-k} \frac{(t-a)^{\alpha+k}}{\Gamma(\alpha+k+1)} \\ &= \sum_{k=0}^m \frac{\Gamma(m+1)(a-d)^{m-k} (t-a)^{\alpha+k}}{\Gamma(m-k+1)\Gamma(\alpha+k+1)}, t \in \mathbb{R} \end{aligned} \quad (62)$$

where we have made use of the descending Pochhammer symbol in eqn. (8) and the identity in eqn. (10).

**Corollary:** Consider the hypothesis of theorem. If the lower limit  $t=a \in U \subset \Omega$ , where  $U$  is an interval where the power function is analytic, then calculating its RLFI can be done by operating on its Taylor series expansions.

**Proof:** Just consider the Taylor series expansion of  $f(t)=(t-d)$  at  $t=a$  and operate  ${}_a J_t^\alpha$  on it. Due to the uniform convergence, we can integrate term by term obtaining the expressions listed on theorem.

**Remark:** It is easily verifiable, that each of the expressions in eqns. (27) and (31) reduces to the expected integer order expressions when choosing  $\alpha=0$  or  $\alpha=1$  and, as an example, we will show this in eqn. (27) (the others are similar). Indeed, setting  $\alpha=0$  in eqn. (27) we directly get the Taylor series expansion of  $f(t)=(t-d)^\beta$  centered on  $t=a$ , while setting  $\alpha=1$  in eqn. (27) we get

$$[{}_a J_t (x-d)^\beta](t) = \sum_{k=0}^{\infty} \frac{\Gamma(\beta+1)(a-d)^{\beta-k} (t-a)^{k+1}}{\Gamma(\beta-k+1)\Gamma(k+2)} \quad (63)$$

On the other hand, since the power function is analytic in the interval in consideration, then

$$\begin{aligned} \frac{(t-d)^{\beta+1} - (a-d)^{\beta+1}}{\beta+1} &= \int_a^t (x-d)^\beta dx \\ &= \int_a^t \sum_{k=0}^{\infty} \binom{\beta}{k} (a-d)^{\beta-k} (x-a)^k dx \\ &= \sum_{k=0}^{\infty} \binom{\beta}{k} (a-d)^{\beta-k} \int_a^t (x-a)^k dx \\ &= \sum_{k=0}^{\infty} \binom{\beta}{k} (a-d)^{\beta-k} \frac{(t-a)^{k+1}}{k+1} \end{aligned} \quad (64)$$

and clearly, in eqn. (63) equals in eqn. (64) since  $(k+1)\Gamma(k+1)=\Gamma(k+2)$ .

Before ending this section, there's one final observation that we want to call attention. All expressions listed on theorem for the power functions  $f(t)=(t-d)^\beta$  are only valid when we choose the lower limit  $a \neq d$ , which guarantees the convergence of the series in their respective intervals of definition  $([a, a + \frac{\epsilon}{2}])$  or  $[d^+, d^+ + \epsilon)$ , with  $\epsilon = |d-a| > 0$ . The

only exception to this is the case when  $\beta=m \in \mathbb{N}_0$ , where if we set  $a=d$  (that means  $\epsilon=0$ ) in eqn. (29) it reduces to

$$[{}_a J_t^\alpha (x-d)^m](t) = \frac{\Gamma(m+1)(t-a)^{\alpha+m}}{\Gamma(\alpha+m+1)}, a, t \in \mathbb{R}, \quad (65)$$

which agrees with the usual formula in eqn. (1). But this is to be expected. In one hand, in eqn. (29) is a finite sum and its convergence doesn't depend on, on the other hand we know that the only class of power functions that are analytic everywhere in its domain of definition (which in such case includes the point  $t=d$ ) are the polynomials. For all other values of the index  $\beta$ , these functions are not analytic at  $t=d$  even if  $t=d$  belongs to its domain (e.g., consider  $f(t)=(t-d)^{\frac{1}{2}}$ ).

### Riemann-Liouville Differentiation of Power Functions

The result for the RLFD comes as a corollary of theorem.

#### Corollary

Let  $f(t)=(t-d)^\beta$ ,  $d, \beta \in \mathbb{R}$  and suppose  $t \in \Omega \subset \mathbb{R}$ , where  $\Omega$  is an interval where  $f$  is properly defined as real valued function. Then

$$(I) [{}_a D_t^\alpha (x-d)^\beta](t) = \sum_{k=0}^{\infty} \frac{\Gamma(\beta+1)(a-d)^{\beta-k} (t-a)^{-\alpha+k}}{\Gamma(\beta-k+1)\Gamma(-\alpha+k+1)}, t \in [a, a + \frac{\epsilon}{2}] \quad (66)$$

$$(II) [{}_a D_t^\alpha (x-d)^\beta](t) = \sum_{k=0}^{\infty} \frac{\Gamma(\beta+1)\epsilon^{\beta-k} (t-d^+)^{-\alpha+k}}{\Gamma(\beta-k+1)\Gamma(-\alpha+k+1)}, t \in [d^+, d^+ + \epsilon) \quad (67)$$

Particularly, if  $\beta=m \in \mathbb{N}_0$ , then

$$[{}_a D_t^\alpha (x-d)^m](t) = \sum_{k=0}^m \frac{\Gamma(m+1)(a-d)^{m-k} (t-a)^{k-\alpha}}{\Gamma(m-k+1)\Gamma(k+1-\alpha)}, a, t \in \mathbb{R} \quad (68)$$

while if  $\beta=-m$  with  $m \in \mathbb{N}$ , then we can use alternatively the following expressions as well

$$[{}_a D_t^\alpha (x-d)^m](t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(m+k)(a-d)^{-m-k} (t-a)^{-\alpha+k}}{\Gamma(m)\Gamma(-\alpha+k+1)} \quad (69)$$

$$[{}_a D_t^\alpha (x-d)^{-m}](t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(m+k)\epsilon^{-(m+k)} \epsilon^{-m-k} (t-d^+)^{-\alpha+k}}{\Gamma(m)\Gamma(-\alpha+k+1)} \quad (70)$$

with  $t \in [a, a + \frac{\epsilon}{2}]$  and  $t \in [d^+, d^+ + \epsilon)$ , respectively and where in all cases,  $\epsilon = |d-a| > 0$ .

#### Proof

We can use a similar tedious analysis as in theorem or simply realize that by the definition of the RLFD for  $0 < \alpha < 1$  in eqn. (22) we have

$$[{}_a D_t^\alpha (x-d)^\beta](t) = \frac{d}{dt} [{}_a J_t^{1-\alpha} (x-d)^\beta](t) \quad (71)$$

then it suffices to differentiate the corresponding expressions obtained for  $[{}_a J_t^{1-\alpha} (x-d)^\beta](t)$  using the results of theorem (Appendix A).

#### Corollary

Consider the hypothesis of corollary. If the lower limit  $t=a \in U \subset \Omega$ , where  $U$  is an interval where the power function is analytic, then calculating its RLFD can be done by operating on its Taylor series expansions.

#### Proof

Same as Corollary.

**Remark**

The expressions in eqns. (66) and (67) also reduce to the expected integer order formulas when setting  $\alpha=0$  and  $\alpha=1$  and the proofs can be done in a similar way as in remark.

**Summarizing the Results**

We have then the following expressions for the RLFI of order  $0 < \alpha < 1$  for power functions of any order.

$$\begin{aligned}
 [{}_a J_t^\alpha (x-d)^m](t) &= \sum_{k=0}^m \frac{\Gamma(m+1)(a-d)^{m-k}(t-a)^{\alpha+k}}{\Gamma(m-k+1)\Gamma(\alpha+k+1)}, m \in \mathbb{N}_0, t \in \mathbb{R}, \\
 [{}_a J_t^\alpha (x-d)^\beta](t) &= \sum_{k=0}^\infty \frac{\Gamma(\beta+1)(a-d)^{\beta-k}(t-a)^{\alpha+k}}{\Gamma(m-k+1)\Gamma(\alpha+k+1)}, \beta \in \mathbb{R} \setminus \mathbb{N}_0, t \in \Omega_1, \\
 [{}_{d^+} J_t^\alpha (x-d)^\beta](t) &= \sum_{k=0}^\infty \frac{\Gamma(\beta+1) \epsilon^{\beta-k} (t-d^+)^{-\alpha+k}}{\Gamma(\beta-k+1)\Gamma(\alpha+k+1)}, \beta \in \mathbb{R} \setminus \mathbb{N}_0, t \in \Omega_2
 \end{aligned}$$

Where  $\Omega_1 = [a, a + \frac{|d-a|}{2}]$  and  $\Omega_2 = [d^+, d^+ + \epsilon)$ . Particularly, if  $\beta = -m$  with  $m \in \mathbb{N}$ , then we can use alternatively these expressions as well

$$\begin{aligned}
 [{}_a J_t^\alpha (x-d)^{-m}](t) &= \sum_{k=0}^\infty \frac{(-1)^k \Gamma(m+1)(a-d^+)^{-(m-k)}(t-a)^{\alpha+k}}{\Gamma(m-k+1)\Gamma(\alpha+k+1)}, t \in \Omega_1, \\
 [{}_{d^+} J_t^\alpha (x-d)^{-m}](t) &= \sum_{k=0}^\infty \frac{(-1)^k \Gamma(m+k) \epsilon^{-(m-k)} (t-d^+)^{-\alpha+k}}{\Gamma(m)\Gamma(\alpha+k+1)}, t \in \Omega_2
 \end{aligned}$$

Now for the RLFD of order  $0 < \alpha < 1$  for power functions of any order, we have

$$\begin{aligned}
 [{}_a D_t^\alpha (x-d)^m](t) &= \sum_{k=0}^m \frac{\Gamma(m+1)(a-d)^{m-k}(t-a)^{k-\alpha}}{\Gamma(m-k+1)\Gamma(k+1-\alpha)}, m \in \mathbb{N}_0, t \in \mathbb{R} \\
 [{}_a D_t^\alpha (x-d)^\beta](t) &= \sum_{k=0}^\infty \frac{\Gamma(\beta+1)(a-d)^{\beta-k}(t-a)^{-\alpha+k}}{\Gamma(\beta-k+1)\Gamma(-\alpha+k+1)}, \beta \in \mathbb{R} \setminus \mathbb{N}_0, t \in \Omega_1, \\
 [{}_{d^+} D_t^\alpha (x-d)^\beta](t) &= \sum_{k=0}^\infty \frac{\Gamma(\beta+1) \epsilon^{\beta-k} (t-d^+)^{-\alpha+k}}{\Gamma(\beta-k+1)\Gamma(-\alpha+k+1)}, \beta \in \mathbb{R} \setminus \mathbb{N}_0, t \in \Omega_2
 \end{aligned}$$

particularly, if  $\beta = m$  with  $m \in \mathbb{N}$ , then we can use alternatively these expressions as well

$$\begin{aligned}
 [{}_a D_t^\alpha (x-d)^{-m}](t) &= \sum_{k=0}^m \frac{(-1)^k \Gamma(m+1)(a-d)^{-(m-k)}(t-a)^{-\alpha+k}}{\Gamma(m-k+1)\Gamma(k+1-\alpha)}, t \in \Omega_1, \\
 [{}_{d^+} D_t^\alpha (x-d)^{-m}](t) &= \sum_{k=0}^\infty \frac{(-1)^k \Gamma(m+k) \epsilon^{-(m-k)} (t-d^+)^{-\alpha+k}}{\Gamma(m)\Gamma(-\alpha+k+1)}, t \in \Omega_2,
 \end{aligned}$$

Finally, we point out that in the appendix we have related the expressions for  $[{}_a J_t^\alpha (x-d)^\beta](t), [{}_a D_t^\alpha (x-d)^\beta](t), [{}_{d^+} J_t^\alpha (x-d)^\beta](t)$  and  $[{}_{d^+} D_t^\alpha (x-d)^\beta](t)$  in terms of the hypergeometric function.

**Conclusion**

In this work, we have shown that it is possible to calculate the RLFI and RLFD of order  $0 < \alpha < 1$  of power functions  $(t-*)^\beta$  with being any real value and we are able to express the results in terms of function series of the type  $(t-*)^{\pm\alpha+k}$  and that such expressions can be related to the famous hypergeometric functions of the Mathematical-Physics. We have also observed that since the Riemann Liouville formulations of the fractional integral and differential operators are actually defined in terms of the notion of a definite integral, then the series obtained are convergent on a proper neighborhood of the lower limit, therefore careful attention must be taken to check if the lower limit belongs (or not) to the original functions domain. In fact, the whole problem of

“not being able to integro differentiate the power functions of index strictly less than  $-1$ ” is not really distinct than being (or not) able to integrate ordinarily power functions with a singularity in the lower limit. Recall that in ordinary calculus (integer orders), the integrals final result of achieving or not a valid expression actually depends on “how strong is the singularity in the lower limit versus how large is the order of the n-fold integral, so for the sake of instigating future discussions and works, this strongly suggest further investigation when setting the order with values greater than unity. It is also interesting to look if one can obtain a better formulation of the notions of an -primitive and therefore something as a *fractional indenite integral of order  $\alpha$* .”

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**References**

- Ross B (1970) Fractional Calculus. Math Mag 50: 115-122.
- Ross B (1974) Fractional Calculus and its Applications: Proceedings of the International Conference, New Haven. Springer Verlag, New York.
- Ross B (2006) A Brief History and Exposition of the Fundamental Theory of Fractional Calculus. Lecture Notes in Mathematics 457: 1-36.
- Oliveira EDP, Machado JAT (2014) A Review of Definitions for Fractional Derivatives and Integral. Math Prob Ing ID 238459.
- Caputo M, Fabrizio M (2015) A New Definition of Fractional Derivative without Singular Kernel. Prog Fract Differ Appl 1 2: 73-85.
- Losada J, Nieto JJ (2015) Properties of a New Fractional Derivative without Singular Kernel, Prog Fract Differ Appl 1 2: 87-92.
- Atangana A (2016) On the New Fractional Derivative and Application to Nonlinear Fischer’s Reaction-Discussion Equation. Appl Math Comp 273: 948-956.
- Vanterler J, Sousa C, Capelas de Oliveira E (2017) On the -Hilfer Fractional Derivative.
- Oliveira DSE, Oliveira ECD (2017) Hilfer-Katugampola Fractional Derivatives. Comput Appl Math 37: 3672-3690.
- Sousa JVDC and Oliveira ECD (2017) Mittag-Leffler Functions and the Truncated V-fractional Derivative.
- Yang XJ, Machado JAT, Baleanu D (2017) Anomalous Discussion Models with General Fractional Derivatives Within the Kernels of the Extended Mittag-Leffler Type Functions. Romanian Report in Physics 69: 1-19.
- Yang XJ (2017) Fractional Derivative of Constant and Variable Orders Applied to Anomalous Relaxation Models in Heat-Transfer Problems. Thermal Science 21: 1161-1171.
- Atangana A, Baleanu D (2016) New Fractional Derivative with Non-Local and Non-Singular Kernel: Theory and Application to Heat Transfer Model. Thermal Science 20: 763-769.
- Yang XJ, Machado JAT, Nieto JJ (2017) A New Family of the Local Fractional Partial Differential Equations. Fundamenta Informaticae 151: 63-75.
- Yang XJ, Machado JAT, Cattani C, Gao F (2016) On a Fractal LC-Electric Circuit Modeled by Local Fractional Calculus. Comm Nonl Sci NUm Simulat 47, 200-206.
- Gao F, Yang XJ (2016) Local Fractional Euler.s Method for the Steady Heat-Conduction Problem. Thermal Science 20: S735-S738.
- Plata ARG, Oliveira ECD (2017) Variation Interaction Method in the Fractional Burgers Equation. J Applied Nonlinear Dynamic.
- Costa FS, Soares JCA, Plata ARG, Oliveira ECD (2017) On the Fractional Harry Dym Equation. Comput Appl Math 37: 2862-2876.
- Garrappa R, Mainardi F, Maione G (2016) Models of Dielectric Relaxation Based on Completely Monotone Functions. Fract Cal & Appl Anal 19: 1105-1160.

20. Rosa ECAF, Oliveira ECD (2017) Complete Monotonicity of Fractional Kinetic Functions.
21. Capelas de Oliveira E, Menon MJ, Silva PVRG (2017) Analytic Components for the Hadronic Total Cross-Section: Fractional Calculus and Mellin Transform. Cornell University Library.
22. Chen W, Holm S (2004) Fractional Laplacian Time-Space Models for Linear and Nonlinear Lossy Media Exhibiting Arbitrary Frequency Power-Law Dependency. *J Acoust Soc Am* 115: 1424-1430.
23. Korabel N, Zaslavsky GM, Tarasov VE (2007) Coupled Oscillators With Power-Law Interaction and Their Fractional Dynamics Analogues. *Commun Nonlinear Sci Numer Simulat* 12: 1405-1417.
24. Caputo M, Carcione JM, Cavallini F (2011) Wave Simulation in Biologic Media Based on the Kelvin-Voigt Fractional-Derivative Stress-Strain Relation. *Ultrasound Med Biol* 37: 996-1004.
25. Nasholm SP, Holm S (2011) Linking Multiple Relaxation, Power-Law Attenuation, and Fractional Wave Equations. *J Acoust Soc Am* 130: 3038-3045.
26. Ionescu CM, Ionescu FD (2014) Power Law and Fractional Derivative Models Can Measure Analgesia. *IEEE International Conference on Automation, Quality and Testing, Robotics*.
27. Kilbas AA, Srivastava HM, Trujillo JJ (2006) *Theory and Applications of Fractional Deferential Equations*. Elsevier Science Inc. New York, NY, USA.
28. Diethelm K (2010) *The Analysis of Fractional Deferential Equations. Dynamical Systems & Differential Equations*.
29. Morgado ML, Neville NJ, Lima PM (2013) Analysis and Numerical Methods for Fractional Deferential Equations with Delay. *J Comput Appl Math* 252: 159-168.
30. Lavoie JL, Osler TJ, Tremblay R (1976) Fractional Derivatives and Special Functions. *SIAM Review* 18: 240-268.
31. Andriambololona R, Hanitriarivo R, Ranaivoson T, Raboanary R (2013) Two Definitions of Fractional Derivatives of Power Functions. *Pure Appl Math J* 2: 10-19.
32. Khalil R, Al Horani M, Yousef A, Sababheh M (2014) A New Definition of Fractional Derivative. *J Comput Appl Math* 264: 65-70.
33. Ortigueira MD, Tenreiro Machado JA (2015) What is a Fractional Derivative? *J Comput Phys* 293: 4-13.
34. Podlubny I (1999) *Fractional Deferential Equations*. Academic Press, San Diego.
35. Rodrigues FG, Capelas de Oliveira E (2015) Confluent Hypergeometric Equation Via Fractional Calculus. *J Phys Math* 6: 1-4.
36. Oldham KB, Spanier J (1974) *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*. Elsevier Science.
37. Magnus W, Oberhettinger F, Soni RP (1966) *Formulas and Theorems for the Special Functions of Mathematical Physics. Theoretical, Mathematical & Computational Physics*.
38. Tricomi FG, Ederlyi A (1951) The Asymptotic Expansion of a Ratio of Gamma Functions. *Pac J Math* 1: 133-142.