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A Tranversality Condition of Codemesion One Submanifolds

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Abstract

In this article we study the transversality of two hypersurfaces M^n and P^n of a given Riemannian manifold $\bar{M}^{n+1}$ along the boundary Σ^{n-1} of M^n and the ellipticity of the Newton tranformations, provided that P^n is totally geodesic.

Keywords: Newton transformations; Symmetric functions; Elliptic operators

Introduction

In this article, we wish to derive a condition of transversality of two hypersurfaces M^n and P^n of $\bar{M}^{n+1}$ along the boundary ∂M^n; provided that $\partial M^n \subset P^n$. This condition is given by the transversality of the classical Newton transformation T_r. In particular we proof that at a point p of the boundary ∂M^n and for every $1 \leq r \leq n-1$ we have:

$$\langle T_r \nu, \nu \rangle = \rho^r \sigma_r \tag{1}$$

Where $\rho = \langle \xi, \nu \rangle, \sigma_r$ is the r-th symmetric function of the principal curvatures of the inclusion $\partial M^n \subset P^n$ with respect to the outward unit normal vector field ν normal to ∂M^n, and ξ is the vector field normal to P^n in $\bar{M}^{n+1}$.

Relation (1) shows that the ellipticity of the Newton transformation T_r, for some $1 \leq r \leq n-1$ on M^n, implies the transversality of the hypersurfaces Mn and Pn along ∂M^n. A similar formula of eqn. (1) was also obtained [1] by the author and Benalili in context of pseudo-Riemannian spaces. It is to emphasize the importance of the application of Newton transformations in intrinsic Riemannian geometry [2-8].

Preliminaries

In this section, we will recall some properties of the Newton transformations and we will show how our method works.

Newton transformations

Let E be an n-dimensional real vector space, End(E) be the vector space of endomorphisms of E, and $(A_1,A_2) \in End(E) \times End(E)$.

For $\alpha \in \{1,2\}$ define the musical functions a|: $\mathbb{N}^2 \to \mathbb{N}^2$ and $\alpha^{\#}: \mathbb{N}^2 \to \mathbb{N}^2$ by:

$$\begin{cases} 1_{|,}(i_1,i_2) = (i_1 - 1, i_2) \\ 2_{|,}(i_1,i_2) = (i_1, i_2 - 1) \end{cases}$$

and

$$\begin{cases} 1_{|,}(i_1,i_2) = (i_1 - 1, i_2) \\ 2_{|,}(i_1,i_2) = (i_1, i_2 - 1) \end{cases}$$

It is clear that α|, is the inverse map of $\alpha^{\#}$.

The generalized Newton transformations of (A_1,A_2) is a system of endomorphisms $T_{(i,j)}=T(_{i,j)}\ (A_1,A_2)$, that satisfies the following recursive relations

$T_{(0,0)}=I$ where 0=(0,00 0)

$T_{(i,j)}=\sigma_{(i,j)}I-A_1T_{(i-1,j)}-A_2T_{(i,j-2)}$ where $i + j \geq 1$

where σ(I,j) are the coefficients of the Newton polynomial $P(A_1,A_2)$: $\mathbb{R}^2 \to \mathbb{R}$ of (A_1,A_2), given by

$$P_{(A_1,A_2)}(t) = \det(I + t_1A_1 + t_2A_2) = \sum_{i+j\leq n} \sigma_{(i,j)} t_1^i t_2^j$$

I is the identity map on E.

If we replace the couple (A_1,A_2) by a one endormorphism0020A, then we recover the definition of the classical Newton transformations and the elementary symmetric functions introduced [7].

A geometric configuration

We establish some algebraic formulas which will be useful in the next section.

Let A be a symmetric (n-1) × (n-1) matrix and consider the n × n-matrix of the block form

$$\tilde{A} = \begin{pmatrix} A & B \\ B^{\mathrm{T}} & c \end{pmatrix}$$

where c is a constant. Let us compare symmetric functions of $\tilde{A}$ with symmetric functions of A. We have

$$P_{\tilde{A}}(t) = \det\left(\begin{pmatrix} I_{n-1} + tA & tB \\ tB^{\mathrm{T}} & 1 + tc \end{pmatrix}\right)$$

$=(1+tc-t^2B^T(I_{n-1}+tA)^{-1}B))\det(I_{n-1}+tA)$

$=f(t)P_A(t)$

where $f(t)=1 + tc-t^2B^T (I_{n-1} + tA)^{-1}B$

Recall that

$$P_{\tilde{A}}(t) = \sum_{j=0}^{n} \sigma j(\tilde{A}) t^j$$

Hence

$$r!\sigma_r(\tilde{A}) = \frac{d^r}{dt^r} P_{\tilde{A}}(0) = \sum_{j=0}^{r} \binom{r}{j} \frac{d^{r-j}}{dt^{r-j}} P_{A|_{\Sigma}}^{(r-j)}(0) f^{(j)}(0)$$

$$= \sum_{j=0}^{r} \binom{r}{j} (r-j)! \sigma_{r-j}(A|_{\Sigma}) f^{(j)}(0)$$

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It is not difficult to see that

f (0)=1,f'(0)=c and $f^{(j)}(0)=(-1)^{j-1}j!B^TA|^{j-2}_{\Sigma}B$ for j≥2.

Therefore

$$\sigma_r\left(\tilde{A}\right)=\sigma_r\left(A|_{\Sigma}\right)+c\sigma_{r-1}\left(A|_{\Sigma}\right)-\sum_{j=2}^{r}(-1)^j\left(B^TA|_{\Sigma}^{j-2}B\right)\sigma_{r-j}\left(A|_{\Sigma}\right) \quad (2)$$

Let us now move to symmetric functions of two matrices. Notice first that

$$\sigma_r\left(\tilde{A}+\lambda I_n\right)=\sum_{j=0}^{r}\binom{n-j}{r-j}\lambda^{r-j}\sigma_j\left(\tilde{A}\right) \quad (3)$$

Indeed, $P_{\tilde{A}+\lambda I_n}(t)=(1+t(\alpha_1+\lambda)...(1+t(\alpha_n+\lambda)$ if $\alpha_1\ldots,\alpha_n$ are the eigen values of A. Notice moreover that

$$P_{\tilde{A}+\lambda I_n}(t)=\det\left(I_n+t\tilde{A}+s\lambda I_n\right)$$

$=(1+t\alpha_1+s\lambda)\ldots(1+t\alpha_n+s\lambda)$

$$\sigma_{(k,l)}\left(\tilde{A},\lambda I_n\right)=\binom{n-k}{l}\lambda^l\sigma_k\left(\tilde{A}\right) \quad (4)$$

Hence

$$\sigma_r\left(\tilde{A}+\lambda I_n\right)=\sum_{j=0}^{r}\sigma_{(j,r-j)}\left(\tilde{A},\lambda I_n\right)$$

The Main Results

LetM^n be an hypersurface of $\bar{M}^{n+1}$ of boundary ∂M. Assume the boundary $\Sigma^{n-1}=\partial M$ is an n-1 submanifold of $P^n\subset\bar{M}^{n+1}$. Then we have the inclusions.

$\Sigma^{n-1}\subset M^n\subset\bar{M}^{n+1}$ and $\Sigma^{n-1}\subset P^n\subset\bar{M}^{n+1}$

Denote the corresponding shape operators, respectively, by

A_Σ, A, $A_{\Sigma,P}$, A_P:

In our consideration we will need only A_Σ, $A_{P|\Sigma}$, A. More precisely we will use

A_Σ, $A_{P|\Sigma}$, A

First two are represented by square matrices of dimension (n-1) whereas the last one by a square matrix of dimension n. The intrinsic geometry of Σ^{n-1} in M^n is coded in the pair (A_Σ,$A_{P|\Sigma}$) and the geometry of $M^n\subset\bar{M}^{n+1}$ is given by A. Therefore we will use the following Newton transformation and the generalized Newton transformations.

$T_r=T_r(A|_\Sigma)$ and $T_{(k,l)}=T_{(k,l)}(A_\Sigma,A_{P|\Sigma})$

and corresponding symmetric functions

$\sigma_r=\sigma_r(A|_\Sigma)$ and $\sigma_{(k,l)}=\sigma_{(k,l)}(A_\Sigma,A_P|_\Sigma)$.

Denote by ν the unit normal vector to Σ^{n-1} in M^n, by N the unit normal vector to with respect to the inclusion $M^n\subset\bar{M}^{n+1}$, ξ the unit normal vector of $P^n\subset\bar{M}^{n+1}$ and η is the unit normal vector to $\Sigma^{n-1}\subset P^n$ and (e_1,...e_{n-1}) is a local orthonormal basis of $T\Sigma^{n-1}$.The only geometric considerations involved are the ones which lead to the formulas [2].

$$\langle N,\nu\rangle=\langle\xi,N\rangle,\langle\eta,N\rangle=-\langle\xi,\nu\rangle$$

$$\left\langle A|_\Sigma e_i,e_j\right\rangle=-\left\langle A_{\Sigma ei,ej}\right\rangle\langle\xi,\nu\rangle+\langle A_Pei,ej\rangle\langle\xi,\mathrm{N}\rangle$$

Suppose now that (e_1,…e_{n-1}) is a frame basis of $T\Sigma^{n-1}$consists of the engenvectors of $A|_\Sigma$ i.e $A|_\Sigma e_i=\gamma_i e_i$ then

$$A|\Sigma=-\langle\xi,\nu\rangle A_\Sigma+\langle\xi,N\rangle A_{P|\Sigma}$$

Assuming P^n is totally umbilical in $\bar{M}^{n+1}$, we have $A|_\Sigma=\lambda I_{T\Sigma n-1}$, hence

$$A|\Sigma=-\langle\xi,\nu\rangle A_\Sigma+\lambda\langle\xi,N\rangle I_{T\Sigma^{n-1}} \quad (5)$$

For brevity put $\rho=-\langle\xi,\nu\rangle$ and $\mu=\langle\xi,N\rangle$

The goal in this part is to show that

$$\sigma r\left(A|\Sigma\right)=\sum_{k+l=r}\rho^k\mu^l\sigma(k,l) \quad (6)$$

Or in another terms

$$\langle T_r\nu,\nu\rangle=\sum_{k+l=r}\rho^k\mu^l\sigma(k,l) \quad (7)$$

When Pn is totally umbilical in $\bar{M}^{n+1}$

By formula (5), we get

$$\sigma_r\left(A|\Sigma\right)=\sigma_r\left(\rho A_\Sigma+\mu\lambda I_{T\Sigma^{n-1}}\right)$$

$$=\sum_{k+l=r}\sigma_{(k,l)}\left(\rho A_\Sigma,\lambda\mu I_{T\Sigma^{n-1}}\right)$$

$$=\sum_{k+l=r}\rho^k\mu^l\sigma(k,l)\rho^k$$

Now we will recover the results [2] that is to say

$$\langle T_r\nu,\nu\rangle=\sigma r\left(A|_\Sigma\right) \quad (8)$$

Denote by $\tilde{A}$ the matrix of A with the respect of the basis (e_1,…, e_{n-1}, ν) and by $A|_\Sigma$ the restriction of A to Σ^{n-1}.

$$\tilde{A}=\begin{pmatrix}A|_\Sigma & B\\ B^{\mathrm{T}} & c\end{pmatrix} \text{ where } B=\begin{pmatrix}\langle A\nu,e_1\rangle\\ \cdot\\ \cdot\\ \cdot\\ \langle A\nu,e_{1-1}\rangle\end{pmatrix} \text{ and } c=\langle A\nu,\nu\rangle.$$

By the recurrence formula for T_r we have,

$$\langle T_r\nu,\nu\rangle=\sigma_r\left(\tilde{A}\right)-\left\langle T_{r-1}\nu,\tilde{A}\nu\right\rangle$$

$$=\sigma_r\left(\tilde{A}\right)-c\left\langle T_{r-1}\nu,\tilde{A}\nu\right\rangle-\sum_{i=1}^{n-1}\langle T_{r-1}\nu,e_i\rangle\left\langle\tilde{A}\nu,e_i\right\rangle$$

Taking into account of the relation (2), to show (8) amounts to show that

$$\sum_{i=1}^{n-1}\langle T_k\nu,e_i\rangle\left\langle\tilde{A}\nu,e_i\right\rangle=\sum_{j=2}^{k+1}(-1)^{j-1}\left(B^{\mathrm{T}}A^{j-2}B\right)\sigma_{k+1-j} \quad (9)$$

And assuming that (e_i)i=1,…, n-1 is a basis consisting of eigenvectors of $A|_\Sigma$with eigenvalues ($\gamma_i)_{i=1}$,…n-1, it turns out to show that for fixed i=1,…, n-1

$$\langle T_k\nu,e_i\rangle\left\langle\tilde{A}\nu,e_i\right\rangle=\sum_{j=2}^{k+1}(-1)^{j-1}\left(b_i\gamma_i^{j-2}b_i\right)\sigma_{k+1-j}\left(A|_\Sigma\right) \quad (10)$$

For every k≥1, where B^T="(bi)i

Now, since

$$\left\langle\nu,\tilde{A}e_i\right\rangle=\left\langle\nu,\sum_{j=1}^{n-1}\left\langle\tilde{A}e_i,e_j\right\rangle e_j+\left\langle\tilde{A}e_i,\nu\right\rangle\nu\right\rangle$$

$$=\left\langle e_i, \tilde{A}\nu\right\rangle = b_i$$

We need to show that

$$\left\langle T_k\nu, e_i\right\rangle = \sum_{j=2}^{k+1}(-1)^{j-1}\left(b_i\gamma_i^{j-2}b_i\right)\sigma_{k+1-j}\left(A|_{\Sigma}\right) \tag{11}$$

We procced by induction. First we have

$$\left\langle T_1\nu, e_i\right\rangle = \left\langle \sigma_1\left(\tilde{A}\right)\nu - \tilde{A}\nu, e_i\right\rangle = -b_i$$

Which shows that equality () is fulfilled for the first term

Suppose now that, for any $1 \le k \le r-1$, the relation (11) holds and calculate

$$\left\langle T_{k+1}\nu, e_i\right\rangle = \left\langle \sigma_{k+1}\left(\tilde{A}\right)\nu - \tilde{A}T_k\nu, e_i\right\rangle$$

$$= -\left\langle T_k\nu, \tilde{A}e_i\right\rangle = -\left\langle T_k\nu, \sum_{j=1}^{n-1}\left\langle \tilde{A}e_i, e_j\right\rangle e_j + \left\langle \tilde{A}e_i, \nu\right\rangle \nu\right\rangle$$

$$= -\sum_{j=1}^{n-1}\left\langle \tilde{A}e_i, e_j\right\rangle\left\langle T_k\nu, e_j\right\rangle - \left\langle \tilde{A}e_i, \nu\right\rangle\left\langle T_k\nu, \nu\right\rangle$$

$$= \sum_{j=2}^{k+2}(-1)^{j-1}b_i\gamma_i^{j-2}\sigma_{k+1-j}\left(A|_{\Sigma}\right) - b_i\left\langle T_k\nu, \nu\right\rangle$$

$$= \sum_{j=2}^{k+2}(-1)^{j-1}b_i\gamma_i^{j-2}\sigma_{k+2-j}\left(A|_{\Sigma}\right)$$

Which proves eqn. (11).

As a consequence of formula (8), we have formula (9).

Theorem 1: theorem Let $\bar{M}^{n+1}$ be an (n+1)-Riemannaian manifolds and $P^n \subset \bar{M}^{n+1}$ oriented totally umbilical subn manifolds of $\bar{M}^{n+1}$. Denote by $\sum^{n-1} \subset P^n$ an (n-1)-compact hypersurfaces of P^n. Let Ψ: Mn→ $\bar{M}^{n+1}$ be a oriented connected and compact hypersurface of $\bar{M}^{n+1}$ with boundary ∑n-1=Ψ(∂M), then along the boundary ∂M, we have

$$\left\langle T_r\nu, \nu\right\rangle = \sigma r\left(A|_{\Sigma}\right) \tag{12}$$

Corollary 1: With the conditions of the above theorem and assuming that $P^n \subset \bar{M}^{n+1}$ is totally geodesic, then for every r with length $r \le n-1$, we have

$$\left\langle T_r\nu, \nu\right\rangle = \rho_r\sigma_r\left(A_{\Sigma}\right)$$

It suffices to use (12) with μ=0.

Transversality

The formula for the Newton transformations implies the relation between transversality of M^n and P^n and ellipticity of T_r provided that P^n is totally geodesic in $\bar{M}^{n+r}$ [9-12].

Theorem 2: With the conditions in Corollary 1 the submanifolds M^n and P^n are transversal along ∂M provided that for some r of length $1 \le r \le n-1$; the Newton transformations T_u is positive definite on M^n.

Proof. Saying that M^n and P^n are not transversal means that there exist p∈∂M such that

$\rho = \left\langle \eta, N\right\rangle = 0$ at p.

Therefore, if we suppose that for all $p \in \bar{M}^{n+1}$, T_r is positive definite, then by Corollary 1, ρ(p) ≠0. Thus

$\left\langle \eta, N\right\rangle \neq 0$.

Hence M^n and P^n are transversal.

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