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A Note on Integration of Trigonometric Functions

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Abstract

In this paper, we derive equivalent equations of integration of secant and cosecant functions. Furthermore, we derive reduction formula for integration of product of integer powers of cosine and sine functions.

Keywords: Trigonometric identities; Integration by parts formula; Integration by substitution formula; Chain rule

Introduction

This paper consists of integration of some trigonometric functions and reduction formula of the product of integer powers of cosine and sine functions. We want to include integration of secant and cosecant functions because integration of these functions is not trivial. Most of authors of calculus books applied the sum of secant and tangent function to evaluate integration of secant function. In this paper, we apply Integration by Substitution Formula to evaluate integration of secant and cosecant functions. Moreover, we derive the reduction formula of integration of the product of integer power of cosine and sine functions.

The product of integer power of trigonometric functions is the product of integer power of cosine and sine functions. Thus, one can evaluate integration of the product of integer power of trigonometric functions by applying the reduction formula of integration of the product of integer power of cosine and sine functions. In general, we can evaluate integration of the product of integer power of trigonometric functions.

Trigonometric Identities

The following trigonometric identities are well known [1].

$$\cos^2 x + \sin^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

Derivative of Real Valued Functions

Definition 1: Suppose that the function f: [a, b] → R. If x_0 is an element of [a, b], then f is said to be differentiable at $x=x_0$ if

$$f'(x_0) = \lim_{h\to 0}[\frac{f(x_0+h)-f(x_0)}{h}]$$

exists [2].

Definition 2: If f is differentiable on its domain, then it is said to be differentiable [3].

Derivative of cosine and sine functions

Let f (x)=sin x. Then the derivative of f at a point x is given by $f'(x)=\cos x$ .It is known that the derivative of cos x is – sin x.

Derivative of product of two functions

If f and g are differentiable functions of x, then $\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}(f(x)) + f(x)\frac{d}{dx}(g(x))$.

Chain rule

Theorem 3.1 (The Chain Rule): Suppose that f is differentiable at $x=x_0$ and g is differentiable at f (x_0). Then the composite function h, defined by h(x)=g(f (x)), is differentiable at $x=x_0$, with $h'(x_0) = g'(f(x_0))f'(x_0)$ [2].

Integration of Real Valued Functions

Integration by parts formula

Let's consider the following equation.

$$\int u\,dv = uv - \int v\,du, \qquad (6.1)$$

Where u=f (x) and v=g(x) are differentiable functions of x.

The equation in equation 6.1 is the integration by parts formula [4].

Integration by substitution formula

Let's consider the following equation.

$$\int f(g(x))g'(x)dx = \int f(u)du, \qquad (6.2)$$

Where u=g(x).

The equation in equation 6.2 is the Substitution Formula [5].

Integration of Secant and Cosecant Functions

We know that

$$\cos^2 x + \sin^2 x = 1 \qquad (6.3)$$

∀ real number x.

It follows that

$$\cos x = \pm\sqrt{1-\sin^2 x} \qquad (6.4)$$

and

$$\sin x = \pm\sqrt{1-\cos^2 x} \qquad (6.5)$$

∀ real number x.

Let's choose x=arcsin u. Clearly sin (arcsin u)=u and cos (arccos u)=u.

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Therefore,

$$\cos(\arcsin u) = \pm\sqrt{1-u^2} \tag{6.6}$$

And

$$\sin(\arccos u) = \pm\sqrt{1-u^2}. \tag{6.7}$$

Claim: To derive an equivalent equation for integration of secant function.

$$\int \sec x dx = \int \frac{1}{\cos x} dx$$

Let's apply Integration by Substitution Formula. Let x=arcsin u.

This implies that sin x=u. Differentiating both sides of the equation sin x=u with respect to x, we have

$$\frac{d(\sin x)}{dx} = \frac{du}{dx}$$

Thus, $\frac{du}{dx} = \cos x = \cos(\arcsin u) = \pm\sqrt{1-u^2}$.

Now we consider two cases.

Case 1:

suppose that $cos x = \sqrt{1-sin^2 x}$

Thus,

$$\int \sec x dx = \int \frac{1}{\cos x} dx = \int \frac{1}{\cos(\arcsin u)\sqrt{1-u^2}} du$$

$$= \int \frac{1}{\sqrt{1-u^2}\sqrt{1-u^2}} du = \int \frac{1}{\sqrt{1-u^2}\sqrt{1-u^2}} du$$

$$= \int \frac{1}{1-u^2} du = \frac{1}{2}[\int \frac{1}{1+u} du + \int \frac{1}{1-u} du]$$

$$= \frac{1}{2}[\ln|1+u| - \ln|1-u|] + c = \frac{1}{2}[\ln\frac{|1+u|}{|1-u|}] + c$$

$$= \ln\sqrt{\frac{|1+u|}{|1-u|}} + c = \ln\sqrt{\frac{|1+u|^2}{|1+u||1-u|}} + c$$

$$= \ln\sqrt{\frac{|1+u|^2}{|(1+u)(1-u)|}} + c = \ln\sqrt{\frac{|1+u|^2}{|1-u \quad|}} + c$$

$$= \ln\sqrt{\frac{|1+\sin x|^2}{|1-\sin x^2|}} + c = \ln\sqrt{\frac{|1+\sin x|^2}{|\cos x|^2}} + c$$

$$= \ln\sqrt{(\frac{|1+\sin x|}{|\cos x|})^2} + c = \ln\frac{|1+\sin x|}{|\cos x|} + c$$

$$= \ln|\sec x + \tan x| + c$$

Therefore, $\int \sec x dx = \ln|\sec x + \tan x| + c.$

Case 2:

Suppose that $\cos x = -\sqrt{1-\sin^2 x}$ Then $\frac{du}{dx} = -\sqrt{1-u^2}$

Thus

$$\int \sec x dx = \int \frac{1}{\cos x} dx = -\int \frac{1}{\cos(\arcsin u)\sqrt{1-u^2}} du$$

$$= (-)(-)\int \frac{1}{\sqrt{1-u^2}\sqrt{1-u^2}} du = \int \frac{1}{\sqrt{1-u^2}\sqrt{1-u^2}} du$$

$$= \int \frac{1}{1-u^2} du = \frac{1}{2}[\int \frac{1}{1+u} du + \int \frac{1}{1-u} du]$$

$$= \frac{1}{2}[\ln|1+u| - \ln|1-u|] + c = \frac{1}{2}[\ln\frac{|1+u|}{|1-u|}] + c$$

$$= \ln\sqrt{\frac{|1+u|}{|1-u|}} + c = \ln\sqrt{\frac{|1+u|^2}{|1+u||1-u|}} + c$$

$$= \ln\sqrt{\frac{|1+u|^2}{|(1+u)(1-u)|}} + c = \ln\sqrt{\frac{|1+u|^2}{|1-u^2|}} + c$$

$$= \ln\sqrt{\frac{|1+\sin x|^2}{|1-\sin x^2|}} + c = \ln\sqrt{\frac{|1+\sin x|^2}{|\cos x|^2}} + c$$

$$= \ln\sqrt{(\frac{|1+\sin x|}{|\cos x|})^2} + c = \ln\frac{|1+\sin x|}{|\cos x|} + c$$

$$= \ln|\sec x + \tan x| + c$$

Therefore, $\int \sec x dx = \ln|\sec x + \tan x| + c$.

Therefore, considering both cases case 1 and case 2, we have $\int \sec x dx = \ln|\sec x + \tan x| + c \cdot$

Claim: To derive an equivalent equation for integration of cosecant function.

$$\int \csc x dx = \int \frac{1}{\sin x} dx$$

Let's apply Integration by Substitution Formula. Let x=arccos u.

This implies that cos x=u. Differentiating both sides of the equation cos x=u with respect to x, we have

$$\frac{d(\cos x)}{dx} = \frac{du}{dx}$$

Thus, $\frac{du}{dx} = -\sin x = -\sin(\arccos u) = -(\pm)\sqrt{1-u^2}$.

Now we consider two cases.

Case 1:

Suppose that $\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-u^2}$. Then $\frac{du}{dx} = -\sqrt{1-u^2}$.

Thus,

$$\int \csc x dx = \int \frac{1}{\sin x} dx = -\int \frac{1}{\sin(\arccos u)\sqrt{1-u^2}} du$$

$$= -\int \frac{1}{\sqrt{1-u^2}\sqrt{1-u^2}} du = -\int \frac{1}{1-u^2} du$$

$$= -\frac{1}{2}[\int \frac{1}{1+u} du + \int \frac{1}{1-u} du] = -\frac{1}{2}[\ln|1+u| - \ln|1-u|] + c$$

$$= -\frac{1}{2}[\ln\frac{|1+u|}{|1-u|}] + c = -\ln\sqrt{\frac{|1+u|}{|1-u|}} + c$$

$$= -\ln\sqrt{\frac{|1+u|^2}{|1+u||1-u|}} + c = -\ln\sqrt{\frac{|1+u|^2}{|(1+u)(1-u)|}} + c$$

$$= -\ln\sqrt{\frac{|1+u|^2}{|1-u^2|}} + c = -\ln\sqrt{\frac{|1+\cos x|^2}{|1-\cos x^2|}} + c$$

$$= -\ln\sqrt{\frac{|1+\cos x|^2}{|\sin x|^2}} + c = -\ln\sqrt{(\frac{|1+\cos x|}{|\sin x|})^2} + c$$

$$= -\ln\frac{|1+\cos x|}{|\sin x|} + c$$

$$= -\ln|\csc x + \cot x| + c$$

Therefore, $\int csc\ x\ dx = -ln|\ csc\,x + \ cot\ x| + c$.

Case 2:

Suppose that $\sin x = -\sqrt{1-\cos^2 x} = -\sqrt{1-u^2}$. Then $\frac{du}{dx} = \sqrt{1-u^2}cc$

Thus,

$$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{1}{\sin(\arccos u)\sqrt{1-u^2}} du$$

$$= -\int \frac{1}{\sqrt{1-u^2}\sqrt{1-u^2}} du = -\int \frac{1}{1-u^2} du$$

$$= -\frac{1}{2}[\int \frac{1}{1+u} du + \int \frac{1}{1-u} du] = -\frac{1}{2}[\ln|1+u| - \ln|1-u|] + c$$

$$= -\frac{1}{2}[\ln\frac{|1+u|}{|1-u|}] + c = -\ln\sqrt{\frac{|1+u|}{|1-u|}} + c$$

$$= -\ln\sqrt{\frac{|1+u|^2}{|1+u||1-u|}} + c = -\ln\sqrt{\frac{|1+u|^2}{|(1+u)(1-u)|}} + c$$

$$= -\ln\sqrt{\frac{|1+u|^2}{|1-u^2|}} + c = -\ln\sqrt{\frac{|1+\cos x|^2}{|1-\cos x^2|}} + c$$

$$= -\ln\sqrt{\frac{|1+\cos x|^2}{|\sin x|^2}} + c = -\ln\sqrt{(\frac{|1+\cos x|}{|\sin x|})^2} + c$$

$$= -\ln\frac{|1+\cos x|}{|\sin x|} + c$$

$$= -\ln|\csc x + \cot x| + c$$

Therefore, $\int \csc x dx = -\ln|\csc x + \cot x| + c$. Therefore, considering both cases case1 and case 2, we have $\int \csc x dx = -\ln|\csc x + \cot x| + c$.

Reduction formula for integration of product of integer power of cosine and sine functions

For m=−1 and n=1,

$$\int \cos^m x \sin^n x dx = \int \tan x dx$$

$$= \int \frac{\sin x}{\cos x} dx$$

$$= -\ln|\cos x| + c \tag{6.8}$$

For m=-n,

$$\int \cos^m x \sin^n x dx = \int \tan^n x dx$$

$$= \int \tan^{n-2} x \tan^2 x dx$$

$$= \int \tan^{n-2} x[1-\sec^2 x]dx$$

$$= \int \tan^{n-2} x dx - \int \tan^{n-2} x[\sec^2 x]dx \tag{6.9}$$

Let's apply Integration by Substitution Formula.

Let tan x=u. Then du=sec² dx Thus, we have

$$\int \tan^{n-2} x[\sec^2 x]dx = \int u^{n-2} du$$

$$= \frac{u^{n-1}}{n-1}$$

$$= \frac{\tan^{n-1} x}{n-1} \tag{6.10}$$

Here $n \neq 1$. If n=1, see 6.8. Therefore, for m=-n and $n \neq 1$, from equations 6.9 and 6.10, we have

$$\int \cos^m x \sin^n x dx = \int \tan^{n-2} x dx - \frac{\tan^{n-1} x}{n-1} \tag{6.11}$$

Theorem 6.1: Let m and n be integers. Then for $m \neq -n$ and $m \neq 2-n$,

$$\int \cos^m x \sin^n x dx = \frac{[n-1+(2-m-n)\cos^2 x]\cos^{m-1} x \sin^{n-1} x}{(m+n-2)(m+n)}$$

$$+ \frac{(m-1)(n-1)}{(m+n-2)(m+n)} \int \cos^{m-2} x \sin^{n-2} x dx$$

Proof of Theorem 6.1:

Step 1

Let's find the derivative of $\cos^m x \sin^{n-1} x$ with respect to x.

$$\frac{d}{dx}[(\cos^m x)(\sin^{n-1} x)] = (\frac{d}{dx}(\cos^m x))(\sin^{n-1} x)$$

$$+(\frac{d}{dx}(\sin^{n-1} x))(\cos^m x)$$

$$= [m\cos^{m-1} x][\frac{d}{dx}(\cos x)][\sin^{n-1} x]$$

$$+[(n-1)\sin^{n-2} x][\frac{d}{dx}(\sin x)](\cos^m x)$$

$$= -m(\cos^{m-1} x)(\sin x)(\sin^{n-1} x) + (n-1)(\sin^{n-2} x)(\cos x)(\cos^m x)$$

$$= -m(\cos^{m-1} x)(\sin^n x) + (n-1)(\sin^{n-2} x)(\cos^{m+1} x)$$

$$= [\cos^{m-1} x \sin^{n-2} x][(n-1)\cos^2 x - m\sin^2 x]$$

$$= [\cos^{m-1} x \sin^{n-2} x][(n-1)(1-\sin^2 x) - m\sin^2 x]$$

$$= [(\cos^{m-1} x)(\sin^{n-2} x)][(n-1) + (1-m-n)\sin^2 x]$$

$$= (n-1)(\cos^{m-1} x)(\sin^{n-2} x) + (1-m-n)(\cos^{m-1} x)(\sin^n x)$$

Step 2

Let's find the derivative of $(\cos^{m-1} x)(\sin^{n-2} x)$ with respect to x.

$$\frac{d}{dx}[(\cos^{m-1} x)(\sin^{n-2} x)] = (\frac{d}{dx}(\cos^{m-1} x))\sin^{n-2} x$$

$$+(\frac{d}{dx}(\sin^{n-2}x))\cos^{m-1}x$$
$$=[(m-1)\cos^{m-2}x][\frac{d}{dx}(\cos x)][\sin^{n-2}x]$$
$$+[(n-2)\sin^{n-3}x][\frac{d}{dx}(\sin x)][\cos^{m-1}x]c$$
$$=(1-m)(\cos^{m-2}x)(\sin x)(\sin^{n-2}x)$$
$$+(n-2)(\sin^{n-3}x)(\cos x)(\cos^{m-1}x)$$
$$=(1-m)(\cos^{m-2}x)(\sin^{n-1}x)+(n-2)(\sin^{n-3}x)(\cos^{m}x)$$
$$=[(\cos^{m-2}x)(\sin^{n-3}x)][(n-2)\cos^2 x+(1-m)\sin^2 x]$$
$$=[(\cos^{m-2}x)(\sin^{n-3}x)][(n-2)(\cos^2 x)+(1-m)(1-\cos^2 x)]$$
$$+[(\cos^{m-2}x)(\sin^{n-3}x)][(1-m)+(m+n-3)(\cos^2 x)]$$
$$=(1-m)(\cos^{m-2}x)(\sin^{n-3}x)+(m+n-3)(\cos^{m}x)(\sin^{n-3}x)$$

Step 3

$$\int\cos^m x\sin^n x dx=\int\cos^m x\sin^{n-1}x\sin x dx$$
$$=-\cos^m x\sin^{n-1}x\cos x$$
$$+\int\cos x[(n-1)\cos^{m-1}x\sin^{n-2}x+(1-m-n)\cos^{m-1}x\sin^n x]dx$$
$$=-\cos^{m+1}x\sin^{n-1}x$$
$$+(n-1)\int\cos^m x\sin^{n-2}x dx+(1-m-n)\int\cos^m x\sin^n x dx$$

Now, observe that

$$\int\cos^m x\sin^n x dx=\frac{-\cos^{m+1}x\sin^{n-1}x+(n-1)\int\cos^m x\sin^{n-2}x dx}{m+n} \quad (6.12)$$

If we choose m=0 and $n\neq 0$ for equation 6.12, then equation in equation 6.12 is reduction formula for sine function. That is, for $n\neq 0$,

$$\int\sin^n x dx=\frac{-\cos x\sin^{n-1}x+(n-1)\int\sin^{n-2}x dx}{n}.$$

If m=2 – n for equation 6.12, then see equation 6.11.

Step 4

$$\int\cos^m x\sin^{n-2}x dx=\int\cos^{m-1}x\sin^{n-2}x\cos x dx$$
$$=\cos^{m-1}x\sin^{n-1}x$$
$$-\int\sin x[(1-m)\cos^{m-2}x\sin^{n-3}x+(m+n-3)\cos^m x\sin^{n-3}x]dx$$
$$=\cos^{m-1}x\sin^{n-1}x$$
$$+(m-1)\int\cos^{m-2}x\sin^{n-2}x dx+(3-m-n)\int\cos^m x\sin^{n-2}x dx$$

Now, observe that

$$\int\cos^m x\sin^{n-2}x dx=\frac{\cos^{m-1}x\sin^{n-1}x+(m-1)\int\cos^{m-2}x\sin^{n-2}x dx m+n-2}{m+n-2} \quad (6.13)$$

If we choose n=2 and $m\neq 0$, then equation in equation 6.13 is reduction formula for cosine function. That is, for $m\neq 0$,

$$\int\cos^m x dx=\frac{\cos^{m-1}x\sin x+(m-1)\int\cos^{m-2}x dx}{m}.$$

Therefore, from equations in 6.12 and 6.13, we get

$$\int\cos^m x\sin^n x dx=\frac{[n-1+(2-m-n)\cos^2 x]\cos^{m-1}x\sin^{n-1}x}{(m+n-2)(m+n)}$$
$$+\frac{(m-1)(n-1)}{(m+n-2)(m+n)}\int\cos^{m-2}x\sin^{n-2}x dx$$

Hence proved

Conclusion

In general, we observed the following results.

Let m and n be integers. Then

1. For $m\neq -n$ and $m\neq 2-n$,

$$\int\cos^m x\sin^n x dx=\frac{[n-1+(2-m-n)\cos^2 x]\cos^{m-1}x\sin^{n-1}x}{(m+n-2)(m+n)}$$
$$+\frac{(m-1)(n-1)}{(m+n-2)(m+n)}\int\cos^{m-2}x\sin^{n-2}x dx$$

2. For m=-n and $n\neq 1$,

$$\int\cos^m x\sin^n x dx=\int\tan^{n-2}x dx-\frac{\tan^{n-1}x}{n-1}$$

3. For m=-1 and n=1,

$$\int\cos^m x\sin^n x dx=\int\tan x dx$$
$$=\int\frac{\sin x}{\cos x}dx$$
$$=-\ln|\cos x|+c$$

4. For m=-n and $n\neq 1$, we have

$$\int\cos^m x\sin^n x dx=\int\tan^{n-2}x dx-\frac{\tan^{n-1}x}{n-1}$$

5. For $m\neq 0$,

$$\int\cos^m x dx=\frac{\cos^{m-1}x\sin x+(m-1)\int\cos^{m-2}x dx}{m}$$

6. For $n\neq 0$,

$$\int\sin^n x dx=\frac{-\cos x\sin^{n-1}x+(n-1)\int\sin^{n-2}x dx}{n}$$

7. $\int\csc x dx=-\ln|\csc x+\cot x|+c$

8. $\int\sec x dx=\ln|\sec x+\tan x|+c$

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