Journal of Physical Mathematics
ISSN: 2090-0902
Alharbi et al., J Phys Math 2019, 10:4
Research Article
Open Access

An Extended Jacobian Elliptic Function Expansion Approach to the Generalized Fifth Order KdV Equation

Alharbi AR[1], Almatrafi MB[1] and Abdelrahman MAE[1,2]*

[1]Department of Mathematics, College of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt

Abstract

In the present study, the well-known extended Jacobian elliptic function expansion approach is successfully employed to reveal the exact travelling wave solutions of the generalized fifth order KdV (GKdV) equation. The travelling wave solutions are derived in forms rational, hyperbolic and exponential functions. The achieved results are correctly confirmed by presenting some 2D and 3D figures. The proposed technique is concise and effective, and simply applicable mathematical concept for constructing the exact solutions of most nonlinear wave equations emerged in some natural phenomena.

Keywords: Extended jacobian elliptic function expansion method; Generalized fifth order KdV equation; Solitary wave solutions

Mathematics subject classification (2010): 35A20, 35A99, 83C15, 65L12, 65L50, 65M50, 65N06, 65N50, 35Q51, 65Z05

Introduction

A massive number of dynamical problems appeared in mathematical physics, engineering, biology, nuclear physics, mechanics, optics and chemistry can be often modelled by using nonlinear partial differential equations (NPDEs) [1-8]. Thus, it is necessary to construct the exact or numerical solutions of NPDEs. In recent decades, there has been an utterly immense interest in establishing theoretical techniques by which one can find the travelling wave solutions. On other words, obtaining the exact solutions of NPDEs has become one of the broadest scientific subjects. Some specialists discovered some methods such as the Jacobi elliptic function method [9,10], tanh-sech method [11,12], homogeneous balance method [13,14], exp-function method [15,16], Hirotaâ€™s bilinear transformation scheme [17], Riccati-Bernoulli sub-ODE method [18,19], sine-cosine method [20-22], F-expansion method [23,24], extended tanh-method [25], $(\frac{G'}{G})-$ expansion method [26,27] and many others. It should be observed that the mentioned methods are not applicable for some NPDEs.

In this paper, we aim to use the Jacobi elliptic function method to understand the travelling wave solutions for the following GKdV equation [28,29] from a theoretical standpoint:

$$G_t + \mu_0 G G_{xxx} + \mu_1 G_x G_{xx} + \frac{\mu_2}{3}\left(G^3\right)_x + G_{xxxxx} = 0. \tag{1}$$

Where μ_0, μ_2 and μ_2 are arbitrary constants which their values will modify the GKdV equation characteristics. Eq. (1) is investigated by some experts. For instance, the authors in a study [30] utilized the inverse scattering method to present the solution of Eq. (1) when μ_0=-10, μ_1=-20 and μ_2=30 Seadawy et al. [31] expressed the solutions and stability of Eq. (1) for μ_0=-15, μ_2=45 and μ_1 was assumed to be constant. In particular, μ_1 was taken by -15 and 75/2. Various cases of solutions were presented according to the values of $\mu_{1.}$ When $\mu_0=\mu_1$ then Eq. (1) is formed as

$$G_t + \mu_1\left(G G_{xx}\right)_x + \frac{\mu_2}{3}\left(G^3\right)_x + G_{xxxxx} = 0. \tag{2}$$

For example, the standard SK equation is given by Eq. (2) when $\mu_1=\mu_2=5$. It was solved by using the exp-function method [32]. Ali [33] employed the generalized exp(-ϕ(ζ)) expansion method to extract the exact wave solutions of the standard SK equation. In a study [34], the invariant, symmetry and exact solutions of Eq. (2) for μ_2=15 and μ_2=45 are developed by using Lie symmetry analysis method.

The extended Jacobian elliptic function expansion method [9,10] is described and utilized to extract the exact wave solutions of Eq. (1) and Eq. (2). It is the critical purpose of this analysis to show the validity and accuracy of the proposed technique in obtaining the solutions. The considered method played a prominent role in nonlinear evolution equations is a reliable, accurate, powerful and effective mathematical tool for finding the exact solutions of several NPDEs.

The outline of this article is given as follows. In Section 2, the proposed approach is deeply described and suggested some possible solutions. Section 3 is mainly devoted to analyse the GKdV equation and extract some relevant exact solutions. The travelling wave solutions are presented in various cases. Finally, Section 4 concludes this work and shows the essential results.

Analysis of the Extended Jacobian Elliptic Function Expansion Method

An extensive description of the extended Jacobian elliptic function expansion approach [9,10] is illustrated in this section. We present the proposed mathematical concept in some steps listed as follows.

• Consider a given NPDEs in x,t on the form

$$L(p, p_t, p_x, p_{tt}, p_{xx}, \ldots) = 0, \tag{3}$$

and seek a solution on the form

$$p = p(\zeta),\ \ \zeta = k(x - ct). \tag{4}$$

***Corresponding author:** Mahmoud AE Abdelrahman, Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt, Tel: 00201094379921; E-mail: mahmoud.abdelrahman@mans.edu.eg

Received October 10, 2019; **Accepted** November 25, 2019; **Published** December 10, 2019

Citation: Alharbi AR, Almatrafi MB, Abdelrahman MAE (2019) An Extended Jacobian Elliptic Function Expansion Approach to the Generalized Fifth Order KdV Equation. J Phys Math 10: 310.

Here, k and c represent the wave number and wave speed, respectively.

• Using Eq. (4), Eq. (3) is simply converted to the following ODE:

$$Q(p, p_\zeta, p_{\zeta\zeta}, p_{\zeta\zeta\zeta}, ...) = 0, \quad (5)$$

Where Q is a polynomial in p(ζ), and $p_\zeta = \frac{dp}{d\zeta}$, $p_{\zeta\zeta} = \frac{d^2p}{d\zeta^2}$, and so on.

• The considered scheme presents the solutions on the form

$$p(\zeta) = a_0 + \sum_{k=1}^{N} f_j^{k-1}(\zeta)\left[a_k f_j(\zeta) + b_k g_j(\zeta)\right], \; j = 1,2,3,..., \quad (6)$$

with

$$\begin{aligned} &f_1(\zeta) = sn\zeta, \quad g_1(\zeta) = cn\zeta, \\ &f_2(\zeta) = sn\zeta, \quad g_2(\zeta) = dn\zeta, \\ &f_3(\zeta) = ns\zeta, \quad g_3(\zeta) = cs\zeta, \\ &f_4(\zeta) = ns\zeta, \quad g_4(\zeta) = ds\zeta, \\ &f_5(\zeta) = sc\zeta, \quad g_5(\zeta) = nc\zeta, \\ &f_6(\zeta) = sd\zeta, \quad g_6(\zeta) = nd\zeta, \end{aligned} \quad (7)$$

where the Jacobian elliptic sine, cosine and tan functions are denoted by snζ, cnζ, dnζ, respectively. The relationships of these functions are provided as follows:

$$\begin{aligned} &ns\zeta = \frac{1}{sn\zeta}, nc\zeta = \frac{1}{cn\zeta}, nd\zeta = \frac{1}{dn\zeta}, sc\zeta = \frac{cn\zeta}{sn\zeta}, \\ &cs\zeta = \frac{sn\zeta}{cn\zeta}, ds\zeta = \frac{dn\zeta}{sn\zeta}, sd\zeta = \frac{sn\zeta}{dn\zeta}, \end{aligned} \quad (8)$$

which obey the laws

$$\begin{aligned} &sn^2\zeta + cn^2\zeta = 1, dn^2\zeta + m^2 sn^2\zeta = 1, ns^2\zeta = 1 + cs^2\zeta, \\ &ns^2\zeta = m^2 + ds^2\zeta, sc^2\zeta + 1 = nc^2\zeta, m^2 sd^2 + 1 = nd^2\zeta. \end{aligned} \quad (9)$$

Here, 0 ≤ m ≤ 1 is a modulus. It is to be noted that the relevant derivatives of the Jacobi elliptic functions are illustrated as follows:

$$sn'\zeta = cn\zeta dn\zeta, cn'\zeta = -sn\zeta dn\zeta, dn'\zeta = -m^2 sn\zeta cn\zeta, \quad (10)$$

$$ns'\zeta = -ds\zeta cs\zeta, ds'\zeta = -cs\zeta ns\zeta, cs'\zeta = -ns\zeta ds\zeta, \quad (11)$$

$$sc'\zeta = nc\zeta dc\zeta, nc'\zeta = sc\zeta dc\zeta, cd'\zeta = cd\zeta nd\zeta, nd'\zeta = m^2 sd\zeta cd\zeta. \quad (12)$$

• The value of N is evaluated by balancing the highest order linear term with nonlinear term as illustrated in the equations

$$D\left[\frac{d^k p}{d\zeta^k}\right] = N + k, \quad D\left[p^{r_0}\left(\frac{d^k p}{d\zeta^k}\right)^{r_1}\right] = r_0 N + r_1(N + k). \quad (13)$$

• Substituting N into Eq. (6) leads to periodic solutions. As m→1, the functions snζ, cnζ, and dnζ, are renamed by tanhζ, sechζ, sechζ, respectively. Therefore, the solutions take the forms:

$$p(\zeta) = a_0 + \sum_{j=1}^{N} tanh^{j-1}(\zeta)\left[a_j tanh(\zeta) + b_j sech(\zeta)\right], \quad (14)$$

$$p(\zeta) = a_0 + \sum_{j=1}^{N} coth^{j-1}(\zeta)\left[a_j coth(\zeta) + b_j coth(\zeta)\right], \quad (15)$$

$$p(\zeta) = a_0 + \sum_{j=1}^{N} tan^{j-1}(\zeta)\left[a_j tan(\zeta) + b_j sec(\zeta)\right], \quad (16)$$

$$p(\zeta) = a_0 + \sum_{j=1}^{N} cot^{j-1}(\zeta)\left[a_j cot(\zeta) + b_j csc(\zeta)\right]. \quad (17)$$

It has been confirmed that the extended Jacobian elliptic function expansion technique is more dependable and appropriate than the mentioned schemes in a study [35].

Results and Discussion

In this section, the exact travelling wave solutions of GKdV equation are discussed and obtained. Eq. (1) can be easily written as

$$G_t + \mu_1\left(GG_{xx}\right)_x + \delta\left(G_x^2\right)_x + \frac{\mu_2}{3}\left(G^3\right)_x + G_{xxxxx} = 0, \quad (18)$$

where $\delta = \frac{\mu_0 - \mu_1}{2}$. The wave transformation

$$G(x,t) = G(\zeta), \quad \zeta = x - wt, \quad (19)$$

is used with Eq. (18), where w is the speed of the wave, to arise the following ODE:

$$-wG_\zeta + \mu_1\left(GG_{\zeta\zeta}\right)_\zeta + \delta\left(G_\zeta^2\right)_\zeta + \frac{\mu_2}{3}\left(G^3\right)_\zeta + G_{\zeta\zeta\zeta\zeta\zeta} = 0. \quad (20)$$

We now balance the highest order derivative $G_{\zeta\zeta\zeta\zeta\zeta}$ and non-linear term G^3 to end up with m=2. Since m=2, the proposed technique introduces the exact solutions on the form

$$G(\zeta) = a_0 + a_1 sn(\zeta) + b_1 cn(\zeta) + a_2 sn(\zeta)^2 + b_2 sn(\zeta)cn(\zeta), \quad (21)$$

Where a_0, a_1, b_1, a_2, b_2 are some constants shown later. From (21) we have

$$\begin{aligned} G_\zeta = &a_1 cn(\zeta)dn(\zeta) + 2a_2 cn(\zeta)dn(\zeta)sn(\zeta) - b_1 dn(\zeta)sn(\zeta) \\ &+ b_2 dn(\zeta) - 2b_2 dn(\zeta)sn(\zeta)^2, \end{aligned} \quad (22)$$

$$\begin{aligned} G_{\zeta\zeta} = &2a_2 - b_1 cn(\zeta) - a_1 sn(\zeta) - a_1 m^2 sn(\zeta) - 4b_2 cn(\zeta)sn(\zeta) \\ &- b_2 m^2 cn(\zeta)sn(\zeta) - 4a_2 sn(\zeta)^2 - 4a_2 m^2 sn(\zeta)^2 + 2b_1 m^2 \\ &cn(\zeta)sn(\zeta)^2 + 2a_1 m^2 sn(\zeta)^3 + 6b_2 m^2 cn(\zeta)sn(\zeta)^3 + 6a_2 m^2 sn(\zeta)^4, \end{aligned} \quad (23)$$

$$\begin{aligned} G_{\zeta\zeta\zeta\zeta} = &-8a_2 - 8a_2 m^2 + b_1 cn(\zeta) + 4b_1 m^2 cn(\zeta) + a_1 sn(\zeta) \\ &+ 14a_1 m^2 sn(\zeta) + a_1 m^4 sn(\zeta) + 16b_2 cn(\zeta)sn(\zeta) \\ &+ 44b_2 m^2 cn(\zeta)sn(\zeta) + b_2 m^4 cn(\zeta)sn(\zeta) + 16a2sn(\zeta)^2 \\ &+ 104a2m^2 sn(\zeta)^2 + 16a2m^4 sn(\zeta)^2 - 20b_1 m^2 cn(\zeta)sn(\zeta)^2 \\ &- 8b_1 m^4 cn(\zeta)sn(\zeta)^2 - 20a_1 m^2 sn(\zeta)^3 - 20a_1 m^4 sn(\zeta)^3 \\ &- 120b_2 m^2 cn(\zeta)sn(\zeta)^3 - 60b_2 m^4 cn(\zeta)sn(\zeta)^3 \\ &- 120a2m^2 sn(\zeta)^4 - 120a2m^4 sn(\zeta)^4 + 24b_1 m^4 cn(\zeta)sn(\zeta)^4 \\ &+ 24a_1 m^4 sn(\zeta)^5 + 120b_2 m^4 cn(\zeta)sn(\zeta)^5 + 120a2m^4 sn(\zeta)^6. \end{aligned} \quad (24)$$

Setting $\mu_0 = \mu_1$, substituting equations (21)-(24) into Eq. (20) and solving the algebraic system obtained by equating the coefficients of sn^6, $cnsn^5$, sn^5, $cnsn^4$, sn^4, $cnsn^3$, sn^3, $cnsn^2$, sn^2, cnsn, sn, cn and sn^0 to zero, give the following solutions described in some cases. It is worth noting that the solutions are shown when m→1, and μ_2 is considered to be constant.

First case The relevant constants are illustrated as follows:

$$a_0 = \frac{6\sqrt{5}}{\sqrt{\mu_2}}, a_1 = b_1 = b_2 = 0, a_2 = -\frac{6\sqrt{5}}{\sqrt{\mu_2}}, w = 16, \mu_1 = \sqrt{5\mu_2}, \quad (25)$$

and the first family of equation presented by

$$G(\zeta) = a_0 + a_2 sn(\zeta)^2. \quad (26)$$

Hence, the corresponding solution to Eq. (26) is degenerated by

$$G_1(x,t) = \frac{6\sqrt{5}}{\sqrt{\mu_2}} - \frac{6\sqrt{5}}{\sqrt{\mu_2}} tanh(x - wt)^2, w = 16, \mu_1 = \sqrt{5\mu_2}. \quad (27)$$

Figure 1 shows 2D and 3D graphs for the solution of Eq (27) when x_0=20, μ_2=45, w=16.

Second case In this case, we determined the constants as follows:

$$a_0=\frac{3\sqrt{5}\mp\sqrt{21}}{\sqrt{\mu_2}},\ a_1=b_1=b_2=0,\ a_2=-\frac{6\sqrt{5}}{\sqrt{\mu_2}},\ w=\frac{8\left(3\sqrt{5}\pm\sqrt{21}\right)}{3\sqrt{5}\mp\sqrt{21}},\ \mu_1=\sqrt{5\mu_2}. \quad (28)$$

The second family of equation is given by

$$G(\zeta)=a_0+a_2 sn(\zeta)^2. \quad (29)$$

Thus, the solution of Eq. (29) is expressed by

$$G_2(x,t)=\frac{3\sqrt{5}\mp\sqrt{21}}{\sqrt{\mu_2}}-\frac{6\sqrt{5}}{\sqrt{\mu_2}}tanh(x-wt)^2,\ w=\frac{8\left(3\sqrt{5}\pm\sqrt{21}\right)}{3\sqrt{5}\mp\sqrt{21}},\ \mu_1=\sqrt{5\mu_2}, \quad (30)$$

Figure 2 shows 2D and 3D plots for the solution of Eq. (30) when $x_0=10$, $\mu_2=45$.

Third case Here, the constants are solved as follows:

$$a_0=\frac{9\sqrt{5}-\sqrt{21}}{4\sqrt{\mu_2}},\ a_1=b_1=0,\ a_2=-\frac{3\sqrt{5}}{\sqrt{\mu_2}},\ b_2=-\frac{3i\sqrt{5}}{\sqrt{\mu_2}},\ w=\frac{\sqrt{105}+11}{8},\ \mu_1=\sqrt{5\mu_2}. \quad (31)$$

Then, the third family of equation is

$$G(\zeta)=a_0+a_2 sn(\zeta)^2+b_2 sn(\zeta)cn(\zeta). \quad (32)$$

Hence, Eq. (32) has the following solution:

$$G_3(x,t)=\pm\frac{9\sqrt{5}-\sqrt{21}}{4\sqrt{\mu_2}}\pm\frac{3\sqrt{5}}{\sqrt{\mu_2}}tanh(x-wt)^2\pm\frac{3i\sqrt{5}}{\sqrt{\mu_2}}tanh(x-wt)sech(x-wt), \quad (33)$$

where $\mu_1=\pm\sqrt{5\mu_2}$, $w=\frac{\sqrt{105}+11}{8}$.

Figure 3 shows the plot of the real part of Eq. (33) when $x_0=20$, $\mu_2=45$.

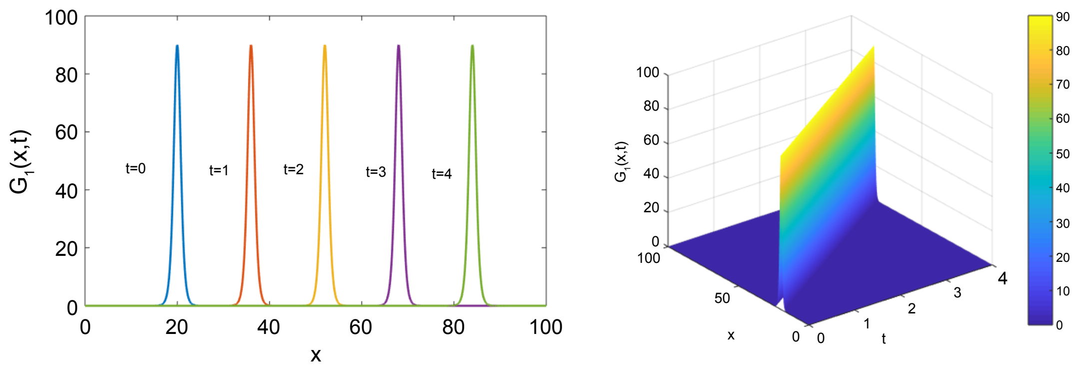

Figure 1: 2D and 3D graphs for the solution of Eq (27) when $x_0=20$, $\mu_2=45$, w=16.

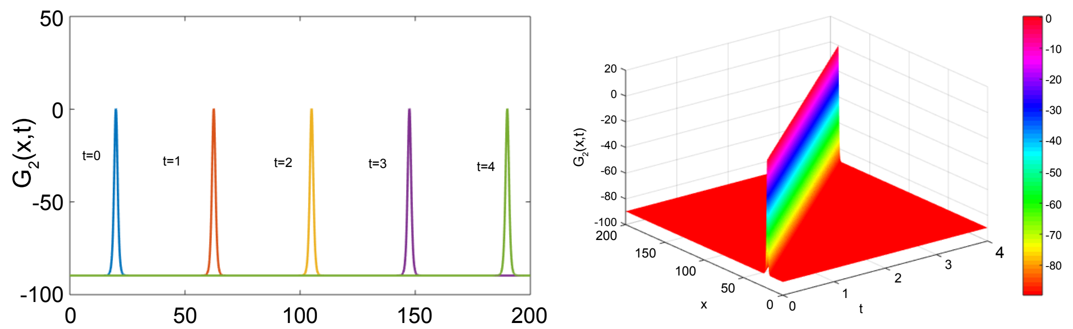

Figure 2: 2D and 3D plots for the solution of Eq. (30) when $x_0=10$, $\mu_2=45$.

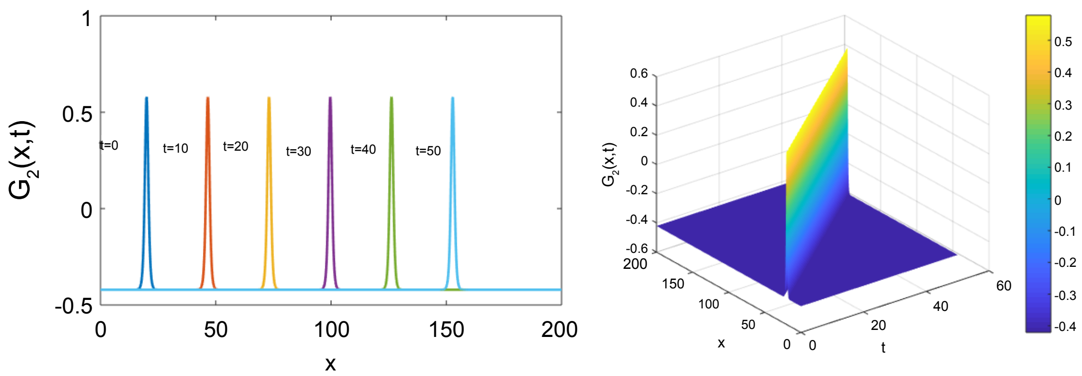

Figure 3: The plot of the real part of Eq. (33) when $x_0=20$, $\mu_2=45$.

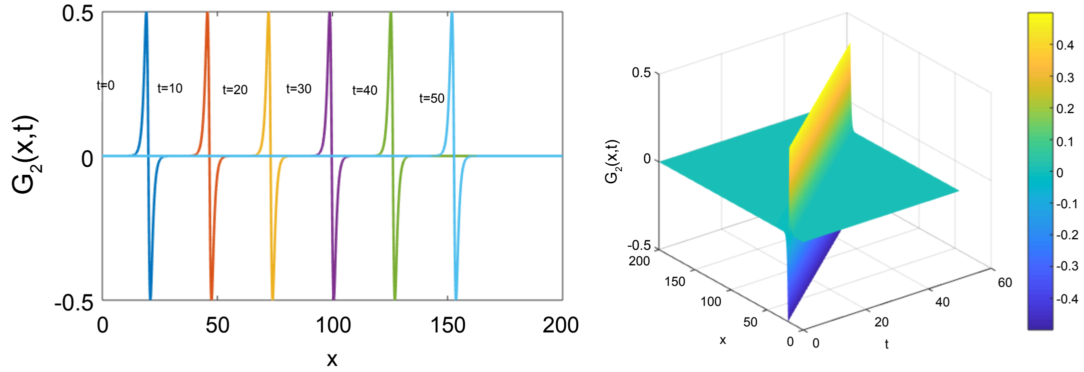

Figure 4: The plot of the imaginary part of Eq. (33) under the values x_0=20, μ_2=45.

Figure 4 shows the plot of the imaginary part of Eq. (33) under the values x_0=20, μ_2=45.

Conclusion

This article has been written to apply the extended Jacobian elliptic function expansion technique on the GKdV equation. The exact travelling wave solutions of Eq. (18) has been surely determined. Some complex solutions are also given as can be seen in above figure. The 2D and 3D figures emphasize the validity of the method. The proposed approach, which can be applied to other NPDEs, gives valuable, advantageous and successful results.

References

1. Abdelrahman MAE (2017) On the shallow water equations. Z Naturforsch 72: 873-879.
2. Abdelrahman MAE (2018) Cone-grid scheme for solving hyperbolic systems of conservation laws and one application. Comp Appl Math 37: 3503-3513.
3. Abdelrahman MAE, Sohaly MA (2018) The development of the deterministic nonlinear PDEs in particle physicsto stochastic case. Results in Physics 9: 344-350.
4. Rekhviashvili S, Pskhu A, Agarwal P, Jain S (2019) Application of the fractional oscillator model to describe damped vibrations. Turk J Phys 43: 236-242.
5. El-Sayed AA, Agarwal P (2019) Numerical solution of multiterm variable order fractional differential equations via shifted Legendre polynomials. Math Methods Appl Sci 42: 3978-3991.
6. Jain S, Agarwal P (2019) On new applications of fractional calculus. Boletim da Sociedade Paranaense de Matemtica 37: 113-118.
7. Jain S, Agarwal P, Kilicman A (2018) Pathway fractional integral operator associated with 3m-parametric Mittag-Leffler functions. Inte J Applied Comp Math 4: 115.
8. Jain S, Agarwal P, Kymaz IO, Cetinkaya A (2018) Some composition formulae for the MSM fractional integral operator with the multi-index Mittag-Leffler functions. AIP Conference Proceedings 1926.
9. Dai CQ, Zhang JF (2006) Jacobian elliptic function method for nonlinear differential difference equations. Chaos Solutions Fractals 27: 1042-1049.
10. Fan E, Zhang J (2002) Applications of the Jacobi elliptic function method to special-type nonlinear equations. Phys Lett A 305: 383-392.
11. Malfliet W, Hereman W (1996) The tanh method: Exact solutions of nonlinear evolution and wave equations. Phys Scr 54: 563-568.
12. Wazwaz AM (2004) The tanh method for travelling wave solutions of nonlinear equations. Appl Math Comput 154: 714-723.
13. Fan E, Zhang H (1998) A note on the homogeneous balance method. Phys Lett A 246: 403-406.
14. Wang ML (1996) Exct solutions for a compound KdV-Burgers equation. Phys Lett A 213: 279-287.
15. He JH, Wu XH (2006) Exp-function method for nonlinear wave equations, Chaos Solitons Fractals 30: 700-708.
16. Aminikhad H, Moosaei H, Hajipour M (2009) Exact solutions for nonlinear partial differential equations via Exp-function method. Numer Methods Partial Differ Equations 26: 1427-1433.
17. Hirota R (1971) Exact solution of the KdV equation for multiple collisions of solutions. Phys Rev Lett 27: 1192-1194.
18. Abdelrahman MAE, Sohaly MA, Alharbi A (2019) The new exact solutions for the deterministic and stochastic (2+1)-dimensional equations in natural sciences. J Taibah Uni 13: 834-843.
19. Abdelrahman MAE, Sohaly MA (2017) Solitary waves for the nonlinear Schro¨dinger problem with the probability distribution function in stochastic input case. Eur Phys J Plus 132: 339.
20. Abdelrahman MAE (2018) A note on Riccati-Bernoulli sub-ODE method combined with complex trans- form method applied to fractional differential equations. Nonlinear Engineering Modeling and Application 7: 279-285.
21. Wazwaz AM (2005) Exact solutions to the double sinh-Gordon equation by the tanh method and a variable separated ODE. method Comput Math Appl 50: 1685-1696.
22. Yan C (1996) A simple transformation for nonlinear waves Phys Lett A 224: 77-84.
23. Ren YJ, Zhang HQ (2006) A generalized F-expansion method to find abundant families of Jacobi elliptic function solutions of the (2+1)-dimensional Nizhnik-Novikov-Veselov equation. Chaos Solitons Fractals 27: 959-979.
24. Zhang JL, Wang ML, Wang YM, Fang ZD (2006) The improved F-expansion method and its applications. Phys Lett A 350: 103-109.
25. Fan E (2000) Extended tanh-function method and its applications to nonlinear equations. Phys Lett A 277: 212-218.
26. Wang ML, Zhang JL, Li XZ (2008) The (G'/G)-expansion method and travelling wave solutions of nonlinear evolutions equations in mathematical physics. Phys Lett A 372: 417-423.
27. Zhang S, Tong JL, Wang W (2008) A generalized (G'/G)-expansion method for the mKdv equation with variable coefficients. Phys Lett A 372: 2254-2257.
28. Whitham GB (1974) Linear and Nonlinear Waves. John Wiley New York.
29. Seadawy AR (2015) Fractional solitary wave solutions of the nonlinear higher-order extended KdV equation in a stratified shear flow: Part I. Comput Math Appl 70: 345-352.
30. Sawada K, Kotera T (1974) A method for finding N-soliton solutions of the KdV equation and KdV-like equation, Prog Theor Phys 51: 1355-1367.
31. Seadawy AR, Luc D, Yue C (2017) Travelling wave solutions of the generalized nonlinear fifth-order KdV water wave equations and its stability. J Taibah Univ Sci 11: 623-633.

32. Matinfar M, Aminzadesh M, Nemati M (2014) Exp-Function Method for the Exact Solutions of Sawda-Kotera Equation. INDIAN J PURE AP MAT 45: 111-119.

33. Ali MY, Hafez MG, Chowdury MKH, Akter MT (2016) Analytical and traveling wave solutions to the fifth order standard Sawada-Kotera equation via the generalized exp(φ(ζ))-expansion method. J Applied Mathe Phy 4: 262-271.

34. Chepngetich W, Oduorv OM, Rotich JK (2019) Lie symmetry solutions of Sawada-Kotera equation. J Advances Mathe 17: 1-11.

35. Liu S, Fu Z, Liu S, Zhao Q (2001) Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys Lett A 289: 69-74.