An example of noncommutative deformations [1]

Eivind ERIKSEN

Oslo University College, Postboks 4, St. Olavs plass, N-0130 Oslo, Norway
E-mail: eeriksen@hio.no

Abstract

We compute the noncommutative deformations of a family of modules over the first Weyl algebra. This example shows some important properties of noncommutative deformation theory that separates it from commutative deformation theory.

2000 MSC: 14D15; 13D10

1 Introduction

Let k be an algebraically closed field and let A be an associative k-algebra. For any left A-module M, there is a commutative deformation functor $\mathsf{Def}_M : \mathsf{l} \to \mathsf{Sets}$ defined on the category l of local Artinan commutative k-algebras with residue field k. We recall that for an object $R \in \mathsf{l}$, a deformation of M over R is a pair (M_R, τ), where M_R is an A-R bimodule (on which k acts centrally) that is R-flat, and $\tau : M_R \otimes_R k \to M$ is an isomorphism of left A-modules. Moreover, $(M_R, \tau) \sim (M'_R, \tau')$ as deformations in $\mathsf{Def}_M(R)$ if there is an isomorphism $\eta : M_R \to M'_R$ of A-R bimodules such that $\tau = \tau' \circ (\eta \otimes 1)$.

In [2], Laudal introduced noncommutative deformations of modules. For any finite family $\mathcal{M} = \{M_1, \dots, M_p\}$ of left A-modules, there is a noncommutative deformation functor $\mathsf{Def}_{\mathcal{M}} : \mathsf{a}_p \to \mathsf{Sets}$ defined on the category a_p of p-pointed Artinian k-algebras. We recall that an object R of a_p is an Artinian ring R, together with a pair of structural ring homomorphisms $f : k^p \to R$ and $g : R \to k^p$, such that $g \circ f = \mathrm{id}$ and the radical $I(R) = \ker(g)$ is nilpotent. The morphisms of a_p are ring homomorphisms that commute with the structural morphisms.

A deformation of the family $\mathcal{M}$ over R is a $(p+1)$-tuple $(M_R, \tau_1, \dots, \tau_p)$, where M_R is an A-R bimodule (on which k acts centrally) such that $M_R \cong (M_i \otimes_k R_{ij})$ as right R-modules, and $\tau_i : M_R \otimes_R k_i \to M_i$ is an isomorphism of left A-modules for $1 \le i \le p$. By definition,

$$(M_i \otimes_k R_{ij}) = \bigoplus_{1 \le i,j \le p} M_i \otimes_k R_{ij}$$

with the natural right R-module structure, and $k_1, \dots, k_p$ are the simple left R-modules of dimension one over k. Moreover, $(M_R, \tau_1, \dots, \tau_p) \sim (M'_R, \tau'_1, \dots, \tau'_p)$ as deformations in $\mathsf{Def}_{\mathcal{M}}(R)$ if there is an isomorphism $\eta : M_R \to M'_R$ of A-R bimodules such that $\tau_i = \tau'_i \circ (\eta \otimes 1)$ for $1 \le i \le p$.

There is a cohomology theory and an obstruction calculus for $\mathsf{Def}_{\mathcal{M}}$, see Laudal [2] and Eriksen [1]. We compute the noncommutative deformations of a family $\mathcal{M} = \{M_1, M_2\}$ of modules over the first Weyl algebra using the constructive methods described in Eriksen [1].

2 An example of noncommutative deformations of a family

Let k be an algebraically closed field of characteristic 0, let $A = k[t]$, and let $D = \mathrm{Diff}(A)$ be the first Weyl algebra over k. We recall that $D = k\langle t, \partial \rangle / (\partial\, t - t\, \partial - 1)$. Let us consider the

[1] Presented at the 3rd Baltic-Nordic Workshop "Algebra, Geometry, and Mathematical Physics", Göteborg, Sweden, October 11–13, 2007.

family $\mathcal{M} = \{M_1, M_2\}$ of left D-modules, where $M_1 = D/D \cdot \partial \cong A$ and $M_2 = D/D \cdot t \cong k[\partial]$. We shall compute the noncommutative deformations of the family $\mathcal{M}$.

In this example, we use the methods described in Eriksen [1] to compute noncommutative deformations. In particular, we use the cohomology $\mathsf{YH}^n(M_j, M_i)$ of the Yoneda complex

$$YC^p(M_j, M_i) = \prod_{m \geq 0} \mathrm{Hom}_D(L_{m,j}, L_{m-p,i})$$

for $1 \leq i, j \leq 2$, where $(L_{*,i}, d_{*,i})$ is a free resolution of M_i, and an obstruction calculus based on these free resolutions. We recall that $\mathsf{YH}^n(M_j, M_i) \cong \mathrm{Ext}^n_D(M_j, M_i)$.

Let us compute the cohomology $\mathsf{YH}^n(M_j, M_i)$ for $n = 1, 2$, $1 \leq i, j \leq 2$. We use the free resolutions of M_1 and M_2 as left D-modules given by

$$0 \leftarrow M_1 \leftarrow D \xleftarrow{\cdot\partial} D \leftarrow 0, \quad 0 \leftarrow M_2 \leftarrow D \xleftarrow{\cdot t} D \leftarrow 0$$

and the definition of the differentials $YC^0(M_j, M_i) \to YC^1(M_j, M_i) \to YC^2(M_j, M_i) = 0$ in the Yoneda complex, and obtain

$$\begin{aligned} &\mathsf{YH}^1(M_1, M_1) \cong \mathrm{Ext}^1_D(M_1, M_1) = 0, && \mathsf{YH}^1(M_1, M_2) \cong \mathrm{Ext}^1_D(M_1, M_2) = k \cdot \xi_{21} \\ &\mathsf{YH}^1(M_2, M_1) \cong \mathrm{Ext}^1_D(M_2, M_1) = k \cdot \xi_{12}, && \mathsf{YH}^1(M_2, M_2) \cong \mathrm{Ext}^1_D(M_2, M_2) = 0 \end{aligned}$$

The base vector ξ_{ij} is represented by the 1-cocycle given by $D \xrightarrow{\cdot 1} D$ in $YC^1(M_j, M_i)$ when $i \neq j$. Since $YC^2(M_j, M_i) = 0$ for all i, j, it is clear that $\mathsf{YH}^2(M_j, M_i) \cong \mathrm{Ext}^2_D(M_j, M_i) = 0$ for $1 \leq i, j \leq 2$.

We conclude that $\mathsf{Def}_{\mathcal{M}}$ is unobstructed. Hence, in the notation of Eriksen [1], the pro-representing hull H of $\mathsf{Def}_{\mathcal{M}}$ is given by

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \cong \begin{pmatrix} k[[s_{12}s_{21}]] & \langle s_{12} \rangle \\ \langle s_{21} \rangle & k[[s_{21}s_{12}]] \end{pmatrix}$$

where $\{s_{ij} = \xi^*_{ij}\}$ is a basis of $\mathrm{Ext}^1_D(M_j, M_i)^*$ dual to the basis $\{\xi_{ij}\}$ of $\mathrm{Ext}^1_D(M_j, M_i)$ for $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$. We write $\langle s_{12} \rangle = H_{11} \cdot s_{12} \cdot H_{22}$ and $\langle s_{21} \rangle = H_{22} \cdot s_{21} \cdot H_{11}$.

In order to describe the versal family $\mathcal{M}_H$ of left D-modules defined over H, we use M-free resolutions in the notation of Eriksen [1]. In fact, the D-H bimodule $\mathcal{M}_H$ has an M-free resolution of the form

$$0 \leftarrow \mathcal{M}_H \leftarrow \begin{pmatrix} D\widehat{\otimes}_k H_{11} & D\widehat{\otimes}_k H_{12} \\ D\widehat{\otimes}_k H_{21} & D\widehat{\otimes}_k H_{22} \end{pmatrix} \xleftarrow{d^H} \begin{pmatrix} D\widehat{\otimes}_k H_{11} & D\widehat{\otimes}_k H_{12} \\ D\widehat{\otimes}_k H_{21} & D\widehat{\otimes}_k H_{22} \end{pmatrix} \leftarrow 0$$

where $d^H = (\cdot\partial)\widehat{\otimes} e_i - (\cdot 1)\widehat{\otimes} s_{12} - (\cdot 1)\widehat{\otimes} s_{21} + (\cdot t)\widehat{\otimes} e_2$. This means that for any $P, Q \in D$, we have that $d^H(P \otimes e_1) = (P \cdot \partial)\widehat{\otimes} e_1 - (P \cdot 1)\widehat{\otimes} s_{21}$ and $d^H(Q \otimes e_2) = (Q \cdot t)\widehat{\otimes} e_2 - (Q \cdot 1)\widehat{\otimes} s_{12}$.

We remark that there is a natural algebraization S of the pro-representing hull H, given by

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \cong \begin{pmatrix} k[s_{12}s_{21}] & \langle s_{12} \rangle \\ \langle s_{21} \rangle & k[s_{21}s_{12}] \end{pmatrix}$$

In other words, S is an associative k-algebra of finite type such that the J-adic completion $\widehat{S} \cong H$ for the ideal $J = (s_{12}, s_{21}) \subseteq S$. The corresponding algebraization $\mathcal{M}_S$ of the versal family $\mathcal{M}_H$ is given by the M-free resolution

$$0 \leftarrow \mathcal{M}_S \leftarrow \begin{pmatrix} D \otimes_k S_{11} & D \otimes_k S_{12} \\ D \otimes_k S_{21} & D \otimes_k S_{22} \end{pmatrix} \xleftarrow{d^S} \begin{pmatrix} D \otimes_k S_{11} & D \otimes_k S_{12} \\ D \otimes_k S_{21} & D \otimes_k S_{22} \end{pmatrix} \leftarrow 0$$

with differential

$$d^S = (\cdot\partial) \otimes e_i - (\cdot 1) \otimes s_{12} - (\cdot 1) \otimes s_{21} + (\cdot t) \otimes e_2$$

We shall determine the D-modules parameterized by the family $\mathcal{M}_S$ over the noncommutative algebra S — this is much more complicated than in the commutative case. We consider the simple left S-modules as the points of the noncommutative algebra S, following Laudal [3], [4]. For any simple S-module T, we obtain a left D-module $M_T = \mathcal{M}_S \otimes_S T$. Therefore, we consider the problem of classifying simple S-modules of dimension $n \geq 1$.

Any S-module of dimension $n \geq 1$ is given by a ring homomorphism $\rho : S \to \mathrm{End}_k(T)$, and we may identify $\mathrm{End}_k(T) \cong M_n(k)$ by choosing a k-linear base $\{v_1, \dots, v_n\}$ for T. We see that S is generated by e_1, s_{12}, s_{21} as a k-algebra (since $e_2 = 1 - e_1$), and there are relations

$$s_{12}^2 = s_{21}^2 = 0, \quad e_1^2 = e_1, \quad e_1 s_{12} = s_{12}, \quad s_{21} e_1 = s_{21}, \quad s_{12} e_1 = e_1 s_{21} = 0$$

Any S-module of dimension n is therefore given by matrices $E_1, S_{12}, S_{21} \in M_n(k)$ satisfying the matric equations

$$S_{12}^2 = S_{21}^2 = 0, \quad E_1^2 = E_1, \quad E_1 S_{12} = S_{12}, \quad S_{21} E_1 = S_{21}, \quad S_{12} E_1 = E_1 S_{21} = 0$$

The S-modules represented by (E_1, S_{12}, S_{21}) and (E_1', S_{12}', S_{21}') are isomorphic if and only if there is an invertible matrix $G \in M_n(k)$ such that $G E_1 G^{-1} = E_1'$, $G S_{12} G^{-1} = S_{12}', G S_{21} G^{-1} = S_{21}'$. Using this characterization, it is a straight-forward but tedious task to classify all S-modules of dimension n up to isomorphism for a given integer $n \geq 1$.

Let us first remark that for any S-module of dimension $n = 1$, ρ factorizes through the commutativization k^2 of S. It follows that there are exactly two non-isomorphic simple S-modules of dimension one, $T_{1,1}$ and $T_{1,2}$, and the corresponding deformations of $\mathcal{M}$ are

$$M_{1,i} = \mathcal{M}_S \otimes_S T_{1,i} \cong M_i \quad \text{for } i = 1, 2$$

This reflects that M_1 and M_2 are rigid as left D-modules.

We obtain the following list of S-modules of dimension $n = 2$, up to isomorphism. We have used that, without loss of generality, we may assume that E_1 has Jordan form:

$$E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.1}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.2}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.3}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.4}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.5}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_{21} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \quad \text{for } a \in k^* \tag{2.6}$$

We shall write $T_{2,1} - T_{2,5}$ and $T_{2,6,a}$ for the corresponding S-modules of dimension two. Notice that $T_{2,6,a}$ is simple for all $a \in k^*$, while $T_{2,1} - T_{2,5}$ are extensions of simple S-modules of dimension one. In fact, $T_{2,1} \cong T_{1,2}^2$, $T_{2,2} \cong T_{1,1} \oplus T_{1,2}$ and $T_{2,3} \cong T_{1,1}^2$ are trivial extensions, while $T_{2,4}$ is a non-trivial extension of $T_{1,2}$ by $T_{1,1}$ and $T_{2,5}$ is a non-trivial extension of $T_{1,1}$ by $T_{1,2}$. The deformations of $\mathcal{M}$ corresponding to the simple modules $T_{2,6,a}$ are given by $M_{2,6,a} = \mathcal{M}_S \otimes_S T_{2,6,a}$ for $a \in k^*$. In fact, one may show that $M_{2,6,a} \cong D/D \cdot (t\partial - a)$ for any $a \in k^*$. In particular, $M_{2,6,a}$ is a simple D-module if $a \notin \mathbb{Z}$, and in this case $M_{2,6,a} \cong M_{2,6,b}$ if and only if $a - b \in \mathbb{Z}$. Furthermore, $M_{2,6,-1} \cong D/D \cdot \partial\ t$, $M_{2,6,n} \cong M_1$ for $n = 1, 2, \ldots$, and $M_{2,6,-n} \cong M_2$ for $n = 2, 3, \ldots$.

We obtain the following list of S-modules of dimension $n = 3$, up to isomorphism. We have used that, without loss of generality, we may assume that E_1 has Jordan form:

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.3}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.4}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.5}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.6}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for } b \in k^* \tag{2.7}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.8}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.9}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.10}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.11}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad S_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} \quad \text{for } c \in k^* \tag{2.12}$$

We shall write $T_{3,1} - T_{3,6}$, $T_{3,7,b}$, $T_{3,8} - T_{3,11}$, and $T_{3,12,c}$ for the corresponding S-modules of dimension three. Notice that all S-modules of dimension three are extensions of simple S-modules of dimension one and two, so there are no simple S-modules of dimension $n = 3$.

In fact, $T_{3,1} \cong T_{1,2}^3$, $T_{3,2} \cong T_{1,1}^3$, $T_{3,3} \cong T_{1,1} \oplus T_{1,2}^2$, $T_{3,8} \cong T_{1,1}^2 \oplus T_{1,2}$, $T_{3,4} = T_{2,4} \oplus T_{1,2}$, $T_{3,5} \cong T_{2,5} \oplus T_{1,2}$, $T_{3,7,b} \cong T_{2,6,b} \oplus T_{1,2}$ for all $b \in k^*$, $T_{3,9} \cong T_{1,1} \oplus T_{2,4}$, $T_{3,10} \cong T_{1,1} \oplus T_{2,5}$, and $T_{3,12,c} \cong T_{1,1} \oplus T_{2,6,c}$ for all $c \in k^*$ are trivial extensions, while $T_{3,6}$ is a non-trivial extension of $T_{1,2}$ by $T_{2,5}$ and $T_{3,11}$ is a non-trivial extension of $T_{1,1}$ by $T_{2,4}$.

We remark that there are no simple S-modules of finite dimension $n \geq 3$. In fact, if T is a simple S-module, then $\rho : S \to \mathrm{End}_k(T)$ is a surjective ring homomorphism. This implies that $\mathrm{End}_k(T) \cong M_n(k)$ can be generated by $E_1 = \rho(e_1)$, $S_{12} = \rho(s_{12})$ and $S_{21} = \rho(s_{21})$ as a k-algebra. To see that this is impossible, notice that we may choose a k-base of T such that

$$E_1 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad S_{12} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad S_{21} = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$$

where $0 \leq r \leq n$, X is a $r \times (n-r)$-matrix and Y is a $(n-r) \times r$ matrix. If $r = 0$ or $r = n$, then $X = Y = 0$, and this leads to a contradiction, because $M_n(k)$ is not generated by diagonal matrices when $n > 1$. Moreover, $r \geq 2$ leads to a contradiction, because $M_r(k) \subseteq M_n(k)$ is not generated by I_r and XY. Similarly $n-r \geq 2$ leads to a contradiction, because $M_{n-r}(k) \subseteq M_n(k)$ is not generated by I_{n-r} and YX. We conclude that $n = r+(n-r) = 1+1 = 2$, a contradiction.

Finally, we remark that the commutative deformation functor $\mathsf{Def}_M : \mathsf{l} \to \mathsf{Sets}$ of the direct sum $M = M_1 \oplus M_2$ has pro-representing hull $(H = k[[s_{12}, s_{21}]], M_H)$, and an algebraization $(S = k[s_{12}, s_{21}], M_S)$. It is not difficult to find the family M_S in this case. In fact, for any point $(\alpha, \beta) \in \operatorname{Spec} S = \mathbf{A}_k^2$, the left D-module $M_{\alpha,\beta} = M_S \otimes_S S/(s_{12} - \alpha, s_{21} - \beta)$ is given by

$$\begin{aligned} M_{0,0} &\cong M_1 \oplus M_2 \\ M_{\alpha,0} &\cong D/D \cdot (\partial\, t) \quad \text{for } \alpha \neq 0 \\ M_{\alpha,\beta} &\cong D/D \cdot (t\, \partial - \alpha\beta) \quad \text{for } \beta \neq 0 \end{aligned}$$

We see that we obtain exactly the same isomorphism classes of left D-modules as commutative deformations of $M = M_1 \oplus M_2$ as we obtained as noncommutative deformations of the family $\mathcal{M} = \{M_1, M_2\}$. However, the points of the algebraization S of the pro-representing hull of the noncommutative deformation functor $\mathsf{Def}_{\mathcal{M}}$ give a much better geometric picture of the local structure of the moduli space of left D-modules. In fact, the family of left D-modules parametrized by the points of S contains few isomorphic D-modules, and the simple S-modules have algebraic properties – such as extensions – that reflect the algebraic properties of the corresponding D-modules.

References

[1] E. Eriksen. An introduction to noncommutative deformations of modules. In *"Noncommutative algebra and geometry"*. Papers from the European Science Foundation (ESF) Workshop held at St. Petersburg State University, St. Petersburg, September 2002. C. De Concini, F. Van Oystaeyen, N. Vavilov, and A. Yakovlev, Eds. Lect. Notes Pure Appl. Math. **243** (2006), 90–125.

[2] O. A. Laudal. Noncommutative deformations of modules. Homology, Homotopy Appl. **4** (2002), 357–396.

[3] O. A. Laudal. Noncommutative algebraic geometry. In *"Proceedings of the International Conference on Algebraic Geometry and Singularities.* Sevilla, Spanish, 2001". Rev. Mat. Iberoamericana, **19** (2003), 509–580.

[4] O. A. Laudal. The structure of $\mathrm{Simp}_{<\infty}(A)$ for finitely generated k-algebras A. In *"Computational commutative and non-commutative algebraic geometry"*. Proceeding of the NATO Advanced Research Workshop held in Chishinau, June 6–11, 2004. S. Cojocaru, G. Pfister, and V. Ufnarovski, Eds. NATO Sci. Ser. III: Comput. Syst. Sci. **196** (2005), 3–43.

Received December 12, 2007
Revised March 27, 2008