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A Report on Null Horizons in Relativity

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Abstract

This is a review paper on isolated, distorted and time-dependent null horizons by providing up-date information on their role in black hole physics. Geometry of totally umbilical null hyper surfaces has been used to establish an interrelation between these three types of horizons in a unified manner. Distorted horizons describe the near isolated black holes which are dis- torted by the presence of faraway matter. On the other hand, time-dependent null horizons are modeled by a family of totally umbilical null hyper surfaces. A sketch of the proofs of the most important results is presented together with sufficient related references.

Keywords: Totally umbilical null hyper surfaces; Mean curvature; Null horizons; Black hole

Introduction

It is well-known that null hyper surfaces play an important role in the study of a variety of black hole horizons. Shortly after Einstein's first version of the theory of gravitation was published, in 1916 Karl Schwarzschild computed the gravitational fields of stars using Einstein's field equations. He assumed that the star is spherical, gravitationally collapsed and non-rotating. His solution is called a Schwarzschild solution which is an exact solution of static vacuum fields of the point-mass. Since then, considerable work has been done on black hole physics of time- independent and time-dependent space times. The purpose of this paper is to report up-to-date information on this most active area of black hole physics. To include a large number of results and at the same time not to repeat the known material, we quote all the main results with either sketch or reference for their proof. For easily readable to a large audience, technical details are minimized. Hopefully, this paper will serve as a reference for those working on some aspects of geometry and physics of null horizons and also stimulate further research in black hole physics.

Event Horizons

A boundary of a space time is called an event horizon (briefly denoted by EH) beyond which events cannot affect the observer. An EH is intrinsically a global concept since its definition requires the knowledge of the whole space time to determine whether null geodesics can reach null infinity. EHs have played a key role and this includes Hawking's area increase theorem, black hole thermodynamics, black hole perturbation theory and the topological censorship results. Most important family is the Kerr-Newman black holes. Moreover, an EH always exists in black hole asymptotically flat space time under a weak cosmic censorship condition. We refer Hawking's paper on "Event Horizon" [1], three papers of Hajiceke's work [2-4] on "Perfect Horizons" (later called "non-expanding horizons" by Ashtekar et al. [5]. Galloway [6] has shown that the null hyper surfaces which arise most naturally in space time geometry and general relativity, such as black hole EHs, are in general C^0 but not C^1. His approach has its roots in the well-known geometric maximum principle of E. Hopf, a powerful analytic tool which is often used in the theory of minimal or constant mean curvature hyper surfaces. This principle implies that two different minimal hyper surfaces in a Riemannian manifold cannot touch each other from one side. A published proof of this fact is not available, however, for a special case of Euclidean spaces [7]. To understand Galloway's work, we first recall some features of the intrinsic geometry of a 3-dimensional null hyper surface, say Σ, of a space time (M, g) where the metric g_{ij} has signature (–, +, +, +). Denote by $q_{ij} = \underleftarrow{g_{ij}}$ the intrinsic degenerate induced metric on Σ which is the pull back of g_{ij}, where an under arrow denotes the pullback to Σ. Degenerate q_{ij} has signature (0, +, +) and does not have an inverse in the standard sense, but, in the weaker sense it admits an inverse q^{ij} if it satisfies $q_{ik}\, q_{jm} q^{km} = q_{ij}$. Using this, the expansion $\theta_{(\ell)}$ is defined by $\theta_{(\ell)} = q_{ij}\nabla_i l_j$, where l^i is a future directed null normal to Σ and ∇ is the Levi-Civita connection on M . The vorticity-free Raychaudhuri equation is given by:

$$\frac{d(\theta_{(\ell)})}{ds} = -R_{ij}\ell^i\ell^j - \sigma_{ij}\sigma^{ij} - \frac{\theta^2}{2},$$

where $\sigma_{ij} = \underleftarrow{\nabla}(i\,\ell_j) - \frac{1}{2}\theta_{(\ell)}q_{ij}$ is the shear tensor, s is a pseudo-arc parameter such that l is null geodesic and R_{ij} is the Ricci tensor of M. We say that two null normal ℓ^i and $\tilde{\ell}^i$ belong to the same equivalence class $[\ell]$ if $\tilde{\ell}^i = c\ell^i$ for some positive constant c. Also we need the following form of a second order quasi-linear elliptic operator: Let $\Omega \subset R^n$ be connected open sets and $U \subset R^n \times R \times R^n$. We say that $\mu \in C^2(\Omega)$ is U-admissible if $(x, \mu(x), \partial_\mu) \in U$ for all $x = (x^1, \cdots, x^n) \in U$, where $\partial_\mu = (\partial_1\mu, \cdots, \partial_n\mu)$ and $\partial_i\mu = \partial\mu / \partial x^i$. For a U -admissible $\mu \in C^2(\Omega)$, let

$$Q = Q(\mu) = \sum_{i,j=1}^{n} a^{ij}(x, \mu, \partial\mu)\partial_{ij}\mu + b(x, \mu, \partial\mu),$$

where $a^{ij}, b \in C^1(U)$, $a^{ij} = a^{ji}$, and $\partial_{ij} = \frac{\partial^2}{\partial^j\partial^i}$. Then, Q is a second order quasi-linear elliptic operator if for each $(x, r, p) \in U$, and $\forall \xi = (\xi^i, \cdots, \xi^n) \in R^n$, $\xi \neq 0$,

$$\sum_{i,j=1}^{n} a^{ij}(x, r, p)\xi^i\xi^j > 0.$$

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Now we quote the following classical result on strong maximum principle for second order quasi- linear elliptic PDE's.

Theorem 1: Alexandrov [8] Let $Q = Q(\mu)$ be a second order quasi-linear elliptic operator. Suppose the U-admissible functions $\mu, \nu \in C^2(\Omega)$ satisfy,

(a) $\mu \le \nu$ on Ω and $\mu(x_0)$ for some $x_0 \in \Omega$, and

(b) $\Omega(\nu) \le \Omega(\mu)$ on Ω.

Then, $\mu = \nu$ on Ω.

In year 2000, Galloway [6] proved the following result for smooth null hyper surfaces restricted to the zero mean curvature case and suitable to asymptotically flat space times.

Theorem 2: Let Σ_1 and Σ_2 be smooth null hyper surfaces in a space time manifold M. Suppose,

(1) Σ_1 and Σ_2 meet at $p \in M$ and Σ_2 lies to the future side of Σ_1 near p

(2) The null mean curvature scalars θ_1 of Σ_1, and θ_2 of Σ_2, satisfy $\theta_2 \le 0 \le \theta_1$.

Then Σ_1 and Σ_2 coincide near p and this common null hyper surface has mean curvature $\theta = 0$.

Sketch of the proof: Let Σ_1 and Σ_2 have a common null direction at p and let P be a time like hyper surface in M passing through p and transverse to this direction. Take P so small such that the intersections

$$H_1 = \Sigma_1 \cap P \; and H_2 = \Sigma_2 \cap P$$

are smooth space like hyper surfaces in P, with H_2 to the future side of H_1 near p. These two hyper surfaces may be expressed as graphs over a fixed space like hyper surface H in P, with respect to normal coordinates around H. Precisely, let $H_1 = graph(u_1), H_2 = graph(u_2)$ and suppose

$$\theta(u_i) = \theta_i |_{H_i = graph(u_i)}, \quad i = 1, 2 .$$

By suitable normalizing the null vector fields $\ell_i \in \Gamma(TM_i)$ determining θ_1 and θ_2, respectively, a simple computation shows that

$$\theta(u_i) = Q(u_i) + \quad \text{lower order terms} ,$$

where Q is the mean curvature operator on space like graph over H . Thus θ is a second order quasi-linear elliptic operator. In this case we have:

(1) $u_1 \le u_2$, and $u_1(p) = u_2(p)$.

(2) $\theta(u_2) \le 0 \le \theta(u_1)$.

Then above quoted Alexandrov's Theorem 1 implies that $u_1 = u_2$. Thus H_1 and H_2 agree near p. The null normal geodesics to Σ_1 and Σ_2 in M will then also agree. Consequently, Σ_1 and Σ_2 agree near p, which completes the proof.

Remark: Although above maximum principle theorem is for smooth null hyper surfaces, in reality null hyper surfaces occurring in relativity are the null portions of achronal boundaries as the sets $= \partial I^{\pm}(A), A \subset M$ which are always C^0 hyper surfaces and contain non-differentiable points. For example [6], consider one such set $\Sigma = \partial I^{-}(A)$ where A consists of two disjoint closed disks in the t=0 slice of a Minkowski 3-space. This set can be represented as a merging surface of two truncated cones having a curve of non-differentiable points corresponding to the intersection of the two cones, but, otherwise it is a smooth null hyper surface. The most important feature of these C^0 null hyper surfaces is that they are ruled by null geodesics which are either past or future in extendible and contained in the hyper surface. Precisely, a C^0future null hyper surface is a locally achronal topological hyper surface Σ of M which is ruled by in extendible null geodesics. These null geodesics (entirely contained in Σ) are the null geodesic generators of Σ. The important use of Galloway's Theorem 2 is that it extends to an appropriate manner [6] to C^0 null hyper surfaces. Based on this we quote the following physically meaningful Null Splitting Theorem for null hyper surfaces arising in space time geometry:

Theorem 3: [6] Let (M, g) be a null geodesically complete space time which obeys the null energy condition, $Ric(X, X) \ge 0$ for all null vectors X. If M admits a null line η, then, η is contained in a smooth closed achronal totally geodesic null hyper surface Σ.

Physical model: Consider a 4-dimensional stationary space time (M, g) which is chronological, that is, M admits no closed time like curves. It is known [9] that a stationary M admits a smooth 1-parameter group, say G, of isometries whose orbits are time like curves in M. A static space time is stationary with the condition that its time like Killing vector field, say T , is hyper surface orthogonal, that is, there exists a space like hyper surface orthogonal to T. Our model will be applicable to these both types. Denote by M' the Hausdorff and para compact 3-dimensional Riemannian orbit space of the action G. The projection $\pi : M \to M'$ is a principal **R**-bundle, with the time like fiber G. Let $T = \partial t$ be the non - vanishing time like Killing vector field, where t is a global time coordinate on M'. Then, g induces a Riemannian metric g'_M on M' such that

$$M = \mathbf{R} \times M', \; g = -u^2(dt + \eta)^2 + \overline{\pi}^{\star} g'_M, \quad u^2 = -\overline{g}(T, T) > 0 \cdot$$

where η is a connection 1 - form for the **R**-bundle π. It is known that a stationary space- time (M, g) uniquely determines the orbit data (M', g'_M, u, η) as described above, and conversely. Suppose the orbit space M' has a non-empty metric boundary $\partial M' \ne \varnothing$. Consider the maximal solution data in the sense that it is not extendible to a larger domain $(M', g'_M, u', \eta') \supset (M', g'_M, u, \eta)$ with $u' > 0$ on an extended space time M'. Under these conditions, it is known [9] that in any neighborhood of a point $x \in \partial M'$, either the metric g_M or the connection 1 - form η degenerates, or $u \to 0$. The third case implies that the time like Killing vector T becomes null and, there exists a Killing horizon K H = {u → 0} of M, subject to satisfying the hypothesis of Galloway's Null Splitting Theorem 3. Examples: Minkowaski, De Sitter and anti-de Sitter space times.

Physically, one must find those stationary space times M which are geodesically complete, chronological and their orbit space M' has a non-empty metric boundary $\partial M'$. The last condition is necessary for the existence of null hyper surfaces as EHs of such a space time M. For this purpose, we quote the following result of Anderson.

Theorem 4: [10] Let (M, g) be a geodesically complete, chronological, stationary vacuum space- time. M is the flat Minkowski space $\mathbf{R}_1^4$, or a quotient of Minkowski space by a discrete group Γ of isometries of $\mathbf{R}^3$, commuting with G. In particular, M is diffeomorphic to $\mathbf{R} \times M', d\theta = 0$, with constant u.

Thus, Anderson's above result implies that only a non-flat M will have a non-empty metric boundary of its orbit space. It turns out that asymptotically flat space times are best physical systems for the non-flat stationary space times, many of them do have Killing horizons.

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Distorted horizons in vacuum

Black holes which retain the time-independent character but are non-isolated due to the presence of some external distribution of matter in the neighborhood of a static or stationary black hole are called "Distorted Horizons" which have played an important role in problems involving black holes immersed in external fields or surrounded by matter rings or black hole collisions. The presence of external matter allows the event horizon to be distorted. In 1982, Geroch-Hartle [11] obtained all exact solutions of Einstein's equations that represent static, axisymmetric distorted horizons using a spacetime (M, g) with Weyl metric (see equation (1)). Topologically, the black holes of these distorted horizons are either $S^2 \times R$ for the spherical case or $S^1 \times R \times R$ for the toroidal case. Geroch-Hartle discussed as follows:

Let $(M = \Sigma \times R, g)$ be a space time with Weyl metric g [12] given by

$$g = -e^{2\psi} dt^2 + e^{2(\psi-\gamma)}(d\rho + dz^2) + e^{-2\psi}\rho d\phi^2 \tag{1}$$

where ψ and γ are functions of ρ and z only and Σ is a connected orientable 3-dimensional Riemannian manifold. It is known that all static axisymmetric solutions to Einstein's equations can be expressed by the above form where Σ is orthogonal to a static Killing field. Geroch- Hartle worked on distorted black holes in vacuum, static axisymmetric space times for which the Einstein's equation for ψ is the Laplace equation

$$\psi_{,\rho\rho} + \rho^{-1}\psi_{,\rho} + \psi_{,zz} = 0$$

in Σ orthogonal to the time like static Killing vector field. With a solution for ψ, the second function γ can be obtained by simple integration of the following remaining field equations:

$\gamma_\rho = \rho\{(\psi_{,\rho})^2 - (\psi_{,z})^2\}$ and $\gamma_{,\rho} = 2\rho\psi_{,\rho}\psi_{,z}$.

They observed that spherical and toroidal are the only two possibilities for the topology of horizon cross section C, that is, ϕ^a either has a zero-point in a sphere and Σ must be topologically $R \times S^2$ or it does not vanish on C (resulting in a torus) and Σ must be topologically $R \times S^1 \times S^1$. In the spherical case, when ρ2 → 0 on the horizon one can show that $\Sigma = \overline{\Sigma} - H$, where H is a segment of the axis in Σ with axial Killing field and $\overline{\Sigma}$ is an open neighborhood of this H.

The function ψ may be singular in $\overline{\Sigma}$ at H. The distorted black hole in the neighborhood of the spherical horizon has the same features as the Schwarzschild holes. On the other hand the geometry of distorted black holes of the toroidal horizon is that of a twisted torus with Killing field and in Σ we have flat tori of constant ρ, each with Killing field of period 2π such that these tori converge when ρ → 0 on the horizon. Again we have $\Sigma = \overline{\Sigma} - H$ where ψ in general singular in $\overline{\Sigma}$ at H. However, it is easy to see from the Weyl metric that a solution will be asymptotically flat with static Killing field $\frac{\partial}{\partial t}$ at infinity, if and only if ψ and γ approach zero at infinity. Consider the spherical case. Let 2 m be the length of the segment H on the axis in Σ along with an axisymmetric solution ψ of Laplace's equation in the neighborhood of H with the same value u at the two ends of H. Set $\psi = \psi_s + \psi_d$, a sum of the Schwarzschild and the distorted functions respectively. Consequently, at the horizon H, $\gamma_d = 2\psi_d - 2u$ so its geometry is uniquely determined by one function $\psi_d(z)$ defined for $-m \le z \le m$. The metric can be expressed in Schwarzschild coordinates and it takes the form

$$ds^2 = -e^{2\psi_d}(1-\frac{2m}{r})dt^2 + e^{2(\gamma_d-\psi_d)}\left((1-\frac{2m}{r})^{\ 1}dr^2 + r^2 d\theta^2\right) + e^{-2\psi_d} r^2 \sin^2\theta d\phi^2. \tag{2}$$

Although the Weyl coordinates cover only the region outside the horizon at r=2 m, the space- time can be extended through this horizon by transforming (2) into Eddington-Finkelstein co-ordinates which we now explain. Consider a new coordinate system *(V, r, θ, φ),* where *V=t + r* is an advanced null coordinate and related to the Schwarzschild coordinate time t_s by $V = t_s + r + 2m\ln|\frac{r}{2m} - 1|$. Then, with respect to the system (t, r, θ, φ) above metric transforms into

$$ds^2 = -e^{2\psi_d}(1-\frac{2m}{r})dt^2 + e^{2\psi_d}\frac{4m}{r}dtdr + e^{2(\gamma_d-\psi_d)}(1+\frac{2m}{r})dr^2$$
$$+ r^2\{e^{2(\gamma_d-\psi_d)}d\theta^2 + e^{-2\psi_d} sin^2\theta d\phi^2\}$$

which is non-singular Eddington-Frankelstein metric for all values of r. Let S be the intersection of a hyper surface r = constant with a hyper surface t = constant, which is 2-dimensional space like surface of M with its Riemannian metric

$$\Sigma_H = 4m^2\{e^{2(\gamma_d-\psi_d)}d\theta^2 + e^{-2\psi_d} sin^2\theta d\phi^2\}$$

representing the distorted horizon H at r = 2m. In [11] the authors explained how the above axisymmetric metric would evolve into the metric of a 2-sphere through a sequence of equilibrium states, examined the local and global structures of distorted and undistorted holes.

Open problem: "Geroch-Hartle [11] proposed following problem (still remains unsolved):

Is there any analogous result for more complicated topology? Suppose the matter were slowly moved from the vicinity of the hole to distant region. How would the hole, which could not, presumably, permit this to happen while retaining its horizon-topology, react? Similarly, what would happen if a spinning gyroscope were dropped into a hole with other than spherical or toroidal topology? The hole could not become rotating consistent with its horizon-topology. To what equilibrium state would it finally settle, and how would it radiate to achieve this state?"

Distorted charged black holes

In 2001, Fairhurst-Krishnan [13] extended the vacuum case in [11] to the solutions of Einstein- Maxwell equations for distorted charged black holes using the Weyl metric (1). We review this case as follows: Let the electromagnetic potential be in the form

$$A = \Phi dt + \beta d\phi,$$

where Φ and β are the electromagnetic and magnetic potentials, respectively. They assumed β=0 (results also hold for $\beta \neq 0$). The field equations are

$$\psi_{,\rho\rho} + \rho^{-1}\psi_{,\rho} + \psi_{,zz} = e^{2\psi}(\Phi^2_{,\rho} + \Phi^2_{,z}),$$

$$(\rho\Phi_{,\rho}e^{2\psi})_{,\rho} + (\rho\Phi_{,z}e^{2\psi})_{,z} = 0,$$

$$\gamma_{,\rho} = \{(\psi_{,\rho})^2 - (\psi_{,z})^2 - e^{2\psi}(\Phi^2_{,\rho} + \Phi^2_{,z})\},$$

$$\gamma_z = 2\rho\{\psi_{,\rho}\psi_{,z} - e^{2\psi}\Phi_{,\rho}\Phi_{,z}\} \tag{3}$$

Above field equations being non-linear in ψ, this prevents from "distorting" the known black hole solutions as in the vacuum case of linear Laplace equation. Therefore, they used a mapping [12] which transforms a distorted vacuum solution of a Schwarzschild family to the Reissner- Nordstrom family. Let *(ψ, γ, b, c)* be a quadruple where ψ and γ satisfy the vacuum Einstein equations and b and c are constants. Construct a solution $(\overline{\psi}, \overline{\gamma}, \Phi)$ to the equations (3) as follows: Consider the potential $\overline{\psi}$ in terms of ψ, a and b by

$$e^{-\bar{\psi}} = \frac{e^{-c}}{2}\left\{\left(1+b(b^2-1)^{-1/2}e^{-\psi}\right)+\left(1+b(b^2-1)^{-1/2}e^{\psi}\right)\right\} \cdot \quad (4)$$

Then it follows from the first two equations in (3) that

$$\frac{d^2(e^{2\bar{\psi}})}{d\Phi^2} = 2 \cdot \quad (5)$$

Using (4) and (5), it is easy to verify that $(\bar{\psi}, \bar{\gamma}, \Phi)$ is a solution to the Einstein-Maxwell equations (3) such that the function ψ from which $\bar{\psi}$ and Φ are obtained satisfies the Laplace equation, which allows to solve (3) just as solving the vacuum Laplace equation for ψ and the equation (5) for γ. This allows one to transform any given vacuum solution to a solution of the electrovac Weyl equations, in particular reference to the distorted Schwarzschild black holes. In [13] the authors have shown that these solutions represent distorted Reissner-Nordstrm solutions. Following is a brief on their construction:

Let m and e be the mass and the charge of the black hole. The metric functions $\bar{\psi}$ and $\bar{\gamma}$ are given by

$$\bar{\psi}_{RN} = \frac{1}{2}\ln\left(\frac{B^2-A^2}{(B+m)^2}\right), \bar{\gamma}_{RN} = \frac{1}{2}\ln\left(\frac{B^2-A^2}{B^2-\eta^2}\right).$$

$$A = \sqrt{m^2-e^2}, \quad \Phi = \frac{e}{B+m}$$

where B and η are function of ρ and z given by

$$B = \frac{1}{2}(l_+ + l_-), \quad \eta = \frac{1}{2}(l_+ - l_-)$$
$$l_+ = \sqrt{\rho^2+(z+A)^2}, \quad l_- = \sqrt{\rho^2+(z-A)^2} \quad .$$

Transform in standard Reissner-Nordstrom coordinates *(t, r, θ, φ)* by

$$r = B+m, \quad \cos\theta = \frac{1}{2A}(l_+ - l_-),$$

$$z = B\cos\theta, \quad \rho^2 = (B^2-A^2)\sin\theta \cdot$$

For the distorted Reissner-Nordstrom case we obtain

$$\bar{\psi} = \bar{\psi}_{RN} + \overline{\psi_d},$$

$$e^{2\bar{\psi}} = \Delta(r)e^{2\overline{\psi_d}}, \quad \Delta(r) = \left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right), \quad (6)$$

where we choose b = m/e and c a free parameter. It follows from (6) that

$$\bar{\gamma}_d = 2\bar{\psi}_d - 2(\bar{u}+c)$$

at the horizon of a charged black hole. Using above, the metric of the distorted Reissner- Nordstrom space time is given by

$$ds^2 = -\Delta(r)e^{2\bar{\psi}_d}dt^2 + \frac{e^{2(\bar{\gamma}_d-\bar{\psi}_d)}}{\Delta r}dr^2 + e^{2(\bar{\gamma}_d-\bar{\psi}_d)}r^2 d\theta^2 + e^{-\bar{\psi}_d}r^2\sin^2 d\phi^2 \cdot \quad (7)$$

The horizon is a line segment $\bar{H}$ on the z-axis with |z| ≤ A. Although the Weyl coordinates cover only the region outside the horizon at r = m + A, the spacetime can be extended through this horizon by transforming (7) into Eddington-Finkelstein coordinates for which the metric is regular at *Δ(r = m + A) = 0.* To do this transformation we set $dw = dt + e^{-2(\bar{u}+c)}dr/\Delta(r)$. Following is the transformed metric in *(w, r, θ, φ)* coordinates

$$ds^2 = -\Delta(r)e^{2\bar{\psi}_d}dwdr + e^{2\bar{\psi}_d}\lambda dr^2 + e^{2(\bar{\gamma}_d-\bar{\psi}_d)}r^2 d\theta^2 + e^{-2\bar{\psi}_d}r^2\sin^2 d\phi^2,$$

where $\lambda = (e^{2\bar{\gamma}_d - 4\bar{\psi}_d} - e^{4(\bar{u}+c)})/\Delta(r)$. This is the metric regular at the horizon Δ(r) = 0 whose horizon geometry is given by the metric

$$h_{\bar{H}} = (m+A)^2\left(e^{2(\bar{\gamma}_d-\bar{\psi}_d)}d\theta^2 + e^{-\bar{\psi}_d}\sin^2 d\phi^2\right).$$

These solutions represent a charged black hole distorted by external matter. Also, these solutions (although regular at the horizon) are not asymptotically flat and so the notion of infinity and an event horizon is not applicable for this case. However, these solutions do admit locally defined isolated horizons [14] (see details on isolated horizons in next section). Nevertheless, there is a way [13] to show that (under some reasonable conditions) this solution can be extended to be asymptotically flat, in which case the horizon will be the event horizon.

They discussed the zeroth and first law for these black holes and proved the first law in two different forms, one using the isolated horizon framework and the other using normalization at infinity. They also suggested that the isolated horizon framework provides a clearer interpretation of the first law for these black holes.

Isolated Horizons

Since to actually locate a black hole one needs to know the full space time metric up to the infinite future and even if one locates the event horizon (EH), using it to calculate the physical parameters is extremely difficult. Also EH is too global to be useful in a number of physical situations ranging from quantum gravity to numerical relativity and to astrophysics. In particular, since it refers to infinity, it cannot be used in especially compact space time. Therefore, attempts were made to find a quasi-local concept of a horizon which requires only minimum number of conditions to detect a black hole and study its properties. To achieve this objective, in a 1999 paper [5] Asktekar et al. introduced the following three notions of isolated horizons, namely, non-expanding, weakly and stronger isolated horizons, respectively:

Definition 5: A null hyper surface *(H, q)* of a 4-dimensional space time (M, g) is called a non-expanding horizon (NEH) if

(1) H has a topology $R \times S^2$,

(2) Any null normal ℓ of H has vanishing expansion, $\theta_{(\ell)} = 0$

(3) All equations of motion hold at H and stress energy tensor T_{ij} is such that $-T^i_j\ell^j$ is future- causal for any future directed null normal ℓ^i.

The condition (1) is a restriction on topology of H which guarantees that marginally trapped surfaces are related to a black hole space time. The condition (2) and the energy condition of (3) imply from the Raychaudhuri equation that $\underset{\leftarrow}{T_{ij}\ell^j} = 0$ and $\underset{\leftarrow}{\nabla_{(i}\ell_{j)}} \equiv £_\ell q_{ij} = 0$ on H, which further implies that the metric q_{ij} is time independent. Note that $£_\ell q_{ij} = 0$ on H does not necessarily imply that *l* is a Killing vector of the full metric g_{ij}. In general, there does not exist a unique induced connection on H due to degenerate q_{ij}. However, on an NEH, the property $£_\ell q_{ij} = 0$ implies that the space time connection∇induces a unique (torsion-free) connection say D, on H which is compatible with q_{ij}.

Definition 6: The pair $(H,[\ell])$ is called a weakly isolated horizon (WIH) if H is a NEH and each normal $\ell \in [\ell]$ satisfies

$$(£_\ell D_i - D_i £_\ell)\ell^i = 0$$

Above condition implies that, in addition to the metric q_{ij}, the connection component $D_i\ell^j$ is also time independent for a WIH. Given a NEH, one can always have an equivalence class $[\ell]$ (which is not unique) of null normals such that $(H,[\ell])$ is a WIH. For details on this issue of "Freedom of the choice of ℓ " see Ashtekar-Fairhurst-Krishnan [14].

Definition 7: A WIH $(H,[\ell])$ is called an isolated horizon (IH) if the full connection D is time- independent, that is, if

$$(\pounds_\ell D_i - D_i \pounds_\ell) V^j = 0$$

for arbitrary vector fields V tangent to H.

An IH is stronger notion of isolation as its above condition cannot always be satisfied by a choice of null normal. IHs are quasi-local and do not require the knowledge of the whole space time. Any Killing horizon which is topologically $R \times S^2$ is a trivial example of an IH [15-17] for examples and their physical use.

On the other hand, we know that the isolated horizons model specifically quasi-local equilibrium regimes of black hole space times. However, in nature, black holes are rarely in equilibrium. This led to research on a quasi-local frame work to describe the geometry of the surface of the black hole, not just at its equilibrium state. First attempt in this direction was made by Hayward [18], in 1994, using the frame work of (2 + 2) - formalism, based on the notion of trapped surfaces. He proposed the following notion of future, outer, trapping horizons (FOTH).

Definition 8: A future, outer, trapped horizon (FOTH) is a three manifold Σ, foliated by family of closed 2-surfaces such that (i) one of its future directed null normal, say ℓ , has zero expansion, $\theta_{(\ell)} = 0$; (ii) the other null normal, **k**, has negative expansion $\theta_{(\mathbf{k})} < 0$ and (iii) the directional derivative of $\theta_{(\ell)}$ along **k** is negative; $\pounds_{\mathbf{n}} \theta_{(\ell)} < 0$.

Σ is either space like or null for which $\theta_{(\mathbf{k})} = 0$ and $\pounds_{\mathbf{k}} \theta_{(\ell)} = 0$. After this, Asktekar-Krishnan [19] observed that in dynamical situations Hayward's condition (iii) is not required for most of the key physical results. For this reason, they studied the following quasi-local concept of dynamical horizons (briefly denoted by DH) which model the evolving black holes and their asymptotic states are isolated horizons.

Definition 9: A smooth, 3-dimensional space like submanifold (possibly with boundary) H of a space time is said to be a dynamical horizon (DH) if it can be foliated by a family of closed 2-manifolds such that

1. On each leaf S its future directed null normal ℓ has zero expansion, $\theta_{(\ell)} = 0$

2. And the other null normal, k, has negative expansion $\theta_{(\mathbf{k})} < 0$.

They first required that H be space like everywhere and then studied the case in which portions of marginally trapped surfaces lie on a space like horizon and the remainder on a null horizon. In the null case, H reaches equilibrium for which the shear and the matter flux vanish and this portion is represented by a weakly isolated horizon. The Vaidya metrics are explicit examples of dynamical horizons with their equilibrium states-the isolated horizons. The horizon geometry of DHs is time-dependent. Compared to Hayward's (2 + 2)-formalism, the DH frame work is based on the standard (1 + 3)-formulism and has the advantage that it only refers to the intrinsic structure of H, without any conditions on the evolution of fields in transverse directions to H. DH has provided a new perspective covering all areas of black holes, that is, quantum gravity, mathematical physics, numerical relativity and gravitational wave phenomenology, leading to the underlying unity of the subject. Since in this paper we only focus on null horizons, we refer [19] for a review on dynamical horizons.

Time-Dependent Null Horizons

We know from discussion so far that the event and isolated horizons have a common condition that their future null normal has vanishing expansion, that is, their underlying null hyper surface is totally geodesic in the corresponding space time. Moreover, these horizons are time-independent. However, in reality the black hole has a cosmological background or it is surrounded by a local mass distribution. Also, black hole grows by swallowing star and galactic debris and electromagnetic as well as gravitational radiation. Therefore, it may cease to be time-independent. Thus, there is significant difference in the structure and properties of such fully dynamical black holes from the well-known event and isolated black holes. For this reason we now review some works done on finding realistic models of non-isolated and time-dependent black hole space times. Among other approaches to get such a model, this raises the possibility of using a metric symmetry from Killing to conformal Killing symmetry. Recall a space time (M, g) admits a conformal Killing vector field (briefly denoted by CKV) ℓ if $\pounds_\ell g = 2\phi g$, where φ is a function on M. Using this metric symmetry, we first review two papers of Sultana and Dyer [20,21] as follows:

Conformal killing horizons

In their first paper [20], Sultana and Dyer have studied this problem for those space times which admit a CKV field. They considered a conformal transformation, $G = \Omega^2 g$, to stationary asymptotically flat black hole space times which admits a Killing horizon, say Σ_0 , generated by the Killing vector field ℓ , where Ω is a conformal function. Under such a conformal transformation ℓ is mapped to a CKV. Although space times are asymptotically conformally flat, nevertheless, there can be non-asymptotically flat space times. In this particular paper [20] Sultana-Dyer considered space times admitting a time like CKV which becomes null on a boundary called the conformal stationary limit hyper surface and locally described the time-dependent event horizon by using this boundary, provided that it is a null geodesic hyper surface. Such a horizon is called a conformal Killing horizon, briefly denoted by CKH). They have shown that such a hyper surface of M is null geodesic if and only if the twist of the conformal Killing trajectories on M vanishes. Following is their main result on the extension of the Hawking's strong rigidity theorem [9].

Theorem 10: [20] Let (M, G) be a space time which is conformally related to an analytic black hole space time (M, g), with a Killing horizon Σ_0, such that the conformal factor in $G = \Omega^2 g$ goes to a constant at null infinity. Then the CKH in (M, G) is globally equivalent to the event horizon, provided that the stress energy tensor satisfies the weak energy condition.

Sketch of proof: The global definition of the event horizon and the properties of conformal transformations imply that the global definition of an event is conformally invariant, provided the conformal factor tends to a constant at null infinity. This means that, at the null infinity state, the CKV field reduces to the homothetic vector (HV) field. Since, as opposed to the proper conformal symmetry, the Einstein equations are invariant with respect to homothetic symmetry, the structure of homothetic infinity in (M, G) is preserved. This means that in the manifold (M, g), the event horizon is a Killing horizon; while in the conformal manifold (M, G) it is a CKH which reduces to an event horizon at null infinity that completes the proof.

This paper also contains the case as to what happens when the conformal stationary limit hyper surface does not coincide with the event horizon at infinity. For this case, they have proved a generalized weak

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rigidity theorem which establishes the conformal Killing property of the event horizon and the rigidity of its rotating conformal Killing horizons.

In another paper [21], they gave an example of a dynamical cosmological black hole, which is a space time that describes an expanding black hole in the asymptotic background of the Einstein-de Sitter universe. For this case, the black hole is primordial in the sense that it forms Ab Initio with the big bang singularity and its expanding event horizon is represented by a CKH whose conformal factor goes to a constant at null infinity. The metric representing the black hole space time is obtained by applying a time-dependent conformal transformation on the Schwarzschild metric, such that the result is an exact solution with the matter content described by a perfect fluid and the other a null fluid. They have also studied properties of several physical quantities related to black holes.

Maximum principle for time-dependent null horizons

Observe that Galloway's maximum principle (see Theorem2) was designed for totally geodesic null hyper surfaces suitable only for asymptotically flat space times. Recently, the present author has modified maximum principle suitable for a family of time-dependent totally umbilical null horizons of a large variety of space times not necessarily restricted to asymptotically flat space times, which we present in this section. Recall that a hyper surface (Σ, h) of a space time manifold (M, g) is totally umbilical if and only if there is a smooth function ρ on Σ such that B = ρh, where B denotes the second fundamental form of Σ and h is its induced metric. Σ is proper totally umbilical if and only if ρ is non-zero on entire Σ. In particular, Σ is totally geodesic if and only if B vanishes, i.e., if and only if ρ vanishes on Σ. Let (M, g) be an n-dimensional space time manifold evolved from a space like hyper surface H_t at a coordinate time t to another space like hyper surface H_{t+dt} at coordinate time t + dt whose metric g is given by

$$ds^2_{|g} = g_{ij}dx^i dx^j = (-\lambda^2 + |U|^2)dt^2 + 2\gamma_{ab}U^a dx^b dt + \gamma_{ab}dx^a dx^b \,, \quad (8)$$

where $x^0 = t$, and $x^a (a = 1, \cdots, n-1)$ are spatial coordinates of the hyper surface H_t with γ_{ab} its (n − 1)-metric induced from g, $\lambda = \lambda(t, x^1, \cdots, x^{n-1})$ is the lapse function and U is a space like shift vector. The coordinate time vector $\mathbf{t} = \frac{\partial}{\partial t}$ is such that $g(dt, \mathbf{t}) = 1$. We write

$$\mathbf{t} = \lambda\mathbf{n} + U \,, \text{ with } \mathbf{n}.U = 0 \,,$$

where **n** is the future time like unit vector field. In general, each space like hyper surface H_t intersects a null hyper surface say Σ on some (n − 2)-dimensional sub manifold $\mathbf{S}_t$, that is, $\mathbf{S}_t = \Sigma \cap H_t$. Consider a family $F = ((\Sigma_u),(h_u))$ with $\mathbf{S}_{t,u} = \Sigma_u \cap H_t$, where $\mathbf{S}_{t,u}, u = constant$, is an element of the family $(\mathbf{S}_{t,u})$ and (h_u) the respective family of degenerate metrics of (Σ_u). Let ℓ be the future directed null normal which is not entirely in a member Σ of the family F , but, is defined in some open subset of M around that Σ. Denote $\mathbf{s} \in H_t$ by a unit vector field normal to $\mathbf{S}_t$ defined in some open neighborhood of Σ. Taking $(\mathbf{S}_{t,u})$ a foliation of (Σ_u), the coordinate t can be used as a non-affine parameter along each null geodesic generating each Σ_u. We normalize each member ℓ of F such that it is tangent vector associated with this parameterization of the null generators, i.e., $\ell^i = \frac{dx^i}{dt}$. This means that ℓ is a vector field "dual" to the 1-form dt. Equivalently, the function t can be regarded as a coordinate compatible with ℓ, i.e., $g(dt, l) = \Delta_l t = 1$. Based on this, we choose the following normalization of each $\ell_u \in (\ell_u)$ for some value of u.

$$\ell_u = (\mathbf{n}u + \mathbf{s}u) \,, \text{ where, } \mathbf{s}_u . s_u = 1, \ \mathbf{x} \in T_p(\mathbf{S}_t) \Leftrightarrow \mathbf{s}_u . \mathbf{x} = 0 \,,$$

which implies that each ℓu is tangent to each member of F and it has the property of Lie dragging the family of surfaces $(\mathbf{S}_{t,u})$. Then, we define a transversal vector field $\mathbf{k}_u$ of T_pM not belonging to F expressed as another suitable linear combination of n_u and s_u such that it represents the light rays emitted in the opposite direction, called the ingoing direction, satisfying:

$$g(\ell_u, \mathbf{k}_u) = -1, \quad \mathbf{k}_u = \frac{1}{2}(\mathbf{n}_u - \mathbf{s}_u) \,.$$

We now state and reproduce complete proof of the following modified maximum principle.

Theorem 11: Let Σ_{u_1} and Σ_{u_2} be two members of a family $F = ((\Sigma_u),(h_u))$ of totally umbilical smooth null hyper surfaces of a space time (M, g) whose metric g is given by (8). Suppose

(1) Σ_{u_1} and Σ_{u_2} meet at $p \in M$ and Σu_2 lies to the future side of Σu_1 near p

(2) The mean curvatures θu1 of , and θ_{u_2} of Σ_{u_2}, satisfy, $\theta_{u_2} \le \theta_u \le \theta_{u_1}$ for some u.

Then $\Sigma_{u_1} = \Sigma_{u_2} = \Sigma_u$ near p and θ_u (which may vanish either on a portion of Σ_u or on entire Σ_u) is the mean curvature of this common null hyper surface Σ_u for some u.

Proof: Let $\Sigma \in (\Sigma_u)$ be a null hyper surface of M and $\mathbf{S}_t \in (\mathbf{S}_{t,u})$ a space like hyper surface of Σ. Suppose Σ and H_t meet at a point $p \in M$ properly transversely in $\mathbf{S_t}$. Take a space like hyper surface $\mathbf{V}_t$ of H_t passing through p such that $x = (x_1, \cdots, x_{n-2})$ are its coordinates centered on p. Express $\mathbf{S}_t$ as a graph over $\mathbf{V}_t$, that is, S_t = *graph* μ = *{(μ(x), x)* ∈ Σ ∩ H_t *: x* ∈ V_t*}*, where $\mu \in C^\infty(V_t)$. Let Q(μ) be the mean curvature of S_t = graph μ and G be the Riemannian metric on $\mathbf{V}_t$ whose components are given by $G_{\alpha\beta}(x) = \gamma_{\alpha\beta}(\mu(x), x)$. Then, It is easy to see that the following expression of *Q(μ)* will hold:

$$Q = Q(\mu) = \sum_{\alpha,\beta=1}^{n-2} a^{\alpha\beta}(x, \mu, \partial\mu)\partial_{\alpha\beta}\mu + b(x, \mu, \partial\mu) \,,$$

where μ is a C^2 U -admissible function, $a^{\alpha\beta}$, $b \in C^1(U)$. The operator Q will also satisfy as elliptic operator and, therefore, $Q = Q(\mu)$ is a second order quasi-linear elliptic operator. Take $\ell = (\mathbf{n} + \mathbf{s})$ a future null normal vector field on Σ. Denote by B_H and B_S the second fundamental forms of H_t and S_t, respectively. Then,

$$BH(\mathbf{x},\mathbf{y}) = \langle \nabla_\mathbf{x}\mathbf{n}, \mathbf{y} \rangle, \quad B\mathbf{S}(\mathbf{x},\mathbf{y}) = \langle \overline{\nabla}_\mathbf{x}\mathbf{s}, \mathbf{y} \rangle, \quad \forall \mathbf{x},\mathbf{y} \in T_q\mathbf{S}_t, \quad q \in \mathbf{S}_t$$

and $\overline{\nabla}$ is an induced metric connection on $\mathbf{S}_t$. Then, with respect to an orthonormal basis $\{e_1, \cdots, e_{n-2}\}$ for $T_q\mathbf{S_t}$ the value of θ at q is given by

$$\theta = \sum_{\alpha=1}^{n-2} h(\nabla_\alpha \ell, e_\alpha) = \sum_{\alpha=1}^{n-2} \langle \nabla_\alpha \mathbf{n}, e_\alpha \rangle + \sum_{\alpha=1}^{n-2} \langle \nabla_\alpha \mathbf{s}, e_\alpha \rangle$$

$$= \sum_{\alpha=1}^{n-2} B_H(e_\alpha, e_\alpha) + \sum_{\alpha=1}^{n-2} B_S(e_\alpha, e_\alpha) = Q_H + B_H(\mathbf{s},\mathbf{s}) + Q_\mathbf{S} \,,$$

Let θ(μ) be the null mean curvature of Σ along $\mathbf{S}_t$ = graph μ. It is straightforward to show that $\theta = \theta(\mu)$ is a second order quasi-linear elliptic operator. Now consider Σ_{u_1} and Σ_{u_2} two null hyper surfaces having a common null direction at p and let H_t in M pass through p and transverse to this direction. Take H_t so small such that the intersections

$$\mathbf{S}_{t,u_1} = \Sigma_{u_1} \cap H_t \text{ and } \mathbf{S}_{t,u_2} = \Sigma_{u_2} \cap H_t$$

are smooth space like hyper surfaces with $\mathbf{S}_{t,u_2}$ to the future side of $\mathbf{S}_{t,u_1}$ near p. As explained above, let $\mathbf{S}_{t,u_1} = graph(\mu_{u_1})$, $\mathbf{S}_{t,u_2} = graph(\mu_{u_2})$ and suppose

$$\theta(\mu_{u_i}) = \theta_{u_i} |_{\mathbf{S}_{t,u_i}=\mathrm{graph}(\mu_{u_i})}, \quad i = 1, 2 .$$

Taking two normalized null normal vector fields $\ell_{u_i} = (\mathbf{n}_{u_i} + \mathbf{s}_{u_i}) \in \Gamma(T\Sigma_{u_i})$, determining θ_{u_1} and θ_{u_2}, respectively (as above), a simple computation shows that $\theta(\mu_{u_i}) = Q(\mu_{u_i}) + \text{lower order terms}$, where Q is the mean curvature operator on space like graphs over $\mathbf{V}_{t_{u_i}}$ in H_t. The lower order terms involve the second fundamental forms of H_t and $\mathbf{S}_{t,u_i}$. Thus each θ_{u_i} is a second order quasi-linear elliptic operator. Consequently, using the hypothesis $\theta_{u_2} \le \theta_u \le \theta_{u_1}$ we have:

(1) $\mu_{u_1} \le \mu_{u_2}$, and $\mu_{u_1}(p) = \mu_{u_2}(p)$.

(2) $\theta(\mu_{u_2}) \le \theta_u \le \theta(\mu_{u_1})$.

Then, using the Alexandrov's strong maximal principle it implies that $\mu_{u_1} = \mu_{u_2} = \mu_u$. Thus, Σ_{u_1} and Σ_{u_2} agree near p. The null normal to Σ_{u_1} and Σ_{u_2} in M will then also agree. Therefore, $\Sigma_{u_1} = \Sigma_{u_2} = \Sigma_u$ near p and θ_u (which may vanish when f_u vanishes on Σ_u or on its portion) is the mean curvature of this common null hyper surface Σ_u.

Consequently, Theorem 11 brings in the role of rich geometry of totally umbilical hyper surfaces of a space time manifold instead of an earlier restricted work of Galloway's Theorem 2 on this problem which was only suitable for totally geodesic null hyper surfaces of asymptotically flat space times. Also, the metric (8) of the working space time (M, g) is physically important. For example, Gourgoulhon and Jaramillo [17] on event and isolated horizons used this metric. More- over, this metric includes the Robertson-Walker (RW) space times which are important models both from mathematical and physical point of view and they further include, among others, the Lorentz-Minkowski space time, the Einstein-de-Sitter space time, the Friedman cosmological models and the static Einstein space time.

However, Theorem 11 is limited by the fact that not every such totally umbilical null hyper- surface can evolve into the vanishing mean curvature totally geodesic null hyper surface which arises, in general relativity, such as black hole event and Cauchy horizon. An example is the family of totally umbilical null cones none of its member can evolve into a totally geodesic hyper-surface Duggal [22] needed to link it with Galloway's Theorems 2 and 3. To achieve this important link we quote the following particular case of Theorem 11.

Theorem 12: Duggal [23] Let (M, g) be a space time with its metric given by (8) such that its coordinate time vector $t = \frac{\partial}{\partial t} = \lambda n + U$ is a conformal Killing vector (CKV) field, that is, $£_t g = 2\sigma g$ for some conformal function σ. Suppose $F = \left((\Sigma_u), (h_u), (\ell_u), (\mathbf{k}_u)\right)$ is a family of totally umbilical null hyper surfaces of (M, g) such that the shift space like vector field U of its each member (Σ, h) is given by $U = \lambda \mathbf{s} - \mathbf{v}$ where λ is the lapse function and **v** belonging to its corresponding space like hyper surface $(S_t, \hat{h})$ is a Killing vector (CKV) field. Then,

(a) $£\ell h(X,Y) = \frac{2\sigma}{\lambda} h(X,Y), \quad \forall X, Y \in T\Sigma$.

(b) **t** Reduces to a Killing vector field if and only if Σ ∈ F is totally geodesic in M.

Proof is quite straightforward. The conclusion (a) implies that the Theorem 11 is valid if **t** is a CKV and $\mathbf{v} \in (S_t, \hat{h})$ is Killing. Also, we know that each Σ ∈ F is totally geodesic if and only if its mean curvature θ vanishes. Therefore, the conclusion (b) establishes a link with Galloway's vanishing mean curvature Theorems 2 and 3.

A physical model of time-dependent null horizons

Let (M, g) be a 4-dimensional space time of general relativity with its metric g defined by (8). Consider a family $F = \left((\Sigma_u), (h_u)\right)$ of 3-dimensional null hyper surfaces of (M, g). The "bending" of each Σ in M (with respect to each ℓ) is described by the Weingarten map:

$$W_\ell : T_p(\Sigma) \to T_p(\Sigma), \quad X \to \nabla_X \ell .$$

W_ℓ associates each X of Σ the variation of £ along X, with respect to the space time connection ∇. The second fundamental form, say B_ℓ with respect to null normal ℓ of Σ is the symmetric bilinear form and is related to the Weingarten map by

$$B_\ell(X,Y) = h(W_\ell X, Y) = h(\nabla_X \ell, Y)$$

From above equation and B_ℓ symmetric implies that

$$B_\ell(X,Y) = \frac{1}{2} £_\ell h(X,Y), \quad \forall X, Y \in T\Sigma . \tag{9}$$

If B_ℓ is conformally equivalent to the metric h, then, we say that (Σ, h) is totally umbilical in M if and only if there is a smooth function f on Σ such that

$$B_\ell(X,Y) = fh(X,Y), \quad \forall X, Y \in T(\Sigma) . \tag{10}$$

In two recent papers of Duggal [22,24] a new class of null hyper surfaces of a space time (M, g),with metric (8), was studied using the following definition:

Definition 13: A null hyper surface (Σ, h, ℓ) of a space time (M, g) is called an Evolving Null Horizon (ENH) if

(i) Σ is totally umbilical in (M, g) and may include a totally geodesic portion.

(ii) All equations of motion hold at Σ and energy tensor T_{ij} is such that $T^i_j \ell^j$ is future-causal for any future directed null normal ℓ.

The condition (i) implies from (9) and (10) that $£_\ell h = 2fh$ on Σ, that is, ℓ is a conformal Killing vector field of the metric h, with conformal function 2f. It is important to note is not necessarily a CKV field of the full metric g. The energy condition of (ii) requires that is non-negative for any ℓ, which implies from page 95 of Hawking and Ellis [9] that $\theta_{(\ell)}$ monotonically decreases in time along ℓ, that is, M obeys the null convergence condition, which further means that the null hyper surface (Σ, h) is time-dependent in the region where $\theta_{(\ell)}$ is non-zero and may evolve into a time-independent totally geodesic hyper surface as a model of event or isolated horizon. Thus, above two implications of the Definition 13 clearly show that there exists a Physical Model of a class $F = \left((\Sigma_u), (h_u)\right)$ of a family of totally umbilical null hyper surfaces of (M, g), satisfying the hypothesis of Theorem 11, such that its each member is a time-dependent evolving null horizon(ENH). We refer [22,24] for details on the geometry and physics of evolving null horizons where there are examples of null cones, Monge null hyper surfaces, Einstein static space time and Schwarzschild space time.

Remark: Observe that Theorem 11 on modified maximum principle is an important step forward towards the ongoing physical use of time-dependent (non-isolated) null horizons of a variety of space times and in some cases their relation with the event and isolated horizons. Also examining the similarity and difference between the Theorem 12 with the two papers of (Sultana and Dyer) (see Subsection 4.1) related to common issue of time-dependent null horizons, it is clear that although their result on time-dependent null conformal Killing horizons is similar with the two conclusions of Theorem 12, but, it is only limited to null hyper surfaces of asymptotically flat space times whereas Theorem 12 is applicable to a variety of space times admitting a time like conformal Killing vector field.

Some Related Results on Black Hole Physics

In view of a very large number of excellent papers appearing in this field we present here a brief on some selected papers closely related to the material presented so far.

(1) On the global structure of solutions (primarily related to event horizons) we refer a review article by Chrusciel [25]. His work included quasi-local mass, strong cosmic censorship, non-linear stability, new construction of solutions of the constraint equations, improved under- standing of asymptotic properties of the solutions, existence of solutions with low regularity, and construction of initial data with trapped surfaces or apparent horizons.

(2) Attempts have been made to extend the black hole mechanics by replacing the use of event horizons in stationary space times with isolated horizons in some dynamical space times and in some cases those space times which admit radiation close to black holes. In a paper by Ashtekar- Beetle-Lewandowski [26] they stated that so far such an extension is restricted to non-rotating black holes. In their paper they have filled this important gap by extending the first law to the rotating case.

(3) It is well-known that a maximally rotating Kerr black hole is said to be external. On this issue we refer a paper (with some related references cited therein) of Booth and Fairhurst [27]. These authors have studied three characterizations of extremality. They presented a way how the standard notions for Kerr black hole do not require the horizon to be either stationary or rotationally symmetric. They studied physical implications and applications of these results. In particular, they have examined how close a horizon is to extremality and should be calculable in numerical simulations.

(4) Kunduri and Lucietti [28] have recently studied a new infinite class of near-horizon geometries of null horizons which satisfy Einstein's equations for all odd dimensions greater than five. The symmetry and topology of these solutions is compatible with those of black holes. They have studied those horizon cross manifolds which all possess Sasakian horizon metrics.

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Conflict of Interests

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