A Problem in Thermoelasticity with and without Energy Dissipation

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Abstract
In this work, we are compared between the theory of thermoelasticity with two relaxation times and the theory of thermoelasticity without energy dissipation. To a half-space overlaid by a thick layer of a different material. The upper surface of the layer is taken to be traction free and is subjected to a constant thermal shock. We are used Laplace transform to eliminate the time variable t. The inverses Laplace transforms are obtained by using a numerical method based on Fourier expansion techniques comparison between the predictions of both theories are discussed. Numerical results are computed for the temperature, displacement and stress distributions. The numerical results are represented graphically.

Keywords: GL theory; GN theory; Laplace transform; Half-space; Different material

Introduction
Thermoelasticity's importance is due to its many applications in diverse fields such as geophysics, plasma physics and related topics, especially in the design of nuclear reactors.


Green and Lindsay [9] introduced two different lag times in the stress-strain relations and the entropy expression. This theory is known as the theory of thermoelasticity with two relaxation times. In this theory both the equations of motion and of heat conduction are hyperbolic. The equation of motion differs from that of the coupled thermoelasticity theory. Sinha and Elsibai [10] discussed the reflection of thermoelastic waves at a solid half-space with two thermal relaxation times. Sherief [11,12] solved a state space approach to thermoelasticity with two relaxation times and the fundamental solution for thermoelasticity with two relaxation times and a thermo-mechanical shock problem for thermoelasticity with two relaxation times. Anwar and Sherief [13] studied the boundary integral equation formulation for thermoelasticity with two relaxation times. Song et al. [14] solved the transient disturbance in a half space under thermoelasticity with two relaxation times due to moving internal heat source.

The theory of thermoelasticity without energy dissipation (GN theory) was proposed by Green and Naghdi [15]. The most important aspect of this theory, which is not present in other thermoelasticity, is that this theory does not accommodate dissipation of thermal energy. Abd El-Latif and Khader [16] obtained the exact solution of thermoelastic problem for a one-dimensional bar without energy Dissipation. Quintanilla [17,18] discussed the spatial behavior in thermoelasticity without energy dissipation and existence in thermoelasticity without energy dissipation. Iesan [19] discussed the theory of thermoelasticity without energy dissipation. Verma and Hasebe [20] solved the dispersion of thermoelastic waves in a plate with and without energy dissipation. Marin and Baleanu [21] discussed the vibrations in thermoelasticity without energy dissipation for micropolar bodies.

Formulation of the Problem
We consider a problem for a half-space overlaid by a thick layer of a different material within the context of the theory of thermoelasticity with two relaxation times and the theory of thermoelasticity without energy dissipation. The x-axis is taken perpendicular to the surfaces of the half-space and of the layer. The layer occupies the region 0 ≤ x ≤ L, while the half-space occupies the region x>L. The upper surface of the layer is taken to be traction free and is subjected to a constant thermal shock.

Due to the physics of the problem, all the functions considered will depend on x and t only. The displacement components have the form

\[ u_i(x,t), u_i=0, \ i=1,2. \]

From now on the suffix 1 denotes quantities in the region of the layer while the suffix 2 denotes quantities in the region of the half-space.

Solution for GL-Theory
The equations of motion have the form

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The generalised equation of heat conduction for both media are given by

\[ k \frac{\partial^2 T}{\partial x^2} = \rho c_v \left( 1 + \tau \frac{\partial}{\partial t} \right) \frac{\partial T_i}{\partial t} + \gamma T_i \frac{\partial u_i}{\partial t} \quad i = 1, 2 \]  

(1)

The constitutive equations have the form:

\[ \sigma_i = \sigma \left( \lambda_i + 2 \mu_i \right) \frac{\partial u_i}{\partial x} - \gamma_i \left( T_{i-1} - T_i + \nu_i \frac{\partial T_i}{\partial t} \right) \]  

(3)

where \( \sigma \) denotes the stress component in medium \( i \).

We also have the Fourier's law of heat conduction, namely

\[ q_i = -k_i \frac{\partial T_i}{\partial x} \]  

(4)

where \( q_i \) is the heat flux in \( x \)-direction in medium \( i \).

Let us introduce the following non-dimensional variables,

\[ x^* = c_i \eta_i x, \quad u_i^* = c_i \eta_i u_i, \quad T_i^* = c_i^2 \eta_i T_i, \quad \tau_i^* = c_i^2 \eta_i \tau_i, \quad \nu_i^* = c_i^2 \eta_i \nu_i, \]  

\[ \sigma_i^* = \frac{\sigma_i}{\lambda_i + 2 \mu_i}, \quad \theta_i^* = \gamma_i \left( T_{i-1} - T_i \right) \frac{c_i}{\kappa_i \eta_i} \left( \lambda_i + 2 \mu_i \right), \quad q_i^* = \frac{q_i}{k_i c_i^2 \eta_i}, \]  

where \( c_i = \sqrt{\lambda_i + 2 \mu_i}, \eta_i = \frac{\rho c_i}{k_i} \)

The governing equations (1-4) in non-dimensional form become (dropping the asterisks for convenience)

\[ \psi_i \frac{\partial^2 u_i}{\partial x^2} - \left( 1 + \nu_i \frac{\partial}{\partial t} \right) \frac{\partial \theta_i}{\partial t} = \xi_i \frac{\partial^2 u_i}{\partial t^2} \]  

(5)

\[ \frac{\partial \theta_i}{\partial x} = \delta_i \left( 1 + \tau_i \frac{\partial}{\partial t} \right) \frac{\partial \theta_i}{\partial t} + \phi_i \frac{\partial u_i}{\partial t} \]  

(6)

\[ \sigma_i \frac{\partial T_i}{\partial x} = \theta_i \left( 1 + \nu_i \frac{\partial}{\partial t} \right) \frac{\partial \theta_i}{\partial t} \]  

(7)

\[ q_i = -\alpha_i \frac{\partial \theta_i}{\partial x} \]  

(8)

where

\[ \psi_i = \frac{\gamma_i (\lambda_i + 2 \mu_i)}{\gamma_i (\lambda_i + 2 \mu_i)} \xi_i = \frac{\gamma_i \eta_i \kappa_i}{\gamma_i (\lambda_i + 2 \mu_i) \eta_i}, \quad \phi_i = \frac{\gamma_i \eta_i \kappa_i}{k_i (\lambda_i + 2 \mu_i) \eta_i}, \quad \alpha_i = \frac{k_i}{k_i} \]  

(9)

The boundary conditions of the problem can be written as:

\[ \sigma_i |_{x=0} = 0, \quad \theta_i |_{x=0} = f(t), \quad q_i |_{x=0} = u_i, \quad \alpha_i |_{q_i = q_i} = \theta_i |_{x=L} = 0 \]  

(10)

The continuity conditions between the two media are:

\[ u_i = u_x, \quad \theta_i = \theta_x, \quad \sigma_i = \sigma_x, \quad q_i = q_x \quad \text{at} \quad x = L \]  

(11)

We assume also that the initial conditions of a problem are homogeneous.

Applying the Laplace transform with parameter \( s \) defined by the relation

\[ \overline{f(s)} = \int f(t) e^{-st} \, dt \]

to both sides of equations 5-8, we obtain

\[ \left( \psi_i \frac{\partial^2}{\partial x^2} - \xi_i s^2 \right) \overline{u_i} = (1 + \nu_i s) \frac{\partial}{\partial x} \left( \frac{\partial \overline{u_i}}{\partial x} \right) \]  

(12)

\[ \frac{\xi_i s^2}{\xi_i s^2} \left( 1 + \tau_i s \right) \overline{\theta_i} = \phi_i \frac{\partial}{\partial x} \left( \frac{\partial \overline{\theta_i}}{\partial x} \right) \]  

(13)

\[ \overline{\sigma_i} = \frac{\gamma_i}{\gamma_i} \left( 1 + \nu_i s \right) \overline{\theta_i} \]  

(14)

\[ \overline{q_i} = -\alpha_i \frac{\partial \overline{\theta_i}}{\partial x} \]  

(15)

Eliminating \( \overline{\theta_i} \) between equations (12) and (13), we get the following fourth order differential equation satisfied by \( \overline{\theta_i} \):

\[ \left( \psi_i D^4 - D^2 \right) \overline{\psi_i \delta_i + \xi_i s^2 + \phi_i \xi_i (1 + \nu_i s)} = 0 \]  

(16)

where \( D = \frac{\partial}{\partial x} \)

Solving equation (16), we obtain for the two different media,

\[ \overline{u_i} = (1 + \nu_i s) \left[ A_k \xi_i e^{k_i x} + A_k \xi_i e^{-k_i x} - A_k \xi_i e^{i\gamma_i x} - A_k \xi_i e^{-i\gamma_i x} \right] \]  

(17a)

\[ \overline{q_i} = (1 + \nu_i s) \left[ k_{i1} \xi_i e^{i\gamma_i x} + k_{i2} \xi_i e^{-i\gamma_i x} \right] \]  

(17b)

where \( A_k, A_{k1}, A_{k2}, A_{k3}, A_{k4} \) are the roots that depend on \( s \) only and \( k_{i1}, k_{i2} \) are the roots with positive real parts of the characteristic equation

\[ \psi_i s^4 - \xi_i s^2 + \phi_i (1 + \nu_i s) \xi_i = 0 \]  

(18)

We note here that we have kept only the terms in equation (17b) that is bounded at infinity.

Similarly, eliminating \( \overline{\theta_i} \) between equations (12), (13) we see that \( \overline{\theta_i} \) satisfies equation 16. Thus, the solution compatible with equation (12) is given by

\[ \overline{u_i} = (1 + \nu_i s) \left[ A_k \xi_i e^{i\gamma_i x} + A_k \xi_i e^{-i\gamma_i x} - A_k \xi_i e^{-k_i x} - A_k \xi_i e^{k_i x} \right] \]  

(19a)

\[ \overline{q_i} = (1 + \nu_i s) \left[ k_{i1} \xi_i e^{i\gamma_i x} + k_{i2} \xi_i e^{-i\gamma_i x} \right] \]  

(19b)

Substituting from equations (17), (19) into equation (14), we obtain

\[ \overline{\sigma_i} = s^2 (1 + \nu_i s) \left[ A_k e^{i\gamma_i x} + A_k e^{-i\gamma_i x} + A_k e^{k_i x} + A_k e^{-k_i x} \right] \]  

(20a)

\[ \overline{q_i} = \gamma_i \xi_i s^2 (1 + \nu_i s) \left[ A_k e^{k_i x} + A_k e^{-k_i x} \right] \]  

(20b)

where \( \gamma_i = \frac{\gamma_i}{2} / \gamma_i \)

Substituting from equations (19) into equation (15), we obtain the heat flux component \( q_i \) in the form

\[ \overline{q_i} = -\frac{k_i}{k_i} (k_{i1} - k_{i2}) \left[ A_k e^{i\gamma_i x} - A_k e^{-i\gamma_i x} \right] - k_{i1} (k_{i1} - k_{i2}) \left[ A_k e^{k_i x} - A_k e^{-k_i x} \right] \]  

(21a)

\[ \overline{q_i} = -\frac{k_i}{k_i} (k_{i1} - k_{i2}) \left[ A_k e^{k_i x} - A_k e^{-k_i x} \right] + k_{i2} (k_{i1} - k_{i2}) \left[ A_k e^{k_i x} - A_k e^{-k_i x} \right] \]  

(21b)

Applying the Laplace transform with parameter \( s \) to both sides of equations (9-11), we get...
Substituting from equations (17), (19), (20) and (21) into equations (22)-(24), we obtain the following system of linear equations in the unknown parameters $A_i, A_k$.

\[
A_1 + A_2 + A_3 + A_4 = 0
\]

\[
\left(k_1 i - s^2\right)\left[A_1 + A_2 + A_3 + A_4\right] + \left[k_2 i - s^2\right]\left[A_1 + A_2 + A_3 + A_4\right] = \frac{\partial}{s}
\]

\[
k_{i,1} A_i e^{k_{i-1}x} + k_{i,2} A_i e^{-k_{i-2}x} - k_{i,3} A_i e^{-k_{i-3}x} + k_{i,4} A_i e^{k_{i-4}x} + \left(\frac{1 + u x}{(1 + u x)}\right) k_{i,1} A_i e^{k_{i-1}x} + k_{i,2} A_i e^{-k_{i-2}x} = 0
\]

\[
\left(k_1 i - s^2\right)\left[A_1 e^{k_{i-1}x} + A_i e^{-k_{i-2}x}\right] + \left[k_2 i - s^2\right]\left[A_1 e^{k_{i-2}x} + A_i e^{-k_{i-3}x}\right] - \left(\psi, k_{i,1} - s\xi \right) A_i e^{k_{i-1}x} + \left(\psi, k_{i,2} - s\xi \right) A_i e^{-k_{i-2}x} = 0
\]

\[
A_i e^{k_{i-1}x} + A_i e^{-k_{i-2}x} + A_i e^{-k_{i-3}x} + A_i e^{k_{i-4}x} = 0
\]

\[
k_{i,1} k_{i,2} \left(k_1 - s^2\right) A_i e^{k_{i-1}x} + k_{i,2} k_{i,3} A_i e^{-k_{i-2}x} + k_{i,4} A_i e^{k_{i-3}x} + k_{i,4} A_i e^{k_{i-4}x} = 0
\]

**Solution for GN-Theory**

The equations of motion have the form

\[
\left(\lambda_i i + 2\mu_i\right) \frac{\partial^2 u_i}{\partial x^2} - \gamma T_i \frac{\partial u_i}{\partial t} = \frac{\partial T_i}{\partial t}, \quad i = 1, 2
\]

The generalized equation of heat conduction for both media are given by

\[
k_i \frac{\partial^2 T}{\partial x^2} = \left(\lambda_i + 2\mu_i\right) \frac{\partial^2 u_i}{\partial x^2} + \gamma T_i \frac{\partial u_i}{\partial t}, \quad i = 1, 2
\]

The constitutive equations take the form

\[
\sigma_i = \left(\lambda_i + 2\mu_i\right) \frac{\partial u_i}{\partial x} - \gamma (T_i - T_0)
\]

We also have the Fourier's law of heat conduction, namely

\[
q_i = -k_i \frac{\partial T_i}{\partial x}
\]

The equations (29)-(32) in non-dimensional form become

\[
\psi \frac{\partial^2 u_i}{\partial x^2} - \frac{\partial \theta_i}{\partial t} = \xi \frac{\partial^2 u_i}{\partial x^2} + \phi \frac{\partial u_i}{\partial t}
\]

\[
C_i \frac{\partial \theta_i}{\partial x} = \frac{\partial^2 u_i}{\partial x^2} + \phi \frac{\partial u_i}{\partial t}
\]

\[
\sigma_i = \left[\xi \left(\frac{\partial u_i}{\partial x}\right) + \theta_i\right]
\]

\[
q_i = -\sigma_i \frac{\partial \theta_i}{\partial x}
\]

where

\[
C_i = \frac{k_i}{\rho c_v e^2}
\]

Solution of the problem in the Laplace transform domain

Applying the Laplace transform with parameter $s$ to both sides of equations (33)-(36), we obtain

\[
\begin{pmatrix}
\psi,\frac{\partial^2 \xi}{\partial{x}^2} - \xi s^2
\end{pmatrix} \bar{u}_i = \frac{\partial \bar{\theta}_i}{\partial x}
\]

\[
\begin{pmatrix}
\psi^2 \frac{\partial^2 \xi}{\partial{x}^2} - \xi s^2
\end{pmatrix} \bar{u}_i = \frac{\partial \bar{\theta}_i}{\partial x}
\]

\[
\sigma_i = \left[\xi \left(\frac{\partial \bar{u}_i}{\partial x}\right) + \bar{\theta}_i\right]
\]

\[
q_i = -\sigma_i \frac{\partial \bar{\theta}_i}{\partial x}
\]

Eliminating $\bar{\theta}_i$, between equations (37), (38), we get the following fourth order differential equation satisfied by $\bar{u}_i$

\[
\begin{pmatrix}
\psi, C_i s^2 D^4 - D^2 [\psi, s^2 + C_i s^2 \xi s^2 + \phi s^2] + \xi s^2
\end{pmatrix} \bar{u}_i = 0
\]

The solution to equation (41) is again of the form in equation (16)

\[
\bar{u}_i = B_{k,1} e^{k_{1,i}x} + B_{k,2} e^{k_{2,i}x} - B_{k,1} e^{-k_{1,i}x} - B_{k,2} e^{-k_{2,i}x}
\]

\[
\bar{u}_2 = -\frac{k_{1,i}}{B_{k,1}} e^{-k_{1,i}x} + \frac{k_{2,i}}{B_{k,2}} e^{-k_{2,i}x}
\]

where $B_{k,1}, B_{k,2}, B_{k,3}$, and $B_{k,4}$ are parameters that depend on $s$ only and $k_{1,i}, k_{2,i}$ are the roots with positive real parts of the characteristic equation

\[
\psi, C_i s^2 k_i^2 - k_i^2 [\psi, s^2 + C_i s^2 \xi s^2 + \phi s^2] + \xi s^2 = 0
\]

\[
\bar{\theta}_i = \left(k_{1,i} - s^2\right) \left[B_{k,1} e^{k_{1,i}x} + B_{k,2} e^{-k_{1,i}x}\right] - \left(k_{2,i} - s^2\right) \left[B_{k,1} e^{k_{2,i}x} + B_{k,2} e^{-k_{2,i}x}\right]
\]

\[
\bar{\theta}_2 = \psi, k_{1,i} - s^2 k_{2,i} B_{k,1} e^{k_{2,i}x} + \psi, k_{2,i} - s^2 k_{1,i} B_{k,2} e^{-k_{2,i}x}
\]

Substituting from equations (43), (46) into equation (39), we obtain

\[
\bar{u}_i = s^2 \theta_i e^{k_{1,i}x} + B_{k,2} e^{k_{2,i}x} + B_{k,1} e^{-k_{1,i}x} + B_{k,2} e^{-k_{2,i}x}
\]

\[
\bar{u}_2 = \psi, k_{1,i} - s^2 k_{2,i} B_{k,1} e^{k_{2,i}x} + \psi, k_{2,i} - s^2 k_{1,i} B_{k,2} e^{-k_{2,i}x}
\]

Substituting from equations (43), (44), (47) and (48) into equations (25), (26) and (27), we obtain the following system of linear equations in the unknown parameters $B_{k,1}, B_{k,2}$,

\[
B_{k,1} + B_{k,2} + B_{k,3} + B_{k,4} = 0
\]

\[
\left(k_{1,i} - s^2\right) \left[B_{k,1} + B_{k,2}\right] + \left(k_{2,i} - s^2\right) \left[B_{k,1} + B_{k,2}\right] = \frac{\partial}{s}
\]

\[
k_{1,i} B_{k,1} e^{k_{1,i}x} + k_{2,i} B_{k,2} e^{k_{1,i}x} - k_{1,i} B_{k,1} e^{-k_{1,i}x} - k_{2,i} B_{k,2} e^{-k_{1,i}x}
\]

\[
+ k_{1,i} B_{k,1} e^{k_{2,i}x} + k_{2,i} B_{k,2} e^{k_{2,i}x} = 0
\]

\[
\left(k_{1,i} - s^2\right) \left[B_{k,1} e^{k_{1,i}x} + B_{k,2} e^{k_{1,i}x}\right] + \left(k_{2,i} - s^2\right) \left[B_{k,1} e^{k_{2,i}x} + B_{k,2} e^{k_{2,i}x}\right]
\]

\[
- \psi, k_{1,i} - s^2 k_{2,i} B_{k,1} e^{k_{2,i}x} + \psi, k_{2,i} - s^2 k_{1,i} B_{k,2} e^{-k_{2,i}x} = 0
\]
Numerical Results and Discussion

During numerical computations, the upper material was taken to be made of Carbon Steel, while that of the half-space was taken to be made of Copper material. The constants of the problem were taken as shown in Table 1.

The computations were carried out for one value of time, namely \( t=0.2 \). The temperature, displacement and stress distributions are shown in Figures 1-3, respectively. Solid line represents the case (GL theory), dotted line represents the case (GN theory). The FORTRAN programming language was used on a personal computer. The accuracy maintained was 4 digits for the numerical program.

We notice that the solution is non-zero only in a region of space adjacent to the upper surface. This region expands with the passage of time. This is different from the solutions obtained using either the uncoupled or the coupled theories of thermoelasticity where the solution is non-zero everywhere due to the infinite speed of wave propagation predicted by these theories. In Figure 1 we observed that the leading wave front has reached the point \( x=1.33 \) for the two theories. The solutions are thus non-zero only in the region \( 0<\xi<1.33 \). The effects of the thermal shock have thus filled the region of the layer and reached the half-space in two theories. But in (GL theory), we find that the dissipation in energy for the (GN theory) no dissipation in energy. In Figure 2, the value of displacement is bigger at the initial, and after that is decay to reached the zero at \( x=1 \) namely (GL theory), in (GN theory) the displacement reached to zero at \( x=1.2 \) namely. In Figure 3, we observed the discontinuity in the stresses at \( x=0.2 \) namely in both theories. In (GL theory), the lased of energy is observed. In all figures the beaver of all functions are different forms the two media but are continues.

References


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Table 1: The Material Constants.


