

A special form of Rund's h-curvature tensor using $R3$ -like Finsler space

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Abstract

The purpose of the present paper is to consider and study a special form of Rund's h-curvature tensor K_{ljk}^i and Berwald's curvature tensor H_{ljk}^i in an $R3$ -like C-reducible Finsler space. In this paper, we modify the Rund's h-curvature tensor K_{ljk}^i to special form by using some special Finsler spaces like C-reducible, $R3$ -like Finsler spaces.

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1 Introduction

Let $F^n = (M^n, F)$ be an n -dimensional Finsler space with the fundamental function $F(x, y)$. The Finsler Γ_v -connection $R\Gamma$, constructed from the Cartan connection $C\Gamma$, is called the Rund connection. Matsumoto defined the curvature tensor $R\Gamma$ and the concept of special form of Rund's curvature tensor K_{ljk}^i [5]. The author [9] have studied Finsler space with Rund's h-curvature tensor K_{ljk}^i of a special form. Here we extend the study of a special form of Rund's h-curvature tensor using $R3$ -like Finsler space and obtain some results. In this paper, the range of indexes varies from 1 to n and the v -covariant and α -covariant derivatives are denoted by $|j$ and $\|j$, respectively.

We use the following notations from [5, 8]:

$$\begin{aligned}
 \text{(a)} \quad & g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2, \quad g^{ij} = (g_{ij})^{-1}, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i} \\
 \text{(b)} \quad & C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}, \quad C_{ij}^k = \frac{1}{2} g^{km} (\dot{\partial}_m g_{ij}) \\
 \text{(c)} \quad & h_{ij} = g_{ij} - l_i l_j, \quad h_j^i = \delta_j^i - l^i l_j \\
 \text{(d)} \quad & C_{lr}^i y^l = 0, \quad h_l^i y^l = 0 \\
 \text{(e)} \quad & y^i L_k = y_k^i L, \quad g_{ij} y^i y^j = F^2, \quad l_i y^i = F \\
 \text{(f)} \quad & L_k = y_k L
 \end{aligned} \tag{1.1}$$

Definition 1.1 (see [1, 2]). A Finsler space F^n ($n > 3$) is called $R3$ -like, if the curvature tensor R_{hijk} is written in the form

$$R_{hijk} = g_{hj} L_{ik} + g_{ik} L_{hj} - g_{hk} L_{ij} - g_{ij} L_{hk} \tag{1.2}$$

where $L = L_{ij} g^{ij}$ is the tensor.

Definition 1.2 (see [2, 3]). A Finsler space F^n is called C-reducible if it satisfies the equation

$$C_{ijk} = (C_i h_{jk} + C_j h_{ki} + C_k h_{ij}) / (n + 1) \quad (1.3)$$

where $C_i = C_{ijk} g^{jk}$.

Definition 1.3 (see [6]). A Finsler space F^n is called P-reducible if the torsion tensor P_{ijk} is written as

$$P_{ijk} = G_i h_{jk} + G_j h_{ki} + G_k h_{ij} \quad (1.4)$$

where $P_{ijk} = C_{ijk|0}$ and $G_i = C_{i|0} / (n + 1)$.

The v -covariant derivative of P-reducible Finsler space is given by [9]

$$P_{jk|l}^i = G_{|l}^i h_{jk} + G_{j|l} h_k^i + G_{k|l} h_j^i \quad (1.5)$$

Definition 1.4 (see [7]). A Finsler space is called Landsberg Finsler space if $C_{ijk|0} = 0$.

We use the following identities from [5, 7, 9]:

$$\begin{aligned} (a) \quad & R_{jk|l}^i - K_{ljk}^i + U_{(jk)} \{ P_{jr}^i P_{kl}^r + P_{klj}^i \} = 0 \\ (b) \quad & R_{ljk}^i y^l = R_{0jk}^i = R_{jk}^i \\ (c) \quad & K_{ljk}^i + K_{jkl}^i + K_{klj}^i = 0 \\ (d) \quad & H_{ljk}^i = R_{jk||l}^i, \\ (e) \quad & H_{ljk}^i + H_{jkl}^i + H_{klj}^i = 0, \\ (f) \quad & K_{ljk}^i = -K_{lkj}^i, \quad K_{lijk} = g_{ir} K_{ljk}^r \\ (g) \quad & h_{ij||k} = 2C_{ijk} - L^{-2} (y_i h_{jk} + y_j h_{ik}) \\ (h) \quad & h_{j||k}^i = -L^{-2} (y_j h_k^i + y^i h_{jk}) \end{aligned} \quad (1.6)$$

where R_{jk}^i is the (v) h-torsion tensor and the suffix '0' means contraction with y^i . The notation $u_{(jk)}$ denotes the interchange on indices j and k and subtraction.

We use the following lemma in the next section.

Lemma 1.5 (see [4]). *If the equation $v_{hi} h_{jk} + v_{ij} h_{hk} + v_{jh} h_{ik} = 0$ holds in F^n , then we have (1) $v_{ij} = 0, (n \geq 4)$ and (2) $v_{ij} = v(m_i n_j - m_j n_i)$, with reference to the Moor frame (l^i, m^i, n^i) , where v is a scalar.*

2 Special form of Rund's h-curvature tensor K_{ljk}^i

Let F^n be a Finsler space with Rund's h-curvature tensor K_{ljk}^i of the special form [9]

$$K_{ljk}^i = U_{(jk)} \{ A_{jk} h_l^i + D_{lk} h_j^i + E_j^i h_{kl} \} \quad (2.1)$$

where $A_{jk}, D_{lj}, E_j^i, F_j^i, G_{lk}$ are Finsler tensor fields.

Consider h-curvature tensor of the form

$$K_{ljk}^i = R_{ljk}^i - C_{lr}^i R_{jk}^r \quad (2.2)$$

Using equations (1.2) and (1.3) in (2.2), we get

$$K_{ljk}^i = \{ (\delta_j^i L_{lk} + g_{lk} L_j^i - \delta_k^i L_{lj} - g_{lj} L_k^i) \\ - (\delta_j^r L_k + y_k L_j^r - \delta_k^r L_j - y_j L_k^r) (C^i h_{lr} + C_l h_r^i + C_r h_l^i) / (n+1) \}$$

By using some Finsler identities, the above equation can be written as

$$K_{ljk}^i = \{ (h_j^i + l^i l_j) L_{lk} + (h_{lk} + l_l l_k) L_j^i - (h_k^i + l^i l_k) L_{lj} - (h_{lj} + l_l l_j) L_k^i \} \\ - \{ C^i h_{lr} \delta_j^r L_k + C_l h_r^i \delta_j^r L_k + C_r h_l^i \delta_j^r L_k + C^i h_{lr} y_k L_j^r C_l h_r^i y_k L_j^r + C_r h_l^i y_k L_j^r \\ - C^i h_{lr} \delta_k^r L_j - C_l h_r^i \delta_k^r L_j - C_r h_l^i \delta_k^r L_j - C^i h_{lr} y_j L_k^r - C_l h_r^i y_j L_k^r - C_r h_l^i y_j L_k^r \} / (n+1)$$

After simplification and the rearrange the terms, we get

$$K_{ljk}^i = \{ h_j^i L_{lk} + h_{lk} L_j^i - h_k^i L_{lj} - h_{lj} L_k^i \} + \{ l^i l_j L h_{lk} + l_l l_k L h_j^i - l^i l_k L h_{lj} - l_l l_j L h_k^i \} \\ - 2 \{ C^i h_{lj} L y_k + C_l h_j^i L y_k + C_j h_l^i L y_k - C^i h_{lk} L y_j - C_l h_k^i L y_j - C_k h_l^i L y_j \} / (n+1) \\ K_{ljk}^i = U_{(jk)} [h_l^i (2C_k L y_j) / (n+1) + h_j^i (L_{lk} + L l_l l_k - 2C_l L y_k) / (n+1) \\ + h_{lk} (L_j^i + L l^i l_j + 2C^i L y_j) / (n+1)]$$

In simple form, the above equation can be written as a special form of (2.1) as

$$K_{ljk}^i = U_{(jk)} (h_l^i A_{jk} + h_j^i D_{lk} + h_{lk} E_j^i) \quad (2.3)$$

where

$$A_{jk} = 2C_k y_j L / (n+1) \\ D_{lk} = (L_{lk} + L l_l l_k - 2C_l L y_k) / (n+1) \\ E_j^i = (L_j^i + L l^i l_j + 2C^i L y_j) / (n+1) \quad (2.4)$$

Thus we state the following.

Theorem 2.1. *In an R3-like, C-reducible Finsler space, the h-curvature tensor reduces to special form of Rund's h-curvature tensor (2.3).*

Now we compare the Rund's curvature tensor and h-curvature tensor. Thus, from (2.1) and (2.2), we have

$$R_{ljk}^i - C_{lr}^i R_{jk}^r = \{ A_{jk} h_l^i + D_{lk} h_j^i + E_j^i h_{kl} - A_{kj} h_l^i - D_{lj} h_k^i - E_k^i h_{jl} \} \quad (2.5)$$

Contracting (2.5) with respect to y^l, y^k and using (1.1d), we get

$$R_{j0}^i = D_{00} h_j^i \quad (2.6)$$

Again contracting (2.6) with respect to i and j , we get

$$R = (n-1) D_{00} \quad (2.7)$$

Now, we will find D_{00} . Consider D_{ij} from the special form (2.3), and contract this with respect to i and j , and by using (1.1e), we have

$$D_{00} = 2LF^2 \quad (2.8)$$

Substituting (2.8) in (2.7), we have

$$R = 2(n-1)LF^2$$

Thus we state the following.

Theorem 2.2. *If the Rund's h-curvature tensor has the special form (2.1), then the scalar curvature of the space is $2(n-1)LF^2$.*

Let us suppose that F^n is R3-like C-reducible Finsler space. Then, by using (1.3) and (2.3), the h-curvature tensor (2.2) can be written as

$$\begin{aligned}
R_{ljk}^i &= K_{ljk}^i + C_{lr}^i R_{jk}^r \\
R_{ljk}^i &= U_{(jk)} \{ A_{jk} h_l^i + D_{lk} h_j^i + E_j^i h_{kl} \} \\
&\quad + \{ (\delta_j^r L_k + y_k L_j^r - \delta_k^r L_j - y_j L_k^r) (C^i h_{lr} + C_l h_r^i + C_r h_l^i) / (n+1) \} \\
R_{ljk}^i &= \{ A_{jk} + (2C_j Ly_k) / (n+1) \} h_l^i - \{ A_{kj} + (2C_k Ly_j) / (n+1) \} h_l^i \\
&\quad + \{ D_{lk} + 2C_l Ly_k / (n+1) \} h_j^i - \{ D_{lj} + 2C_l Ly_j / (n+1) \} h_k^i \\
&\quad + \{ E_j^i - 2C^i Ly_j / (n+1) \} h_{kl} - \{ E_k^i - 2C^i Ly_k / (n+1) \} h_{jl} \\
R_{ljk}^i &= U_{(jk)} [\{ A_{jk} + 2C_j Ly_k / (n+1) \} h_l^i + \{ D_{lk} + 2C_l Ly_k / (n+1) \} h_j^i \\
&\quad + \{ E_j^i - 2C^i Ly_j / (n+1) \} h_{kl}] \\
R_{ljk}^i &= U_{(jk)} \{ Q_{jk} h_l^i + N_{lj} h_k^i + M_j^i h_{kl} \}
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
Q_{jk} &= \{ A_{jk} + 2C_j Ly_k / (n+1) \} \\
N_{lk} &= \{ D_{lk} + 2C_l Ly_k / (n+1) \} \\
M_j^i &= \{ E_j^i - 2C^i Ly_j / (n+1) \}
\end{aligned}$$

Thus we have the following.

Theorem 2.3. *In an R3-like C-reducible Finsler space, if the Rund's h-curvature tensor has the special form (2.3), then the Cartan h-curvature tensor R_{ljk}^i has the special form (2.9).*

Using the special form of Rund's h-curvature tensor K_{ljk}^i in the Bianchi identity (1.6d), we get

$$\begin{aligned}
& (A_{jk} - A_{kj} + D_{kj} - D_{jk}) h_l^i + (A_{kl} - A_{lk} + D_{lk} - D_{kl}) h_j^i \\
& + (A_{lj} - A_{jl} + D_{jl} - D_{lj}) h_k^i = 0
\end{aligned} \tag{2.10}$$

Due to Lemma 1.1, equation (2.10) can be written as

$$A_{jk} - A_{kj} = D_{jk} - D_{kj}$$

Thus we have the following.

Theorem 2.4. *If the Rund's h-curvature tensor K_{ljk}^i is of the special form (2.3), then both the tensor fields A_{ij} and D_{ij} are symmetric simultaneously.*

It is also known that a Finsler space is Landsberg space with $P_{ijk} = C_{ijk/0} = 0$. If F^n is Landsberg, then from (1.6a) and (1.6e), we get

$$R_{jk||l}^i - K_{ljk}^i = 0 \quad \text{or} \quad H_{ljk}^i = K_{ljk}^i$$

Thus we can propose the following.

Corollary 2.5. *If F^n is a Landsberg space and the Rund's h-curvature tensor K_{ljk}^i is of the form (2.3), then Cartan curvature tensor coincides with the Berwald's curvature tensor.*

Now consider h-curvature tensor (1.6a) of the form

$$R_{jk||l}^i = K_{ljk}^i - U_{(jk)} \{P_{jr}^i P_{kl}^r + P_{kl|j}^i\}$$

From equation (1.6e), the above equation can be written as

$$H_{ljk}^i = K_{ljk}^i - U_{(jk)} \{P_{jr}^i P_{kl}^r + P_{kl|j}^i\} \quad (2.11)$$

Suppose F^n is a P-reducible Finsler space, then by using (1.4), (1.5), (2.3), and (2.11), we have

$$\begin{aligned} H_{ljk}^i &= U_{(jk)} \{A_{jk} h_l^i + D_{lk} h_j^i + E_j^i h_{kl}\} - U_{(jk)} \{P_{jr}^i P_{kl}^r + P_{kl|j}^i\} \\ H_{ljk}^i &= \{A_{jk} h_l^i + D_{lk} h_j^i + E_j^i h_{kl} - A_{kj} h_l^i - D_{lj} h_k^i - E_k^i h_{jl}\} \\ &\quad - U_{(jk)} \{(G^i h_{jr} + G_j h_r^i + G_r h_j^i)(G^r h_{kl} + G_k h_l^r + G_l h_k^r) + (G_{k|j} h_l^i + G_{l|j} h_k^i + G_{|j}^i h_{kl})\} \\ H_{ljk}^i &= \{A_{jk} h_l^i + D_{lk} h_j^i + E_j^i h_{kl} - A_{kj} h_l^i - D_{lj} h_k^i - E_k^i h_{jl}\} - (G_{k|j} h_l^i + G_{l|j} h_k^i + G_{|j}^i h_{kl}) \\ &\quad + (G_{j|k} h_l^i + G_{l|k} h_j^i + G_{|k}^i h_{jl}) - \{G^i G_r h_{jr} h_{kl} + G^i G_k h_{jl} + G_j G^r h_r^i h_{kl} + G_j G_l h_k^i \\ &\quad + G_r G^r h_j^i h_{kl} + G_r G_k h_j^i h_l^r + G_r G_l h_j^i h_k^r - G^i G_r h_{kr} h_{jl} - G^i G_j h_{kl} - G_k G^r h_r^i h_{jl} \\ &\quad - G_k G_l h_j^i - G_r G^r h_k^i h_{jl} - G_r G_j h_k^i h_l^r - G_r G_l h_j^i h_k^r\} \\ H_{ljk}^i &= U_{(jk)} \{T_{jk} h_l^i + M_{lk} h_j^i + N_j^i h_{kl}\} \end{aligned} \quad (2.12)$$

where

$$\begin{aligned} T_{jk} &= A_{jk} - g_{k|j} \\ M_{lk} &= D_{lk} + G_{l|k} + G_k G_l - G_r G^r h_{kl} - G_r G_k h_l^r - G_r G_l h_k^r \\ N_j^i &= E_j^i - G_{|j}^i - G^i G^r h_{jr} - G_j G^r h_r^i + G^i G_j \end{aligned}$$

Thus we have the following.

Theorem 2.6. *In a P-reducible Finsler space, and the special form of Rund's h-curvature tensor K_{ljk}^i has the special form of Berwald's curvature tensor, then H_{ljk}^i is of the form (2.12).*

Consider the Bianchi identity

$$H_{ljk}^i + H_{jkl}^i + H_{klj}^i = 0 \quad (2.13)$$

Substituting (2.12) in (2.13), we get

$$\begin{aligned} &[(T_{jk} - T_{kj} + M_{kj} - M_{jk}) h_l^i + (T_{kl} - T_{lk} + M_{lk} - M_{kl}) h_j^i \\ &\quad + (T_{lj} - T_{jl} + M_{jl} - M_{lj}) h_k^i] = 0 \end{aligned} \quad (2.14)$$

Due to Lemma 1.1, equation (2.14) can be written as

$$T_{jk} - T_{kj} = M_{jk} - M_{kj}$$

Thus we have the following.

Theorem 2.7. *If in a Finsler space F^n the h -curvature tensor H_{ljk}^i is of the form (2.12), then the tensor fields T_{jk} and M_{kj} both are simultaneously symmetric.*

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