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A standard example in noncommutative deformation theory [1]

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Abstract

In this paper we generalize the commutative generalized Massey products to the noncommutative deformation theory given by O. A. Laudal. We give an example illustrating the generalized Burnside theorem, one of the starting points in this noncommutative algebraic geometry.

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1 Introduction

In [2], O. A. Laudal defines a noncommutative algebraic geometry based on noncommutative deformation theory, see also [1].

One of the main ingredients in this theory is the following:

Theorem 1.1 (the generalized Burnside theorem)**.** *Let A be a finite dimensional k-algebra, k an algebraically closed field. Consider the family $\mathcal{V} = \{\mathrm{V_i}\}_{\mathrm{i=1}}^{\mathrm{r}}$ of simple A-modules and let $H = (H_{ij})$ be the formal noncommutative moduli of $\mathcal{V}$, then $A \cong O(\mathcal{V}) = (\mathrm{H_{ij}} \otimes_{\mathrm{k}} \mathrm{Hom_k}(\mathrm{V_i}, \mathrm{V_j}))$.*

2 Affine deformations

Definition 2.1. An r-pointed artinian k-algebra is a k-algebra S together with morphisms $k^r \xrightarrow{\iota} S \xrightarrow{\rho} k^r$ such that $\rho \circ \iota = \mathrm{Id}$ and such that $(\ker(\rho))^n = 0$ for some $n > 0$. $\ker(\rho)$ is called the radical of S and denoted $\mathrm{rad}(S)$.

Let $e_i \in k^r$, $1 \leq i \leq r$ be the idempotents. If $S_{ij} = e_i S e_j$ it follows that every r-pointed k-algebra can be written as the matrix algebra $S \cong (S_{ij})$.

Let $V = \{V_1, \ldots, V_r\}$ be a family of right A-modules. Let $S = (S_{ij}) \in \underline{a}_r$ be an r-pointed artinian k-algebra.

Definition 2.2. The deformation functor $\mathrm{Def}_V : \underline{a}_r \to \underline{sets}$ is defined by

$$\mathrm{Def}_V(S) = \{S \otimes_k A\text{-modules } M_S | k_i \otimes_S M_S \cong V_i \text{ and } M_S \cong_k (S_{ij} \otimes_k V_j) = S \otimes_k V\} / \cong$$

Definition 2.3. A morphism $\pi : R \to S$ between to r-pointed artinian k-algebras is called small if $\ker \pi \cdot \mathrm{rad}(R) = \mathrm{rad}(R) \cdot \ker(\pi) = 0$.

Let $M_S \in \mathrm{Def}_V(S)$. Then $M_S \cong_k (S_{ij} \otimes_k V_j)$ and as such it has an obvious structure as left $S = (S_{ij})$-module. The (right) A-module structure is determined by the k-algebra homomorphism $A \xrightarrow{\sigma} End_S(M_S) \Leftrightarrow A \xrightarrow{\sigma} (S_{ij} \otimes_k \mathrm{Hom}_k(V_i, V_j))$ which is completely determined by the morphisms $\sigma_{ij}(a) : V_i \to S_{ij} \otimes_k V_j$. Let M_S be the deformation of V to S given by the

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k-algebra homomorphism $\sigma^S : A \to End_S(S_{ij} \otimes V_j)$ inducing as above $V_i \stackrel{\sigma^S_{ij}(a)}{\to} S_{ij} \otimes_k V_j$. Let $(R_{ij}) = R \stackrel{\pi}{\to} S = (S_{ij})$ be a small morphism. We may lift $\sigma_{ij}(a)$ in the diagram

$$\begin{tikzcd} V_i \arrow[r, "\sigma^R_{ij}(a)"] \arrow[d, "="'] & R_{ij} \otimes_k V_j \arrow[d] \\ V_i \arrow[r, "\sigma^S_{ij}(a)"] & S_{ij} \otimes_k V_j \end{tikzcd}$$

by composing with a section of the right hand vertical map, adding any k-linear morphism $\theta_{ij} : A \to \mathrm{Hom}_k(V_i, I_{ij} \otimes_k V_j)$. Choosing the k-linear lifting σ^R this way, there is a k-linear map $A \to End_R(R_{ij} \otimes_k V_j)$. For this to be an A-module structure commuting with R, we need the conditions; for every $a, b \in A$, $\sigma^R(ab) = \sigma^R(a)\sigma^R(b)$. Because this holds for S, we get an element

$$\psi^R(a, b) = \sigma^R(ab) - \sigma^R(a)\sigma^R(b) \in I_{ij} \otimes_k \mathrm{Hom}_k(V_i, V_j)$$

Because π is a small morphism, we have $I^2 = 0$ and thus $a \cdot \psi^R(b, c) = \sigma^R(a)\psi^R(b, c)$ and $\psi^R(a, b) \cdot c = \psi^R(a, b)\sigma(c)$ implying that ψ^R is a Hochschild 2-cocycle whose class $o(\pi, M_S) \in I_{ij} \otimes_k \mathrm{Ext}^2_A(V_i, V_j)$ is called the obstruction for lifting M_S to R.

Theorem 2.1. *$o(\pi, M_S) = 0$ if and only if there exists a lifting $M_R \in \mathrm{Def}_V(R)$ of M_S. The set of isomorphism classes of such liftings is a torsor under $(I_{ij} \otimes_k \mathrm{Ext}^1_A(V_i, V_j))$.*

Proof. If $0 = o(\pi, M_S)$, then $\psi = d\phi$, and $\sigma' = \sigma + \phi$ is the desired lifting. $\square$

3 Generalized Massey Products

Consider the r-pointed matrix k-algebra $E = k^r\{\varepsilon_{ij}\}$ where all products $\varepsilon_{ij}\varepsilon_{jk} = 0$ for all $1 \leq i, j, k \leq r$, i.e. $E \cong k\{t_{ij}\}/\underline{m}^2$ where $\underline{m}$ is the two-sided ideal generated by $\{t_{ij}\}$. For any covariant functor $F : \underline{a}_r \to \underline{sets}$ $F(E)$ is a k-vector space which is called the tangent space of F. The procategory $\hat{\underline{a}}_r$ is the category of all k-algebras with morphisms $k^r \stackrel{\iota}{\to} R \stackrel{\rho}{\to} k^r$ such that $R/\operatorname{rad}(R)^n \in \underline{a}_r$, for all $n \geq 1$, $\hat{F}(R) = \lim\limits_{\substack{\longleftarrow \\ n}} F(R/(\operatorname{rad}(R))^n)$.

Definition 3.1. A procouple $(\hat{H}, \hat{\xi})$ for F, $\hat{H} \in \hat{\underline{a}}_r$, $\hat{\xi} \in \hat{F}(\hat{H})$, is called a prorepresentable hull, or formal moduli for F, if the induced map $h_{\hat{H}} := \mathrm{Mor}(\hat{H}, -) \to F$ is smooth, and the tangent map $t_{\hat{H}} = \mathrm{Mor}(\hat{H}, E) \to F(E)$ is a bijection.

Definition 3.2 (Non-commutative generalized Massey products)**.** Let $\mathrm{Def}_V : \hat{\underline{a}}_r \to \underline{sets}$ be the deformation functor of the family $\{V_i\}_{i=1}^r$. Then we have an isomorphism as k-vector spaces, $\mathrm{Def}_V(E) \cong (E^1_{ij})$, where $E^1_{ij} = \mathrm{Ext}^1_A(V_i, V_j)$. Let $e^p_{ij} = \dim_k \mathrm{Ext}^p_A(V_i, V_j)$. Let $S_2 = k^r\{t_{ij}(l_{ij})\}/\underline{m}^2$. A sequence of elements $\underline{\alpha} = (\alpha_{ij}(l_{ij})) \in (E^1_{ij})$, $1 \leq l_{ij} \leq e^1_{ij}$ defines a deformation $M_2(\underline{\alpha}) \in \mathrm{Def}(S_2)$. Let B'_2 be the set of all monomials of degree 2 in the $t_{ij}(l_{ij})$ and consider

$$\pi'_2 : R_3 = k^r\{t_{ij}(l_{ij})\}/\underline{m}^3 \to k^r\{t_{ij}(l_{ij})\}/\underline{m}^2 = S_2$$

Then we have that

$$o(M_2(\underline{\alpha}), \pi'_2) = \sum_{\underline{t} \in B'_2} < \underline{\alpha}; \underline{t} > \otimes \underline{t} \in (\mathrm{Ext}^2_A(V_i, V_j) \otimes_k I_{ij})$$

$M_2(\underline{\alpha})$ is then called a defining system for the second order Massey products $< \underline{\alpha}; \underline{t} >$, $\underline{t} \in B_2'$. Choose bases $\{y_{ij}(m_{ij})\}_{m_{ij}=1}^{e_{ij}^2}$ for the dual spaces $\mathrm{Ext}_A^2(V_i, V_j)^*$. Then

$$o(M_2(\underline{\alpha}), \pi_2') = \sum_{\underline{t} \in B_2'} < \underline{\alpha}; \underline{t} > \otimes \underline{t} = \sum_{\underline{t} \in B_2'} y_{ij}^*(m_{ij}) \otimes y_{ij}(m_{ij})(< \underline{\alpha}; \underline{t} >)\underline{t}$$

Put

$$f_{ij}^2(m_{ij}) = \sum_{\underline{t} \in B_2'} y_{ij}(m_{ij})(< \underline{\alpha}; \underline{t} >)\underline{t}$$

Put $S_3 = R_3/(f_{ij}^2)$ and let π_2 be the induced morphism. Choose a monomial basis $B_2 \subseteq B_2'$ for $\ker \pi_2$ and put $\bar{B}_2 = \bar{B}_1 \cup B_2$ where $\bar{B}_1$ is the set of all monomials of degree less than or equal to 1. Then $o(M_2(\underline{\alpha}), S_3) = 0$.

Assume that S_{N-1} has been constructed such that $M_2(\underline{\alpha})$ can be lifted to $M_{N-1}(\underline{\alpha}) \in \mathrm{Def}_V(S_{N-1})$. Also assume that monomial bases B_{N-2}, $\bar{B}_{N-2}$ have been constructed. Put

$$R_N = k^r\{\underline{t}\}/\underline{m}^N + \underline{m}(f_{ij}^{N-1}(m_{ij})) \xrightarrow{\pi_N'} S_{N-1}$$

Write

$$\ker \pi_N' = (f_{ij}^{N-1}(m_{ij}))/\underline{m}(f_{ij}^{N-1}(m_{ij})) \oplus I_N$$

with

$$I_N = \underline{m}^{N-1}/(\underline{m}^N + \underline{m}^{N-1} \cap (f_{ij}^{N-1}(m_{ij}))$$

and pick a monomial basis B_{N-1}' for I_N, where we may assume that for $\underline{t} \in B_{N-1}'$, $\underline{t} = \underline{u} \cdot \underline{s}$ or $\underline{t} = \underline{s} \cdot \underline{u}$ for some $\underline{u} \in B_{N-2}$. Put $\bar{B}_{N-1}' = \bar{B}_{N-2} \cup B_{N-1}'$. Then for every monomial $\underline{u}$ with degree less than N we have a unique relation in R_N

$$\underline{u} = \sum_{\underline{t} \in \bar{B}_{N-1}'} \beta_{\underline{t},\underline{u}}' \underline{t} + \sum_{i,j,m_{ij}} \beta_{\underline{u}}' f_{ij}^{N-1}(m_{ij})$$

and we have that

$$o(M_{N-1}(\underline{\alpha}), \pi_N') = \sum_{i,j,m_{ij}} y_{ij}(m_{ij})^* \otimes f_{ij}^{N-1}(m_{ij}) + \sum_{i,j,m_{ij}} y_{ij}(m_{ij})^* \otimes (\sum_{\underline{t} \in B_{N-1}'} c_{i,j,m_{ij},\underline{t}} \otimes \underline{t})$$

We call $M_{N-1}(\underline{\alpha})$ a defining system for the Massey products

$$< \underline{\alpha}; \underline{t} >= \sum_{i,j,m_{ij}} c_{i,j,m_{ij},\underline{t}} y_{ij}(m_{ij})^* \in \mathrm{Ext}_A^2(V_i, V_j), \quad \underline{t} \in B_{N-1}'$$

To continue, we put

$$f_{ij}^N(m_{ij}) = f_{ij}^{N-1}(m_{ij}) + \sum_{\underline{t} \in B_{N-1}'} y_{ij}(m_{ij})(< \underline{\alpha}; \underline{t} >)\underline{t}$$

and $S_N = R_N/(f_{ij}^N(m_{ij}))$, $\pi_N : S_N \to S_{N-1}$ is the natural morphism. We choose a monomial basis $B_{N-1} \subseteq B_{N-1}'$ for $\ker \pi_N$ and we put $\bar{B}_{N-1} = B_{N-1} \cup \bar{B}_{N-2}$, and we continue by induction.

Theorem 3.1. *The functor* Def_V *has a prorepresenting Hull* $\hat{H}$ *in* $\underline{\hat{a}}_r$, *uniquely determined by a set of matric Massey products*

$$D(i, i_1, i_2, \ldots, i_{n-1}, j) \to \mathrm{Ext}^2_A(V_i, V_j)$$

where $D(i, i_1, i_2, \ldots, i_{n-1}, j)$ *are the defining systems.*

Proof. It follows from Laudal's classical article [3], and it is possible to generalize from Schlessinger [4], that $\hat{H} \cong k^r\{\{\underline{t}\}\}/(f_{ij}(m_{ij}))$, where

$$f_{ij}(m_{ij}) = \sum_{l=0}^{\infty} \sum_{\underline{t} \in B_{N+l}} y_{ij}(m_{ij}) < \underline{x}^*; \underline{t} > \underline{t}$$

and $\underline{x}^*$ is a basis for $(\mathrm{Ext}^1_A(V_i, V_j))^*$, see [1] or [5]. □

4 Example

Consider the 2-pointed k-algebra

$$A = \begin{pmatrix} k[t_{11}] & < t_{12} > /(t_{11} - 1)t_{12} \\ 0 & k \end{pmatrix}$$

This k-algebra has geometric points, i.e simple A-modules, given by the line and the point respectively

$$V_1(a) = \begin{pmatrix} \mathbf{k}(\mathrm{a}) & 0 \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$$

We are going to compute the local formal moduli $\hat{H}_V$, $V = \{V_1(a), V_2\}$ of the modules $V_1(a)$, V_2 for a fixed a, following the algorithm given in [2]. We start by computing the tangent spaces:

In general we have

$$\mathrm{Ext}^1_A(V_i, V_j) = \mathrm{HH}^1(A, \mathrm{Hom}_k(V_i, V_j)) = \mathrm{Der}_k(A, \mathrm{Hom}_k(V_i, V_j))/\operatorname{Ad}$$

where the bi-module structure on $\mathrm{Hom}_k(V_i, V_j)$ is given by $a\phi(v_i) = \phi(av_i)$, $\phi \cdot a(v_i) = \phi(v_i)a$. Notice that by Ad we mean the trivial derivations ad_α, $\alpha \in \mathrm{Hom}_k(V_i, V_j)$.

Any derivation δ is determined on a generator set. In this particular example, we choose as generators

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (t_{11} - a) = \begin{pmatrix} (t_{11} - a) & 0 \\ 0 & 0 \end{pmatrix}, \quad t_{12} = \begin{pmatrix} 0 & t_{12} \\ 0 & 0 \end{pmatrix}$$

(1,1): $\mathrm{Ext}^1_A(V_1(a), V_1(a))$:

$$\begin{aligned} &\delta(e_1) = \delta(e_1^2) = e_1\delta(e_1) + \delta(e_1)e_1 = 2\delta(e_1) \quad \Rightarrow \quad \delta(e_1) = \delta(e_2) = 0 \\ &\delta(t_{11} - a) = \alpha \\ &\delta(t_{12}) = \delta(t_{12}e_2) = \delta(t_{12})e_2 + t_{12}\delta(e_2) = 0 \\ &\mathrm{ad}_\beta(t_{11} - a) = (t_{11} - a)\beta - \beta(t_{11} - a) = 0 \end{aligned}$$

As basis for $\mathrm{Ext}^1_A(V_1(a), V_1(a))$ we choose the one element set $\{\phi_{11} = (t_{11} - a)^\vee\}$.

(1,2): $\mathrm{Ext}^1_A(V_1(a), V_2)$:

$$\delta(e_1) = \alpha, \quad \delta(e_2) = -\alpha$$

$$\delta(t_{11}-a) = \delta((t_{11}-a)e_1) = \delta(t_{11}-a)e_1 + (t_{11}-a)\delta(e_1) = 0$$
$$(a-1)\delta(t_{12}) = \delta(t_{11}t_{12} - t_{12}) = 0$$
$$\mathrm{ad}_\alpha(t_{11}-a) = 0, \quad \mathrm{ad}_\alpha(e_1) = e_1\alpha - \alpha e_1 = \alpha, \quad \mathrm{ad}_\alpha(e_2) = e_2\alpha - \alpha e_2 = -\alpha$$

Thus if $a = 1$ we choose as basis the one point set $\{\phi_{12} = t_{12}^\vee\}$. If $a \neq 1$, then $\mathrm{Ext}^1_A(V_1(a), V_2) = 0$.

(1,i): $\mathrm{Ext}^1_A(V_2, V_i) = 0$ $(i = 1, 2)$ which is trivial.

For the rest we put $a = 1$, that is $V_1 = V_1(1)$ and compute $\hat{H}_{\{V_1,V_2\}}$. Let

$$S = \begin{pmatrix} k[u_{11}] & < u_{12} > \\ 0 & k \end{pmatrix}$$

Then the infinitesimal liftings are given by

$$\phi_2 = \begin{pmatrix} 1 \otimes \cdot a + u_{11} \otimes (t_{11}-1)^\vee & u_{12} \otimes t_{12}^\vee \\ 0 & 1 \otimes \cdot a \end{pmatrix} : A \to (\mathrm{Hom}_k(V_i, S_{2,ij} \otimes V_j))$$

Now $S_2 = S/\operatorname{rad}^2$ and the obstruction for lifting to $R_3 = S/\operatorname{rad}^3$ is

$$o = \begin{pmatrix} u_{11}^2 \otimes (t_{11}-1)^\vee (t_{11}-1)^\vee & u_{11}u_{12} \otimes (t_{11}-1)^\vee t_{12}^\vee \\ 0 & 0 \end{pmatrix}$$

In general, $v^\vee w^\vee = (v \otimes w)^\vee = -d((vw)^\vee)$, so

$$(t_{11}-1)^\vee (t_{11}-1)^\vee = -d((t_{11}-1)^2)^\vee)$$

But $(t_{11}-1)t_{12} = 0$ in A, thus $\overline{o} = \left(\begin{smallmatrix} 0 & u_{11}u_{12} \otimes o_{12} \\ 0 & 0 \end{smallmatrix}\right)$ with $o_{12} \neq 0$. Put $S_3 = S/(\operatorname{rad}^3 + u_{11}u_{12})$. Then we can lift the A-module structure to S_3 by

$$\phi_3 = \begin{pmatrix} 1 \otimes \cdot a + u_{11} \otimes (t_{11}-1)^\vee + u_{11}^2 \otimes ((t_{11}-1)^2)^\vee & u_{12} \otimes t_{12}^\vee \\ 0 & 0 \end{pmatrix}$$

We see that this ϕ_3 can be lifted to ϕ_n on $S_n = S_3/\operatorname{rad}^n$, $n \geq 3$, giving the result

$$\hat{H} = \lim_{\leftarrow} S_n = \begin{pmatrix} k[[u_{11}]] & < u_{12} > /u_{11}u_{12} \\ 0 & k \end{pmatrix} \cong \lim_{\leftarrow} A/\operatorname{rad}^n$$

In general terms this says that A is a scheme for its 1-dimensional simple modules.

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