

A Numeric – Analytic Method for Fractional Order Nonlinear PDE's With Modified Riemann-Liouville Derivative by Means of Fractional Variational Iteration Method

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Abstract

In this article, an approximate analytical solution of fractional order nonlinear PDE's with modified Riemann-Liouville derivative was obtained with the help of fractional variational iteration method (FVIM). It is showed that the solutions obtained by the FVIM are reliable and effective method for strongly nonlinear partial equations with modified Riemann-Liouville derivative. The solutions of our model equation can also be obtained from the known forms of the series solutions.

Keywords: Fractional variational iteration method; Fractional KDV; K (2, 2) and mKDV equation; Riemann-Liouville derivative, Burgers equation, Klein-Gordon equation

Introduction

It is known that various problems in electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, optics, engineering, acoustics, material science and signal processing can be successfully modeled by linear or nonlinear fractional order differential equations. Methods of solutions of problems for fractional differential equations have been studied extensively by many researchers [1-8].

The variational iteration method (VIM), which proposed by Ji-Huan He [9], was successfully applied to autonomous ordinary and partial differential equations and other fields. Ji-Huan He [10] was the first to apply the variational iteration method to fractional differential equations.

Fractional convection–diffusion equation with nonlinear source term solved by Momani and Yildirim [11], space–time fractional advection–dispersion equation by Yildirim and Kocak [12], fractional Zakharov–Kuznetsov equations by Yildirim and Gulkanat [13], integro-differential equation by El-Shahed [14], non-Newtonian flow by Siddiqui et al. [15], fractional PDEs in fluid mechanics by Yildirim [16], fractional Schrödinger equation [17,18]. Different fractional partial differential equations have been studied and solved including the space-time fractional diffusion-wave equation [19-21], the fractional advection-dispersion equation [22,23], the fractional telegraph equation [24], the fractional KdV equation [25] and the linear inhomogeneous fractional partial differential equations [26]. Recently, a new modified Riemann-Liouville left derivative is suggested by G. Jumarie [27-31]. The most recently, a new application of Fractional Variational Iteration Method (FVIM) for solving non-linear fractional coupled- KDV equations with modified Riemann-Liouville derivative performed by Merdan et al. [32]. Roul [33] applied the fractional VIM for obtaining the exact and approximate analytical solutions of time fractional biological population models.

The aim of this paper is to extend the application of the variational iteration method method to solve fractional KDV, K(2,2), mKDV equation and some fractional partial equations in fluid mechanics with modified Riemann-Liouville derivative.

This paper is organized as follows:

In Basic definitions section, we gave definitions related to the

fractional calculus theory briefly. In Fractional variational iteration method section, we define the solution procedure of the fractional variational iteration method. To show in efficiency of this method, we give the implementation of the FVIM for the fractional KDV, K (2,2), mKDV equation and some fractional partial equations in fluid mechanics with modified Riemann-Liouville derivative and numerical results in Applications section. The conclusions are then given at the last.

Basic Definitions

Here, we give some basic definitions and properties of fractional calculus theory which will be used in this paper [1,2,27-37].

Definition 1

Fractional derivative is defined as the following limit form [27-29]

$$f^{(\alpha)} = \lim_{h \rightarrow 0} \frac{\Delta^\alpha [f(x) - f(0)]}{h^\alpha} \quad (1)$$

This definition is close to the standard definition of derivatives (calculus for beginners), and as a direct result, the α th derivative of a constant, $0 < \alpha < 1$, is zero.

Definition 2

The left-sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$ is defined as

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad (2)$$

for $\alpha > 0, x > 0$ and $J_a^0 f(x) = f(x)$.

The properties of the operator J^α can be found in [1,2,34].

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Definition 3

The modified Riemann-Liouville derivative [27-29] is defined as

$${}_0D_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-\tau)^{m-\alpha} (f(\tau) - f(0)) d\tau, \quad (3)$$

where $x \in [0,1]$, $m-1 \leq \alpha < m$ and $m \geq 1$.

Definition 4

Fractional derivative of compounded functions [27-29] is defined as

$$d^\alpha f \equiv \Gamma(1+\alpha) df, \quad 0 < \alpha < 1 \quad (4)$$

Definition 5

The integral with respect to $(dx)^\alpha$ [27-29] is defined as the solution of the fractional differential equation

$$dy \equiv f(x)(dx)^\alpha, \quad x \geq 0, \quad y(0) = 0, \quad 0 < \alpha < 1 \quad (5)$$

Lemma 1

Let $f(x)$ denote a continuous function [27-29] then the solution of the Eq. (5) is defined as

$$y = \int_0^x f(\tau)(d\tau)^\alpha = \alpha \int_0^x (x-\tau)^\alpha f(\tau) d\tau, \quad 0 < \alpha < 1 \quad (6)$$

For example $f(x) = x^\beta$ in Eq. (6) one obtains

$$\int_0^x \tau^\beta (d\tau)^\alpha = \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} x^{\beta+\alpha}, \quad 0 < \alpha < 1 \quad (7)$$

Definition 6

Assume that the continuous function $f: R \rightarrow R, x \rightarrow f(x)$ has a fractional derivative of order $k\alpha$, for any positive integer k and any $\alpha, 0 < \alpha \leq 1$; then the following equality holds, which is

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{k\alpha}}{\alpha k!} f^{(k\alpha)}(x), \quad 0 < \alpha \leq 1. \quad (8)$$

On making the substitution $h \rightarrow x$ and $x \rightarrow 0$ we obtain the fractional Mc-Laurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\alpha k!} f^{(k\alpha)}(0), \quad 0 < \alpha \leq 1. \quad (9)$$

Fractional Variational Iteration Method

To describe the solution procedure of the fractional variational iteration method, we consider the following fractional differential equation [34]:

$$D_t^\alpha u(x,t) = G\left(u(x,t), \frac{\partial u(x,t)}{\partial x}, \frac{\partial^2 u(x,t)}{\partial x^2}, \frac{\partial^3 u(x,t)}{\partial x^3}\right), \quad t > 0, \quad x \in R, \quad (10)$$

$$u(x,0) = f(x), \quad \alpha > 0$$

According to the VIM, we can build a correct functional for Eq. (4) as follows

$$u_{n+1}(x,t) = u_n(x,t) + I^\alpha \left[\lambda(x,t) \left(\frac{\partial^\alpha u_n(x,t)}{\partial t^\alpha} - G\left(u_n(x,t), \frac{\partial u_n(x,t)}{\partial x}, \frac{\partial^2 u_n(x,t)}{\partial x^2}, \frac{\partial^3 u_n(x,t)}{\partial x^3}\right) \right) \right]$$

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \lambda(x,\tau) \left(\frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - G\left(u_n(x,\tau), \frac{\partial u_n(x,\tau)}{\partial x}, \frac{\partial^2 u_n(x,\tau)}{\partial x^2}, \frac{\partial^3 u_n(x,\tau)}{\partial x^3}\right) \right) d\tau \quad (11)$$

Using Eq. (5), we obtain a new correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha+1)} \int_0^t \lambda(x,\tau) \left(\frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - G\left(u_n(x,\tau), \frac{\partial u_n(x,\tau)}{\partial x}, \frac{\partial^2 u_n(x,\tau)}{\partial x^2}, \frac{\partial^3 u_n(x,\tau)}{\partial x^3}\right) \right) (d\tau)^\alpha \quad (12)$$

It is obvious that the sequential approximations $u_k, k \geq 0$ can be established by determining λ , a general Lagrange’s multiplier, which can be identified optimally with the variational theory. The function \tilde{u}_n is a restricted variation which means $\delta \tilde{u}_n = 0$. Therefore, we first designate the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t), n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0 . The initial values are usually used for choosing the zeroth approximation u_0 . With λ determined, then several approximations $u_k, k \geq 0$ follows immediately [35]. Consequently, the exact solution may be procured by using

$$u(x,t) = \lim_{n \rightarrow \infty} u_n(x,t) \quad (13)$$

Applications

In this section, we present the solution of nonlinear fractional partial differential equations as the applicability of FVIM.

Example 1

Consider the fractional KDV equation where $0 < \alpha \leq 1$

$$D_t^\alpha u - 3(u^2)_x + u_{xxx} = 0, \quad (14)$$

With initial conditions

$$u(x,0) = 6x, \quad [36]. \quad (15)$$

Construction the following functional:

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha+1)} \int_0^t \lambda(x,\tau) \left\{ \frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - 6u_n(x,\tau) \frac{\partial u_n(x,\tau)}{\partial x} + \frac{\partial^3 u_n(x,\tau)}{\partial x^3} \right\} (d\tau)^\alpha \quad (16)$$

$$\left\{ \frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - 6u_n(x,\tau) \frac{\partial u_n(x,\tau)}{\partial x} + \frac{\partial^3 u_n(x,\tau)}{\partial x^3} \right\} (d\tau)^\alpha$$

We have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \frac{1}{\Gamma(\alpha+1)} \delta \int_0^t \lambda(x,\tau) \left\{ \frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - 6u_n(x,\tau) \frac{\partial u_n(x,\tau)}{\partial x} + \frac{\partial^3 u_n(x,\tau)}{\partial x^3} \right\} (d\tau)^\alpha \quad (17)$$

$$\left\{ \frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - 6u_n(x,\tau) \frac{\partial u_n(x,\tau)}{\partial x} + \frac{\partial^3 u_n(x,\tau)}{\partial x^3} \right\} (d\tau)^\alpha$$

$$= \delta u_n + \lambda \delta u_n \Big|_{\tau=t} - \frac{1}{\Gamma(\alpha+1)} \int_0^t \frac{\partial^\alpha \lambda(x,\tau)}{\partial \tau^\alpha} \delta u_n(x,\tau) (d\tau)^\alpha$$

Similarly, we can get the coefficients of δu_n to zero:

$$1 + \lambda(x,\tau) \Big|_{\tau=t} = 0, \quad \frac{\partial^\alpha \lambda(x,\tau)}{\partial \tau^\alpha} = 0 \quad (18)$$

The generalized Lagrange multiplier can be identified by the above equations,

$$\lambda(x, t) = -1. \tag{19}$$

Substituting Eq. (19) into Eq. (16) produces the iteration formulation as follows

$$u_{n+1}(x, t) = u_n(x, t) - \frac{1}{\Gamma(\alpha+1)} \int_0^t \left\{ \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} - 6u_n(x, \tau) \frac{\partial u_n(x, \tau)}{\partial x} + \frac{\partial^3 u_n(x, \tau)}{\partial x^3} \right\} (d\tau)^\alpha \tag{20}$$

Taking the initial value $u_0(x, t) = u_0(x, 0) = 6x$, we can derive

$$u_1(x, t) = u_0(x, t) - \frac{1}{\Gamma(\alpha+1)} \int_0^t \left\{ \frac{\partial^\alpha u_0(x, \tau)}{\partial \tau^\alpha} - 6u_0(x, \tau) \frac{\partial u_0(x, \tau)}{\partial x} + \frac{\partial^3 u_0(x, \tau)}{\partial x^3} \right\} (d\tau)^\alpha$$

$$u_1(x, t) = 6x + \frac{6^3 x t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x, t) = 6x + \frac{6^3 x t^\alpha}{\Gamma(\alpha+1)} + \frac{2 \cdot 6^5 x t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{6^7 \Gamma(2\alpha+1) x t^{3\alpha}}{\Gamma^2(\alpha+1) \Gamma(3\alpha+1)}. \tag{21}$$

For the special case $\alpha = 1$ is [36]

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) = 6x(1 + 36t + 36^2 t^2 + 36^3 t^3 + \dots)$$

$$= \frac{6x}{1 - 36t}. \tag{22}$$

Example 2

Consider the fractional K (2,2) equation where $0 < \alpha \leq 1$

$$D_t^\alpha u + (u^2)_x + (u^2)_{xxx} = 0, \tag{23}$$

With initial conditions

$$u(x, 0) = x, [36]. \tag{24}$$

Construction the following functional:

$$u_{n+1}(x, t) = u_n(x, t) + \frac{1}{\Gamma(\alpha+1)} \int_0^t \lambda(x, \tau) \left\{ \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} + 2u_n(x, \tau) \frac{\partial u_n(x, \tau)}{\partial x} \right. \\ \left. 4 \frac{\partial u_n(x, \tau)}{\partial x} \frac{\partial^2 u_n(x, \tau)}{\partial x^2} + 2u_n(x, \tau) \frac{\partial^3 u_n(x, \tau)}{\partial x^3} \right\} (d\tau)^\alpha \tag{25}$$

We have

$$\delta u_{n+1}(x, t) = \delta u_n(x, t) + \frac{1}{\Gamma(\alpha+1)} \delta \int_0^t \lambda(x, \tau) \left\{ \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} + 2u_n(x, \tau) \frac{\partial u_n(x, \tau)}{\partial x} \right. \\ \left. 4 \frac{\partial u_n(x, \tau)}{\partial x} \frac{\partial^2 u_n(x, \tau)}{\partial x^2} + 2u_n(x, \tau) \frac{\partial^3 u_n(x, \tau)}{\partial x^3} \right\} (d\tau)^\alpha$$

$$= \delta u_n + \lambda \delta u_n |_{\tau=t} - \frac{1}{\Gamma(\alpha+1)} \int_0^t \frac{\partial^\alpha \lambda(x, \tau)}{\partial \tau^\alpha} \delta u_n(x, \tau) (d\tau)^\alpha \tag{26}$$

Similarly, we can get the coefficients of δu_n to zero:

$$1 + \lambda(x, t) |_{\tau=t} = 0, \quad \frac{\partial^\alpha \lambda(x, \tau)}{\partial \tau^\alpha} = 0 \tag{27}$$

The generalized Lagrange multiplier can be identified by the above equations,

$$\lambda(x, t) = -1. \tag{28}$$

Substituting Eq. (28) into Eq. (25) produces the iteration formulation as follows

$$u_{n+1}(x, t) = u_n(x, t) - \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} + 2u_n(x, \tau) \frac{\partial u_n(x, \tau)}{\partial x} \right. \\ \left. 4 \frac{\partial u_n(x, \tau)}{\partial x} \frac{\partial^2 u_n(x, \tau)}{\partial x^2} + 2u_n(x, \tau) \frac{\partial^3 u_n(x, \tau)}{\partial x^3} \right\} (d\tau)^\alpha \tag{29}$$

Taking the initial value $u_0(x, t) = u_0(x, 0) = x$, we can derive

$$u_1(x, t) = u_0(x, t) - \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\partial^\alpha u_0(x, \tau)}{\partial \tau^\alpha} + 2u_0(x, \tau) \frac{\partial u_0(x, \tau)}{\partial x} \right. \\ \left. 4 \frac{\partial u_0(x, \tau)}{\partial x} \frac{\partial^2 u_0(x, \tau)}{\partial x^2} + 2u_0(x, \tau) \frac{\partial^3 u_0(x, \tau)}{\partial x^3} \right\} (d\tau)^\alpha$$

$$u_1(x, t) = x - \frac{2xt^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x, t) = x - \frac{2xt^\alpha}{\Gamma(\alpha+1)} + \frac{8xt^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\Gamma(2\alpha+1)8xt^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}. \tag{30}$$

For the special case $\alpha = 1$ is [36]

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) = 6x(1 - 2t + 4t^2 - 8t^3 + \dots)$$

$$= \frac{6x}{1 + 2t}. \tag{31}$$

Example 3

Consider the modified fractional KDV (mKDV) equation where $0 < \alpha \leq 1$

$$D_t^\alpha u + \frac{1}{2}(u^2)_x - u_{xx} = 0, \tag{32}$$

With initial conditions

$$u(x, 0) = x, [36]. \tag{33}$$

Construction the following functional:

$$u_{n+1}(x, t) = u_n(x, t) + \frac{1}{\Gamma(\alpha+1)} \int_0^t \lambda(x, \tau) \left\{ \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} \right. \\ \left. + u_n(x, \tau) \frac{\partial u_n(x, \tau)}{\partial x} - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} \right\} (d\tau)^\alpha \tag{34}$$

We have

$$\delta u_{n+1}(x, t) = \delta u_n(x, t) + \frac{1}{\Gamma(\alpha + 1)} \delta \int_0^t \lambda(x, \tau) \left\{ \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} + u_n(x, \tau) \frac{\partial u_n(x, \tau)}{\partial x} - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} \right\} (d\tau)^\alpha \quad (35)$$

$$= \delta u_n + \lambda \delta u_n \Big|_{\tau=t} - \frac{1}{\Gamma(\alpha + 1)} \int_0^t \frac{\partial^\alpha \lambda(x, \tau)}{\partial \tau^\alpha} \delta u_n(x, \tau) (d\tau)^\alpha$$

Similarly, we can get the coefficients of δu_n to zero:

$$1 + \lambda(x, \tau) \Big|_{\tau=t} = 0, \quad \frac{\partial^\alpha \lambda(x, \tau)}{\partial \tau^\alpha} = 0 \quad (36)$$

The generalized Lagrange multiplier can be identified by the above equations,

$$\lambda(x, t) = -1. \quad (37)$$

Substituting Eq. (37) into Eq. (34) produces the iteration formulation as follows

$$u_{n+1}(x, t) = u_n(x, t) - \frac{1}{\Gamma(\alpha + 1)} \int_0^t \left\{ \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} + u_n(x, \tau) \frac{\partial u_n(x, \tau)}{\partial x} - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} \right\} (d\tau)^\alpha \quad (38)$$

Taking the initial value $u_0(x, t) = u_0(x, 0) = x$, we can derive

$$u_1(x, t) = u_0(x, t) - \frac{1}{\Gamma(\alpha + 1)} \int_0^t \left\{ \frac{\partial^\alpha u_0(x, \tau)}{\partial \tau^\alpha} + u_0(x, \tau) \frac{\partial u_0(x, \tau)}{\partial x} - \frac{\partial^2 u_0(x, \tau)}{\partial x^2} \right\} (d\tau)^\alpha \quad (39)$$

$$u_1(x, t) = x - \frac{xt^\alpha}{\Gamma(\alpha + 1)},$$

$$u_2(x, t) = x - \frac{xt^\alpha}{\Gamma(\alpha + 1)} + \frac{2xt^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{\Gamma(2\alpha + 1)xt^{3\alpha}}{\Gamma^2(\alpha + 1)\Gamma(3\alpha + 1)}.$$

For the special case $\alpha = 1$ is

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) = x(1 - 2t + 4t^2 - 8t^3 + \dots) \quad (40)$$

$$= \frac{x}{1 + 2t}.$$

Example 4

In this example we consider one-dimensional linear inhomogeneous fractional Burgers equation [37] where $0 < \alpha \leq 1, t > 0, x \in \mathbb{R}$

$$D_t^\alpha u + u_x - u_{xx} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \quad (41)$$

With initial conditions

$$u(x, 0) = x^2. \quad (42)$$

Construction the following functional:

$$u_{n+1}(x, t) = u_n(x, t) + \frac{1}{\Gamma(\alpha + 1)} \int_0^t \lambda(x, \tau) \left\{ \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} + \frac{\partial u_n(x, \tau)}{\partial x} - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} \right\} (d\tau)^\alpha \quad (43)$$

$$\left\{ -\frac{2\tau^{2-\alpha}}{\Gamma(3-\alpha)} + 2 - 2x \right\}$$

We have

$$\delta u_{n+1}(x, t) = \delta u_n(x, t) + \frac{1}{\Gamma(\alpha + 1)} \delta \int_0^t \lambda(x, \tau) \left\{ \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} + \frac{\partial u_n(x, \tau)}{\partial x} - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} \right\} (d\tau)^\alpha \quad (44)$$

$$\left\{ -\frac{2\tau^{2-\alpha}}{\Gamma(3-\alpha)} + 2 - 2x \right\}$$

$$= \delta u_n + \lambda \delta u_n \Big|_{\tau=t} - \frac{1}{\Gamma(\alpha + 1)} \int_0^t \frac{\partial^\alpha \lambda(x, \tau)}{\partial \tau^\alpha} \delta u_n(x, \tau) (d\tau)^\alpha$$

Similarly, we can get the coefficients of δu_n to zero:

$$1 + \lambda(x, \tau) \Big|_{\tau=t} = 0, \quad \frac{\partial^\alpha \lambda(x, \tau)}{\partial \tau^\alpha} = 0 \quad (45)$$

The generalized Lagrange multiplier can be identified by the above equations,

$$\lambda(x, t) = -1. \quad (46)$$

substituting Eq. (46) into Eq. (43) produces the iteration formulation as follows

$$u_{n+1}(x, t) = u_n(x, t) - \frac{1}{\Gamma(\alpha + 1)} \int_0^t \left\{ \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} + \frac{\partial u_n(x, \tau)}{\partial x} - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} \right\} (d\tau)^\alpha \quad (47)$$

$$\left\{ -\frac{2\tau^{2-\alpha}}{\Gamma(3-\alpha)} + 2 - 2x \right\}$$

Taking the initial value $u_0(x, t) = u_0(x, 0) = x^2$, we can derive

$$u_1(x, t) = u_0(x, t) - \frac{1}{\Gamma(\alpha + 1)} \int_0^t \left\{ \frac{\partial^\alpha u_0(x, \tau)}{\partial \tau^\alpha} + \frac{\partial u_0(x, \tau)}{\partial x} - \frac{\partial^2 u_0(x, \tau)}{\partial x^2} \right\} (d\tau)^\alpha \quad (48)$$

$$\left\{ -\frac{2\tau^{2-\alpha}}{\Gamma(3-\alpha)} + 2 - 2x \right\}$$

$$= x^2 + t^2,$$

By the similar operations, we have

$$u_2(x, t) = x^2 + t^2, \quad (49)$$

$$\vdots$$

$$u_n(x, t) = x^2 + t^2.$$

Thus, the solution can be given a compact form:

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) = x^2 + t^2. \quad (50)$$

Which is the exact solution of Eq.(41) and in concordance with the conditions in Eq.(42).

Example 5

Consider the following one –dimensional linear inhomogeneous fractional wave equation [37] where $0 < \alpha \leq 1, t > 0, x \in R$

$$D_t^\alpha u + u_x = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x + t \cos x, \quad (51)$$

With initial conditions

$$u(x, 0) = 0. \quad (52)$$

Construction the following functional:

$$u_{n+1}(x, t) = u_n(x, t) + \frac{1}{\Gamma(\alpha+1)} \int_0^t \lambda(x, \tau) \left\{ \begin{array}{l} \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} + \frac{\partial u_n(x, \tau)}{\partial x} \\ - \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sin x - \tau \cos x \end{array} \right\} (d\tau)^\alpha \quad (53)$$

We have

$$\begin{aligned} \delta u_{n+1}(x, t) &= \delta u_n(x, t) + \frac{1}{\Gamma(\alpha+1)} \\ &\delta \int_0^t \lambda(x, \tau) \left\{ \begin{array}{l} \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} + \frac{\partial u_n(x, \tau)}{\partial x} \\ - \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sin x - \tau \cos x \end{array} \right\} (d\tau)^\alpha \\ &= \delta u_n + \lambda \delta u_n \Big|_{\tau=t} - \frac{1}{\Gamma(\alpha+1)} \int_0^t \frac{\partial^\alpha \lambda(x, \tau)}{\partial \tau^\alpha} \delta u_n(x, \tau) (d\tau)^\alpha \end{aligned} \quad (54)$$

Similarly, we can get the coefficients of δu_n to zero:

$$1 + \lambda(x, \tau) \Big|_{\tau=t} = 0, \quad \frac{\partial^\alpha \lambda(x, \tau)}{\partial \tau^\alpha} = 0 \quad (55)$$

The generalized Lagrange multiplier can be identified by the above equations,

$$\lambda(x, t) = -1. \quad (56)$$

Substituting Eq. (56) into Eq. (53) produces the iteration formulation as follows

$$u_{n+1}(x, t) = u_n(x, t) - \frac{1}{\Gamma(\alpha+1)} \int_0^t \left\{ \begin{array}{l} \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} + \frac{\partial u_n(x, \tau)}{\partial x} \\ - \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sin x - \tau \cos x \end{array} \right\} (d\tau)^\alpha \quad (57)$$

Taking the initial value $u_0(x, t) = u_0(x, 0) = 0$, we can derive

$$\begin{aligned} u_1(x, t) &= u_0(x, t) - \frac{1}{\Gamma(\alpha+1)} \\ &\int_0^t \left\{ \begin{array}{l} \frac{\partial^\alpha u_0(x, \tau)}{\partial \tau^\alpha} + \frac{\partial u_0(x, \tau)}{\partial x} \\ - \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sin x - \tau \cos x \end{array} \right\} (d\tau)^\alpha \\ u_1(x, t) &= t \sin x + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} \cos x, \end{aligned} \quad (58)$$

By the similar operations, we have

$$\begin{aligned} u_2(x, t) &= t \sin x + \frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)} \sin x, \\ u_3(x, t) &= t \sin x - \frac{t^{1+3\alpha}}{\Gamma(2+3\alpha)} \cos x, \\ u_4(x, t) &= t \sin x - \frac{t^{1+4\alpha}}{\Gamma(2+4\alpha)} \sin x, \\ u_5(x, t) &= t \sin x + \frac{t^{1+5\alpha}}{\Gamma(2+5\alpha)} \cos x, \\ &\vdots \end{aligned} \quad (59)$$

Canceling the noise terms and keeping the non-noise terms yield the exact solution of Eq. (51) given by

$$u(x, t) = t \sin x, \quad (60)$$

Which is easily confirmed. This formally proved right in [18].

Example 6

We consider the following one –dimensional linear inhomogeneous fractional Klein-Gordon equation [37] where $1 < \alpha \leq 2, t > 0, x \in R$

$$D_t^\alpha u - u_{xx} + u = 6x^3 t + (x^3 - 6x)t^3, \quad (61)$$

With initial conditions

$$u(x, 0) = 0, u_t(x, 0) = 0. \quad (62)$$

Construction the following functional:

$$u_{n+1}(x, t) = u_n(x, t) + \frac{1}{\Gamma(\alpha+1)} \int_0^t \lambda(x, \tau) \left\{ \begin{array}{l} \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} + u_n(x, \tau) \\ - 6x^3 \tau - (x^3 - 6x)\tau^3 \end{array} \right\} (d\tau)^\alpha \quad (63)$$

We have

$$\begin{aligned} \delta u_{n+1}(x, t) &= \delta u_n(x, t) + \frac{1}{\Gamma(\alpha+1)} \\ &\delta \int_0^t \lambda(x, \tau) \left\{ \begin{array}{l} \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} \\ + u_n(x, \tau) - 6x^3 \tau - (x^3 - 6x)\tau^3 \end{array} \right\} (d\tau)^\alpha \\ &= \delta u_n + \lambda \delta u_n \Big|_{\tau=t} - \frac{1}{\Gamma(\alpha+1)} \int_0^t \frac{\partial^\alpha \lambda(x, \tau)}{\partial \tau^\alpha} \delta u_n(x, \tau) (d\tau)^\alpha \end{aligned} \quad (64)$$

Similarly, we can get the coefficients of δu_n to zero:

$$1 + \lambda(x, \tau)|_{\tau=t} = 0, \quad \frac{\partial^\alpha \lambda(x, \tau)}{\partial \tau^\alpha} = 0 \quad (65)$$

The generalized Lagrange multiplier can be identified by the above equations,

$$\lambda(x, t) = -1. \quad (66)$$

substituting Eq. (66) into Eq. (63) produces the iteration formulation as follows

$$u_{n+1}(x, t) = u_n(x, t) - \frac{1}{\Gamma(\alpha + 1)} \int_0^t \left\{ \frac{\partial^\alpha u_n(x, \tau)}{\partial \tau^\alpha} - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} + u_n(x, \tau) \right\} (d\tau)^\alpha \quad (67)$$

$$\left\{ -6x^3\tau - (x^3 - 6x)\tau^3 \right\}$$

Taking the initial value $u_0(x, t) = u_0(x, 0) = 0$, we can derive

$$u_1(x, t) = u_0(x, t) - \frac{1}{\Gamma(\alpha + 1)} \int_0^t \left\{ \frac{\partial^\alpha u_0(x, \tau)}{\partial \tau^\alpha} - \frac{\partial^2 u_0(x, \tau)}{\partial x^2} + u_0(x, \tau) \right\} (d\tau)^\alpha \quad (68)$$

$$\left\{ -6x^3\tau - (x^3 - 6x)\tau^3 \right\}$$

$$= \frac{6x^3 t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{6(x^3 - 6x)t^{3+\alpha}}{\Gamma(4+\alpha)},$$

By the similar operations, we have

$$u_2(x, t) = \frac{6x^3 t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{6(x^3 - 6x)t^{3+\alpha}}{\Gamma(4+\alpha)} + \frac{36xt^{1+2\alpha}}{\Gamma(2+2\alpha)} + \frac{36xt^{3+2\alpha}}{\Gamma(4+2\alpha)} \quad (69)$$

$$- \frac{6x^3 t^{1+2\alpha}}{\Gamma(2+2\alpha)} - \frac{6(x^3 - 6x)t^{3+2\alpha}}{\Gamma(4+2\alpha)},$$

⋮

From Eq. (79), variational iteration method for give the following approximations:

$$u(x, t) = x^3 t^3 + \frac{6(x^3 - 6x)t^5}{3!} + \frac{36xt^5}{5!} - \frac{6x^3 t^5}{5!} + \frac{36xt^7}{7!} \quad (70)$$

$$- \frac{6(x^3 - 6x)t^{3+2\alpha}}{7!} + \dots$$

Canceling the noise terms and keeping the non-noise terms in (70) yield the exact solution of Eq. (61), for the special case $\alpha = 2$,

$$u(x, t) = x^3 t^3, \quad (71)$$

Which is easily verified.

Conclusions

The VIM has been successfully applied to derive explicit numerical solutions for nonlinear problems and ordinary, partial, fractional, integral equations. In this paper, we have discussed modified variational iteration method having integral w.r.t. $(d\tau)^\alpha$ used for the first time by Jumarie. The obtained results indicate that this method is powerful and

meaningful for solving the nonlinear fractional differential equations. Six examples indicate that the results of variational iteration method having integral w.r.t. $(d\tau)^\alpha$ are in excellent agreement with those obtained by classical VIM, GDTM, HPM and Adomian decomposition method which is available in the literature.

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