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A New Number Theory-Algebra Analysis II

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Abstract

The basis of this quaternions algebra. The problem of the $\vec{j}\cdot\vec{k}$ product. 3d (and 4d) product and division in algebraic form; also, the algebraic forms of the product and of the division are differentiable. Questions about the possibility of extend this algebra to more dimensions.

Keywords: Quaternions; Operator theory; Algebra; Tensor methods

Three-Dimensions

A recent publication [1] has extended the concepts of the sum and of the product for 3d-4d numbers as new quaternions. The sum and the product, as defined, are commutative.

Paper [1] implicitly gave the definitions of the norm (or modulus) of a 3d number, of the inverse of a 3d number, and of the conjugate.

Figure 1 gives a 3d space representation; the tern $(\vec{1},\vec{j},\vec{k})$ must be considered a tern of orthogonal unit vectors. $\vec{1}$ is the real unity and can be omitted in the symbolic calculus; so for the 3d space we can write:

$$s = x + \vec{j}\cdot y + \vec{k}\cdot z$$

the conjugate of *s*:

$$\bar{s} = x - \vec{j}\cdot y - \vec{k}\cdot z$$

the norm (or modulus) of *s*:

$$\|s\| = |s| = \sqrt{s\cdot\bar{s}} = \sqrt{x^2+y^2+z^2}$$

the $\frac{1}{s}$ inverse property:

$$s\cdot\frac{1}{s} = s\cdot\frac{\bar{s}}{|s|^2} = \vec{1}$$

the product of *s* for a real constant:

$$\sigma\cdot s = \sigma\cdot x + \vec{j}\cdot\sigma\cdot y + \vec{k}\cdot\sigma\cdot z \qquad \sigma\in\Re$$

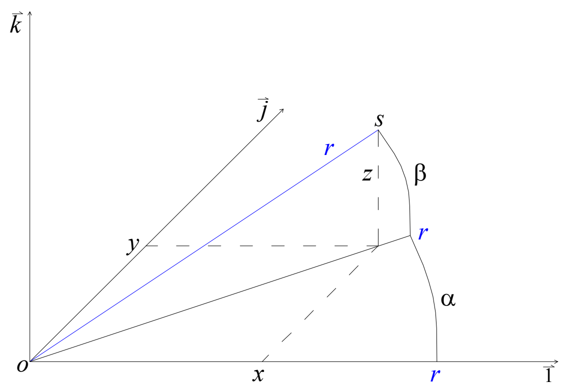

Figure 1: 3d space representation.

the scalar product and the vector product are also well defined (see code 3d - 2.4g in appendix of paper [2]).

3d scalar product:

$$s_a \wedge s_b = x_a\cdot x_b + y_a\cdot y_b + z_a\cdot z_b$$

3d vector product:

$$s_a\times s_b = \det\begin{bmatrix}\vec{1} & \vec{j} & \vec{k}\\ x_a & y_a & z_a\\ x_b & y_b & z_b\end{bmatrix} \quad \text{[operative formula]}$$

in algebraic form:

$$s_a\times s_b = (y_a\cdot z_b - z_a\cdot y_b) + \vec{j}\cdot(z_a\cdot x_b - x_a\cdot z_b) + \vec{k}\cdot(x_a\cdot y_b - y_a\cdot x_b)$$

So, we have the same symbolic of the standard 2d complex numbers.

Paper [2] analyzed some aspects of this algebra, we have seen that this algebra is not distributive, and that this produces some limitations in derivatives and integrals, also we have seen the extended definitions of functions such as sin(*s*) and cos(*s*) may be meaningless.

The problem is because this 3d space is a curved space, the transformations that permit to define the product as a commutative product, are not linear.

Someone could object that the algebraic definition of the $\vec{j}\cdot\vec{k}$ product, in paper [1], is undefined (in polar notation is defined and it is differentiable); in 1843 William Rowan Hamilton has defined the $j\cdot k$ product in an algebraic form but, with that definition, Hamilton created a non-commutative algebra.

I try now to give an answer about the generic algebraic definition of the product between two 3d numbers as defined in paper [1].

Given:

$$s_a = x_a + \vec{j}\cdot y_a + \vec{k}\cdot z_a$$

and

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$s_b = x_b + \vec{j} \cdot y_b + \vec{k} \cdot z_b$

let us develop the algebraic form of the product $s' = s_a \cdot s_b$

For a generic *s* (3d) number we can write:

$r = \sqrt{x^2 + y^2 + z^2}$ $\qquad c = \sqrt{x^2 + y^2}$

if $r \neq 0$ then $\sin(\beta) = \frac{z}{r}$; $\cos(\beta) = \sqrt{1 - \frac{z^2}{r^2}}$

if $c \neq 0$ then $\sin(\alpha) = \frac{y}{c}$; $\cos(\alpha) = \frac{x}{c}$

if *c*=0 then *α*=0, $\beta = \frac{\pi}{2} \cdot sign(z)$; (*x*=*y*=0)

if *r*=0 and *c*=0 then $\beta = 0, \quad \alpha = 0$ (*x*= *y*= *z*=0)

so

$s' = (x_a + \vec{j} \cdot y_a + \vec{k} \cdot z_a) \cdot (x_b + \vec{j} \cdot y_b + \vec{k} \cdot z_b) = x' + \vec{j} \cdot y' + \vec{k} \cdot z'$

the result of the product $s' = s_a \cdot s_b$ in polar notation form is:

$x' = r_a \cdot r_b \cdot \cos(\beta_a + \beta_b) \cdot \cos(\alpha_a + \alpha_b)$

$y' = r_a \cdot r_b \cdot \cos(\beta_a + \beta_b) \cdot \sin(\alpha_a + \alpha_b)$

$z' = r_a \cdot r_b \cdot \sin(\beta_a + \beta_b)$

Definitions:

$r_a = \sqrt{x_a^2 + y_a^2 + z_a^2}$ $\qquad c_a = \sqrt{x_a^2 + y_a^2}$

$r_b = \sqrt{x_b^2 + y_b^2 + z_b^2}$ $\qquad c_b = \sqrt{x_b^2 + y_b^2}$

now, we need to analyze 4 cases:

(1) Generic case: $c_a \neq 0$, $c_b \neq 0$

$$\cos(\beta_a + \beta_b) = \frac{1}{r_a \cdot r_b}(c_a \cdot c_b - z_a \cdot z_b)$$

$$\cos(\alpha_a + \alpha_b) = \frac{(x_a \cdot x_b - y_a \cdot y_b)}{c_a \cdot c_b}.$$

$$\sin(\alpha_a + \alpha_b) = \frac{(x_a \cdot y_b + y_a \cdot x_b)}{c_a \cdot c_b}$$

$$\sin(\beta_a + \beta_b) = \frac{1}{r_a \cdot r_b}(c_a \cdot z_b + z_a \cdot c_b)$$

by substituting:

$$x' = (c_a \cdot c_b - z_a \cdot z_b) \cdot \frac{(x_a \cdot x_b - y_a \cdot y_b)}{c_a \cdot c_b}$$

$$y' = (c_a \cdot c_b - z_a \cdot z_b) \cdot \frac{(x_a \cdot y_b + y_a \cdot x_b)}{c_a \cdot c_b}$$

$$z' = (c_a \cdot z_b + z_a \cdot c_b)$$

(2) If $c_a=0$, $c_b \neq 0$ then $\alpha_a=0$, $\beta_a = \frac{\pi}{2} \cdot sign(z_a)$

($x_a=0$, $y_a=0$)

so in this case:

$$\cos(\beta_a + \beta_b) = -sign(z_a) \cdot \frac{z_b}{r_b}$$

$$\cos(\alpha_a + \alpha_b) = \cos(\alpha_b) = \frac{x_b}{c_b}$$

$$\sin(\alpha_a + \alpha_b) = \sin(\alpha_b) = \frac{y_b}{c_b}$$

$$\sin(\beta_a + \beta_b) = sign(z_a) \cdot \cos(\beta_b) = sign(z_a) \cdot \frac{c_b}{r_b}$$

because in this case $r_a = \sqrt{z_a^2}$

it can be observed that $r_a \cdot sign(z_a) = z_a$

so:

$$x' = -r_a \cdot r_b \cdot sign(z_a) \cdot \frac{z_b}{r_b} \cdot \frac{x_b}{c_b} = -z_a \cdot z_b \cdot \frac{x_b}{c_b}$$

$$y' = -r_a \cdot r_b \cdot sign(z_a) \cdot \frac{z_b}{r_b} \cdot \frac{y_b}{c_b} = -z_a \cdot z_b \cdot \frac{y_b}{c_b}$$

$$z' = r_a \cdot r_b \cdot sign(z_a) \cdot \frac{c_b}{r_b} = z_a \cdot c_b$$

(3) If $c_a \neq 0$, $c_b=0$ then $\alpha_b=0$, $\beta_b = \frac{\pi}{2} \cdot sign(z_b)$

$$x' = -z_a \cdot z_b \cdot \frac{x_a}{c_a}$$

$$y' = -z_a \cdot z_b \cdot \frac{y_a}{c_a}$$

$$z' = z_b \cdot c_a$$

(4) If $c_a=0$ and $c_b=0$ then $\alpha_a=0$, $\alpha_b=0$

$$\beta_a = \frac{\pi}{2} \cdot sign(z_a), \ \beta_b = \frac{\pi}{2} \cdot sign(z_b)$$

($x_a=0$, $y_a=0$); ($x_b=0$, $y_b=0$)

in this case $\beta_a + \beta_b = 0$ or $\pm\pi$

$$x' = -z_a \cdot z_b$$

$$y' = 0$$

$$z' = 0$$

The generic case (1) of the algebraic product of $s_a \cdot s_b$ is differentiable. The other cases are limit case and are differentiable too.

The algebraic form of $\frac{1}{s}$ is quite simple:

given:

$s = x + \vec{j} \cdot y + \vec{k} \cdot z$ $\qquad r = \sqrt{x^2 + y^2 + z^2} \neq 0$

$$s' = \frac{1}{s} = \frac{1}{r^2}(x - \vec{j} \cdot y - \vec{k} \cdot z)$$

and it is differentiable.

Now we can define the algebraic form of the division: $s' = \frac{s_a}{s_b}$

Definitions:

$r_a = \sqrt{x_a^2 + y_a^2 + z_a^2}$ $\qquad c_a = \sqrt{x_a^2 + y_a^2}$

$r_b = \sqrt{x_b^2 + y_b^2 + z_b^2} \neq 0$ $\qquad c_b = \sqrt{x_b^2 + y_b^2}$

it can be observed that

$$\frac{1}{s_b} = \frac{1}{r_b^2} \cdot (x_b - \vec{j} \cdot y_b - \vec{k} \cdot z_b)$$

so

$$s' = \frac{s_a}{s_b} = (x_a + \vec{j} \cdot y_a + \vec{k} \cdot z_a) \cdot \left[\frac{1}{r_b^2} \cdot \left(x_b - \vec{j} \cdot y_b - \vec{k} \cdot z_b \right) \right]$$

again we have to analyze 4 cases:

(1a) Generic case: $c_a \neq 0$, $c_b \neq 0$

$$x' = \frac{1}{r_b^2} \cdot \left(c_a \cdot c_b + z_a \cdot z_b \right) \cdot \frac{\left(x_a \cdot x_b + y_a \cdot y_b \right)}{c_a \cdot c_b}$$

$$y' = \frac{1}{r_b^2} \cdot \left(c_a \cdot c_b + z_a \cdot z_b \right) \cdot \frac{\left(-x_a \cdot y_b + y_a \cdot x_b \right)}{c_a \cdot c_b}$$

$$z' = \frac{1}{r_b^2} \cdot \left(-c_a \cdot z_b + z_a \cdot c_b \right)$$

(2a) If $c_a = 0$, $c_b \neq 0$

$$x' = \frac{1}{r_b^2} \cdot z_a \cdot z_b \cdot \frac{x_b}{c_b}$$

$$y' = -\frac{1}{r_b^2} \cdot z_a \cdot z_b \cdot \frac{y_b}{c_b}$$

$$z' = \frac{1}{r_b^2} \cdot z_a \cdot c_b$$

(3a) If $c_a \neq 0$, $c_b = 0$

in this case it can be observed that $\frac{1}{r_b^2} \cdot z_b = sign(z_b)$

$$x' = \frac{1}{r_b^2} \cdot z_a \cdot z_b \cdot \frac{x_a}{c_a} = sign(z_b) \cdot z_a \cdot \frac{x_a}{c_a}$$

$$y' = \frac{1}{r_b^2} \cdot z_a \cdot z_b \cdot \frac{y_a}{c_a} = sign(z_b) \cdot z_a \cdot \frac{y_a}{c_a}$$

$$z' = -\frac{1}{r_b^2} \cdot z_b \cdot c_a = -sign(z_b) \cdot c_a$$

(4a) If $c_a = 0$, $c_b = 0$

$$x' = \frac{z_a}{z_b}$$

$$y' = 0$$

$$z' = 0$$

The generic case (1a) of the algebraic form of the division $\frac{s_a}{s_b}$ is differentiable. The other cases are limit cases, and are differentiable too, in fact $\frac{s_a}{s_b}$ can be seen as the product of $s_a \cdot \frac{1}{s_b}$ where $|s_b| \neq 0$

Anyway, the objectionable limit case (3a) is:

$$x' = sign(z_b) \cdot z_a \cdot \frac{x_a}{c_a}$$

$$y' = sign(z_b) \cdot z_a \cdot \frac{y_a}{c_a}$$

$$z' = -sign(z_b) \cdot c_a$$

The differential depends on x_a, y_a, z_a and on the sign of z_b; this because we are doing a differential around a fixed s_b point (North Pole or South Pole of the sphere), where, what matters of s_b, is just the sign of z_b.

The algebraic definition of the $\vec{j} \cdot \vec{k}$ product can be seen as defined by the algebraic analysis of above and, in particular, it is the limit case (3) (see appendix B for the solved code of the algebraic definition of the 3d product and division).

Another consequence of this analysis is that, now, it is possible to try to analyze a generic 2° order (3d) equation:

$$a \cdot s^2 + b \cdot s + c = 0 \Leftrightarrow a \cdot s \cdot |s|^2 + b \cdot |s|^2 + c \cdot \bar{s} = 0 \qquad |s|^2 \neq 0$$

where, in general, a, b, and c can be real numbers or 3d numbers.

Because now we have an algebraic differentiable definition of the product and of the division, it is clear that if we have two 3d functions (see paper [2]) such as:

$$f_1(z) = x_1(z) + \vec{j} \cdot y_1(z) + \vec{k} \cdot z_1(z) \qquad z \in R \text{ or } z \in C$$

$$f_2(z) = x_2(z) + \vec{j} \cdot y_2(z) + \vec{k} \cdot z_2(z)$$

Where $x_1(z)$, $y_1(z)$, $z_1(z)$ and $x_2(z)$, $y_2(z)$, $z_2(z)$ are all differentiable functions and they give real results, the product:

$$f(z) = f_1(z) \cdot f_2(z)$$

and the division:

$$f(z) = f_1(z) / f_2(z)$$

are differentiable. This was another open argument of paper [2].

Conclusion

The above analysis has shown it is possible to give an algebraic definition of the product and of the division for 3d numbers as defined in paper [1] and that these algebraic definitions are differentiable.

Same algebraic analysis can also be done for 4d product and division, even that it is a bit more complex (see Figure 2 and appendix A).

For a 4d number we can write:

$$s = x + \vec{j} \cdot y + \vec{k} \cdot z + \vec{h} \cdot t$$

the conjugate:

$$\bar{s} = x - \vec{j} \cdot y - \vec{k} \cdot z - \vec{h} \cdot t$$

and so on for the inverse property, the norm (or modulus) etc.

In paper [2] I gave a proposal generic sum definition.

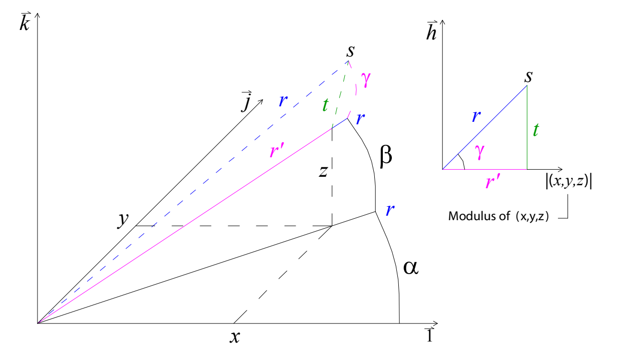

Figure 2: 4d space representation.

The objectionable point was to assign by default the v3space' sign set to 1 in the case that $r'_a = r'_b$ ($r'_a = \sqrt{x_a^2 + y_a^2 + z_a^2}$; $r'_b = \sqrt{x_b^2 + y_b^2 + z_b^2}$); also, it could be questionable the generic 4d scalar product definition. This was a mistake.

The solution is much simpler; the γ angle must be treated in the same way of the β angle; β rotates, but in fact, at the end of calculations is reduced to $|\beta| <= \pi/2$ (see Figure 2). The same must be done for γ, at the end of calculations γ must be reduced to $|\gamma| <= \pi/2$.

So the sum in 4d space is the same of the sum in 3d space (see appendix C) and, because the schema for 4d is the same for 3d, it is obvious that this idea can be extended to more dimensions.

There are no problems to extend the scalar product formula to more dimensions; a last consideration can be done for the extended definition of the 4d vector product.

A 3d number (or a 4d number) can be seen as a vector (*x,y,z*). Let us consider C as the 3d resulting vector product between two 3d vectors A and B; C can be seen as an orthogonal vector whose length is the area of the parallelogram identified by the two non-parallel vectors A and B, so the D 4d resulting vector product between tree 4d vectors A, B and C can be defined as an orthogonal vector to A, B and C whose length is the volume of the solid identified by the tree non-coplanar 4d vectors A, B and C (for simplicity, you can think that A, B and C are 4d numbers whose $\vec{h}$ component value is 0).

The D vector as result of 4d vector product of A, B and C, is given by the following operative formula:

$$D = \det\begin{bmatrix} \vec{1} & \vec{j} & \vec{k} & \vec{h} \\ x_A & y_A & z_A & t_A \\ x_B & y_B & z_B & t_B \\ x_C & y_C & z_C & t_C \end{bmatrix}$$

The formula can be extended to more dimensions. Versus (sign) of D depends on the tern A, B and C, but these are all well-known questions.

Appendix A: 4d numbers analysis

Consider *s* a 4d number:

$$s = x + \vec{j}\cdot y + \vec{k}\cdot z + \vec{h}\cdot t$$

given:

$$r = \sqrt{x^2 + y^2 + z^2 + t^2}$$

$$\sin(\gamma) = \frac{t}{r}$$

$$\cos(\gamma) = \sqrt{1 - \frac{t^2}{r^2}}$$

$$r' = \sqrt{x^2 + y^2 + z^2} \neq 0$$

$$\sin(\beta) = \frac{z}{r'}$$

$$\cos(\beta) = \sqrt{1 - \frac{z^2}{r'^2}}$$

given:

$$c = \sqrt{x^2 + y^2} \neq 0$$

$$\sin(\alpha) = \frac{y}{c}$$

$$\cos(\alpha) = \frac{x}{c}$$

If *c*=0 then $\alpha = 0, \beta = \frac{\pi}{2}\cdot sign(z);\ (x = y = 0)$

if $r' = 0$, then *β*=0, *α*=0, (*x*=*y*=*z*=*0*)

The 4d product between two 4d numbers $s' = s_a \cdot s_b$ is:

$$s' = (x_a + \vec{j}\cdot y_a + \vec{k}\cdot z_a + \vec{h}\cdot t_a)\cdot(x_b + \vec{j}\cdot y_b + \vec{k}\cdot z_b + \vec{h}\cdot t_b)$$

$$s' = s_a \cdot s_b = x' + \vec{j}\cdot y' + \vec{k}\cdot z' + \vec{h}\cdot t'$$

The result in polar notation is:

$$R' = r_a \cdot r_b \cdot |\cos(\gamma_a + \gamma_b)| = |r'_a \cdot r'_b - t_a \cdot t_b| \text{ (see definitions below)}$$

$$t' = r_a \cdot r_b \cdot \sin(\gamma_a + \gamma_b)$$

$$z' = R' \cdot \sin(\beta_a + \beta_b)$$

$$y' = R' \cdot \cos(\beta_a + \beta_b)\cdot\sin(\alpha_a + \alpha_b)$$

$$x' = R' \cdot \cos(\beta_a + \beta_b)\cdot\cos(\alpha_a + \alpha_b)$$

Definitions:

$$r_a = \sqrt{x_a^2 + y_a^2 + z_a^2 + t_a^2} \quad r'_a = \sqrt{x_a^2 + y_a^2 + z_a^2} \quad c_a = \sqrt{x_a^2 + y_a^2}$$

$$r_b = \sqrt{x_b^2 + y_b^2 + z_b^2 + t_b^2} \quad r'_b = \sqrt{x_b^2 + y_b^2 + z_b^2} \quad c_b = \sqrt{x_b^2 + y_b^2}$$

$$\cos(\gamma_a + \gamma_b) = \frac{r'_a \cdot r'_b - t_a \cdot t_b}{r_a \cdot r_b}$$

$$\sin(\gamma_a + \gamma_b) = \frac{r'_a \cdot t_b + r'_b \cdot t_a}{r_a \cdot r_b}$$

now we have to analyze 7 cases:

(1) Generic case: $c_a \neq 0$, $c_b \neq 0$, $r'_b \cdot r'_b \neq 0$

$$r'_a = \sqrt{x_a^2 + y_a^2 + z_a^2} \qquad r'_b = \sqrt{x_b^2 + y_b^2 + z_b^2}$$

$$R' = |r'_a \cdot r'_b - t_a \cdot t_b|$$

$$\cos(\beta_a + \beta_b) = \frac{1}{r'_a \cdot r'_b}(c_a \cdot c_b - z_a \cdot z_b)$$

$$\cos(\alpha_a + \alpha_b) = \frac{(x_a \cdot x_b - y_a \cdot y_b)}{c_a \cdot c_b}$$

$$\sin(\alpha_a + \alpha_b) = \frac{(x_a \cdot y_b + y_a \cdot x_b)}{c_a \cdot c_b}$$

$$\sin(\beta_a + \beta_b) = \frac{1}{r'_a \cdot r'_b}(c_a \cdot z_b + z_a \cdot c_b)$$

$$x' = \frac{R'}{r'_a \cdot r'_b}\cdot(c_a \cdot c_b - z_a \cdot z_b)\cdot\frac{(x_a \cdot x_b - y_a \cdot y_b)}{c_a \cdot c_b}$$

$$y'=\frac{R'}{r'_a \cdot r'_b}\cdot(c_a\cdot c_b - z_a\cdot z_b)\cdot\frac{(x_a\cdot y_b + y_a\cdot x_b)}{c_a\cdot c_b}$$

$$z'=\frac{R'}{r'_a \cdot r'_b}\cdot(c_a\cdot z_b + z_a\cdot c_b)$$

$$t'=r_a\cdot r_b\cdot\sin(\gamma_a+\gamma_b)=r'_a\cdot t_b + r'_b\cdot t_a$$

(2) If $c_a=0$, $c_b\neq 0$, $r'_b\cdot r'_b\neq 0$ then $\alpha_a=0$

$$\beta_a=\frac{\pi}{2}\cdot sign(z_a),\quad (x_a=0, y_a=0)$$

$$x'=-\frac{R'}{r'_a\cdot r'_b}\cdot z_a\cdot z_b\cdot\frac{x_b}{c_b}$$

$$y'=-\frac{R'}{r'_a\cdot r'_b}\cdot z_a\cdot z_b\cdot\frac{y_b}{c_b}$$

$$z'=\frac{R'}{r'_a\cdot r'_b}\cdot z_a\cdot c_b$$

$$t'=r'_a\cdot t_b + r'_b\cdot t_a$$

(3) If $c_a\neq 0$, $c_b=0$, $r'_b\cdot r'_b\neq 0$ then $\alpha_b=0$

$$\beta_b=\frac{\pi}{2}\cdot sign(z_b)\quad (x_b=0, y_b=0)$$

$$x'=-\frac{R'}{r'_a\cdot r'_b}\cdot z_a\cdot z_b\cdot\frac{x_a}{c_a}$$

$$y'=-\frac{R'}{r'_a\cdot r'_b}\cdot z_a\cdot z_b\cdot\frac{y_a}{c_a}$$

$$z'=\frac{R'}{r'_a\cdot r'_b}\cdot z_b\cdot c_a$$

$$t'=r'_a\cdot t_b + r'_b\cdot t_a$$

(4) If $c_a=0$ and $c_b=0$; $r'_a\cdot r'_b\neq 0$ then $\alpha_a=0$, $\alpha_b=0$

$$\beta_a=\frac{\pi}{2}\cdot sign(z_a),\ \beta_b=\frac{\pi}{2}\cdot sign(z_b)$$

$$x_a=0\, y_a=0,\ x_b=0\, y_b=0$$

So $\beta_a+\beta_b=0$ or $\pm\pi$

In this case note that $r'_a\cdot r'_b=z_a\cdot z_b$

$$x'=-\frac{R'}{r'_a\cdot r'_b}\cdot z_a\cdot z_b=-R'\cdot sign(z_a\cdot z_b)$$

$$y'=0$$

$$z'=0$$

$$t'=r'_a\cdot t_b + r'_b\cdot t_a$$

(5) If $r'_a\cdot r'_b=0$; $r'_b\neq 0$; $\alpha_a=0$, $\beta_a=0$

$$x'=R'\cdot\cos(\beta_b)=\frac{R'}{r'_b}\cdot x_b$$

$$y'=R'\cdot\cos(\beta_b)\cdot\sin(\alpha_b)=\frac{R'}{r'_b}\cdot y_b$$

$$z'=R'\cdot\sin(\beta_b)=\frac{R'}{r'_b}\cdot z_b$$

$$t'=r'_a\cdot t_b + r'_b\cdot t_a$$

(6) If $r'_a\cdot r'_b=0$; $r'_a\neq 0$; $\alpha_b=0$, $\beta_b=0$

$$x'=R'\cdot\cos(\beta_a)=\frac{R'}{r'_a}\cdot x_a$$

$$y'=R'\cdot\cos(\beta_a)\cdot\sin(\alpha_a)=\frac{R'}{r'_a}\cdot y_a$$

$$z'=R'\cdot\sin(\beta_b)=\frac{R'}{r'_a}\cdot z_a$$

$$t'=r'_a\cdot t_b + r'_b\cdot t_a$$

(7) If $r'_a\cdot r'_b=0$; $\alpha_a=0$, $\alpha_b=0$; and $\beta_a=\beta_b=0$

in this case $\gamma_a+\gamma_b=0$ or $\pm\pi$; so:

$$x'=|t_a\cdot t_b|$$

$$y'=0$$

$$z'=0$$

$$t'=0$$

The generic case (1) of the 4d algebraic product of $s_a\cdot s_b$ is differentiable. The other cases are limit case and are also differentiable. Limit case (4) may be an objectionable limit case, but is the same questionable problem we have seen above for the division in 3d; the differential depends on the z_a and z_b sign.

The algebraic form of $s'=\frac{1}{s}$:

given:

$$r=\sqrt{x^2+y^2+z^2+t^2}\neq 0$$

$$s'=\frac{1}{s}=\frac{1}{r^2}\left(x-\vec{j}\cdot y-\vec{k}\cdot z-\vec{h}\cdot t\right)$$

and it is differentiable.

The algebraic form of $s'=\frac{s_a}{s_b}$:

Definitions:

$$r_a=\sqrt{x_a^2+y_a^2+z_a^2+t_a^2}$$

$$r'_a=\sqrt{x_a^2+y_a^2+z_a^2}\qquad c_a=\sqrt{x_a^2+y_a^2}$$

$$r_b=\sqrt{x_b^2+y_b^2+z_b^2+t_b^2}\neq 0$$

$$r'_b=\sqrt{x_b^2+y_b^2+z_b^2}\qquad c_b=\sqrt{x_b^2+y_b^2}$$

$$\frac{1}{s_b}=\frac{1}{r_b^2}\cdot\left(x_b-\vec{j}\cdot y_b-\vec{k}\cdot z_b-\vec{h}\cdot t_b\right)$$

$$\frac{s_a}{s_b}=(x_a+\vec{j}\cdot y_a+\vec{k}\cdot z_a+\vec{h}\cdot t_a)\cdot\left[\frac{1}{r_b^2}\cdot\left(x_b-\vec{j}\cdot y_b-\vec{k}\cdot z_b-\vec{h}\cdot t_b\right)\right]$$

Also here we have to analyze 7 cases:

(1a) Generic case: $c_a\neq 0$, $c_b\neq 0$; $r'_a\cdot r'_b\neq 0$

$$r'_a=\sqrt{x_a^2+y_a^2+z_a^2}\qquad r'_b=\sqrt{x_b^2+y_b^2+z_b^2}$$

$$R' = \left| r'_a \cdot r'_b + t_a \cdot t_b \right|$$

$$x' = \frac{1}{r_b^2} \cdot \frac{R'}{r'_a \cdot r'_b} \cdot (c_a \cdot c_b + z_a \cdot z_b) \cdot \frac{(x_a \cdot x_b + y_a \cdot y_b)}{c_a \cdot c_b}$$

$$y' = \frac{1}{r_b^2} \cdot \frac{R'}{r'_a \cdot r'_b} \cdot (c_a \cdot c_b + z_a \cdot z_b) \cdot \frac{(-x_a \cdot y_b + y_a \cdot x_b)}{c_a \cdot c_b}$$

$$z' = \frac{1}{r_b^2} \cdot \frac{R'}{r'_a \cdot r'_b} \cdot (-c_a \cdot z_b + z_a \cdot c_b)$$

$$t' = \frac{1}{r_b^2} \cdot (-r'_a \cdot t_b + r'_b \cdot t_a)$$

(2a) If $c_a = 0$, $c_b \neq 0$, $r'_a \cdot r'_b \neq 0$ then $\alpha_a = 0$

$$\beta_a = \frac{\pi}{2} \cdot sign(z_a); (x_a = 0 \quad y_a = 0)$$

$$x' = \frac{1}{r_b^2} \cdot \frac{R'}{r'_a \cdot r'_b} \cdot z_a \cdot z_b \cdot \frac{x_b}{c_b}$$

$$y' = -\frac{1}{r_b^2} \cdot \frac{R'}{r'_a \cdot r'_b} \cdot z_a \cdot z_b \cdot \frac{y_b}{c_b}$$

$$z' = \frac{1}{r_b^2} \cdot \frac{R'}{r'_a \cdot r'_b} \cdot z_a \cdot c_b$$

$$t' = \frac{1}{r_b^2} \cdot (-r'_a \cdot t_b + r'_b \cdot t_a)$$

(3a) If $c_a \neq 0, c_b = 0, r'_a \cdot r'_b \neq 0$ then $\alpha_b = 0$

$$\beta_b = \frac{\pi}{2} \cdot sign(z_b); \ (x_b = 0 \ y_b = 0)$$

you can observe that in this case $z_b / r'_b = sign(z_b)$

$$x' = \frac{1}{r_b^2} \cdot \frac{R'}{r'_a \cdot r'_b} \cdot z_a \cdot z_b \cdot \frac{x_a}{c_a}$$

$$y' = \frac{1}{r_b^2} \cdot \frac{R'}{r'_a \cdot r'_b} \cdot z_a \cdot z_b \cdot \frac{y_a}{c_a}$$

$$z' = -\frac{1}{r_b^2} \cdot \frac{R'}{r'_a \cdot r'_b} \cdot z_b \cdot c_a$$

$$t' = \frac{1}{r_b^2} \cdot (-r'_a \cdot t_b + r'_b \cdot t_a)$$

(4a) If $c_a = 0$ and $c_b = 0$, $r'_a \cdot r'_b \neq 0$ then $\alpha_a = \alpha_b = 0$

$$\beta_a = \frac{\pi}{2} \cdot sign(z_a), \ \beta_b = \frac{\pi}{2} \cdot sign(z_b)$$

$$(x_a = 0; \ y_a = 0), (x_b = 0; y_b = 0)$$

So $\beta_a - \beta_b = 0$ or $\pm\pi$

$$x' = \frac{1}{r_b^2} \cdot R' \cdot sign(z_a \cdot z_b)$$

$$y' = 0$$

$$z' = 0$$

$$t' = \frac{1}{r_b^2} \cdot (-r'_a \cdot t_b + r'_b \cdot t_a)$$

(5a) If $r'_a \cdot r'_b = 0$; $r'_b \neq 0$; $c_b \neq 0$; $\alpha_a = 0$, $\beta_a = 0$

$$x' = \frac{1}{r_b^2} \cdot \frac{R'}{r'_b} \cdot x_b$$

$$y' = -\frac{1}{r_b^2} \cdot \frac{R'}{r'_b} \cdot y_b$$

$$z' = -\frac{1}{r_b^2} \cdot \frac{R'}{r'_b} \cdot z_b$$

$$t' = \frac{1}{r_b^2} \cdot (-r'_a \cdot t_b + r'_b \cdot t_a)$$

(6a) If $r'_a \cdot r'_b = 0$; $r'_a \neq 0$; $c_a \neq 0$; $\alpha_b = 0$, $\beta_b = 0$

$$x' = \frac{1}{r_b^2} \cdot \frac{R'}{r'_a} \cdot x_a$$

$$y' = \frac{1}{r_b^2} \cdot \frac{R'}{r'_a} \cdot y_a$$

$$z' = \frac{1}{r_b^2} \cdot \frac{R'}{r'_a} \cdot z_a$$

$$t' = \frac{1}{r_b^2} \cdot (-r'_a \cdot t_b + r'_b \cdot t_a)$$

(7a) If $r'_a \cdot r'_b = 0$; $\alpha_a = \alpha_b = 0$ and $\beta_a = \beta_b = 0$ in this case $\gamma_a - \gamma_b = 0$ or $\pm\pi$

$$x' = \left| \frac{t_a}{t_b} \right|$$

$$y' = 0$$

$$z' = 0$$

$$t' = 0$$

The generic case (1a) of the 4d algebraic division s_a/s_b is differentiable. The other cases are limit case and are also differentiable.

Limit case (4a) may be an objectionable limit case, but, again, is the same questionable problem we have seen above for the division in 3d; the differential depends on the z_a and z_b sign.

Appendix B: 3d core visual basic source code

```
‘reference to the code 3d–2.4g in appendix of paper [2]

'The algebric product

Function MulA_3d(a As Complex3d, b As Complex3d) As Complex3d

Dim Ca As Double, Cb As Double, R As Complex3d

If Near0(a.R) = 0 Or Near0(b.R) = 0 Then Go To Set_To_Zero

Ca = Sqr(a.X ^ 2 + a.Y ^ 2)

Cb = Sqr(b.X ^ 2 + b.Y ^ 2)

If Near0(Ca) <> 0 And Near0(Cb) <> 0 Then

'generic case
```

Citation: Sonaglioni L (2016) A New Number Theory-Algebra Analysis II. J Appl Computat Math 5: 289. doi:10.4172/2168-9679.1000289


```
 R.X = (Ca * Cb - a.Z * b.Z) * (a.X * b.X - a.Y * b.Y) / (Ca * Cb)
 R.Y = (Ca * Cb - a.Z * b.Z) * (a.X * b.Y + a.Y * b.X) / (Ca * Cb)
 R.Z = (Ca * b.Z + Cb * a.Z)
 GoTo To_End
End If

If Near0(Ca) = 0 And Near0(Cb) <> 0 Then
 R.X = -a.Z * b.Z * b.X / Cb
 R.Y = -a.Z * b.Z * b.Y / Cb
 R.Z = a.Z * Cb
 GoTo To_End
End If

If Near0(Cb) = 0 And Near0(Ca) <> 0 Then
 R.X = -a.Z * b.Z * a.X / Ca
 R.Y = -a.Z * b.Z * a.Y / Ca
 R.Z = b.Z * Ca
 GoTo To_End
End If

 R.X = -a.Z * b.Z
 R.Y = 0
 R.Z = 0

To_End:
Calc_Vector_Notation R ''reference to 3d sub code…
MulA_3d = R
Exit Function
Set_To_Zero:
 R.X = 0
 R.Y = 0
 R.Z = 0
 R.R = 0
 R.Alfa = 0
 R.Beta = 0
 MulA_3d = R
End Function

'The algebric division
Function DivA_3d(a As Complex3d, b As Complex3d) As Complex3d
 Dim Ca As Double, Cb As Double, R As Complex3d, Rb As Double
 If Near0(a.R) = 0 Or Near0(b.R) = 0 Then GoTo Set_To_Zero

 Rb = 1 / Sqr(b.X ^ 2 + b.Y ^ 2 + b.Z ^ 2)
 Ca = Sqr(a.X ^ 2 + a.Y ^ 2)
 Cb = Sqr(b.X ^ 2 + b.Y ^ 2)

 If Near0(Ca) <> 0 And Near0(Cb) <> 0 Then
  'generic case
  R.X = Rb ^ 2 * (Ca * Cb + a.Z * b.Z) * (a.X * b.X + a.Y * b.Y) / (Ca * Cb)
  R.Y = Rb ^ 2 * (Ca * Cb + a.Z * b.Z) * (-a.X * b.Y + a.Y * b.X) / (Ca * Cb)
  R.Z = Rb ^ 2 * (-Ca * b.Z + Cb * a.Z)
  GoTo To_End
 End If

 If Near0(Ca) = 0 And Near0(Cb) <> 0 Then
  R.X = Rb ^ 2 * a.Z * b.Z * b.X / Cb
  R.Y = -Rb ^ 2 * a.Z * b.Z * b.Y / Cb
  R.Z = Rb ^ 2 * a.Z * Cb
  GoTo To_End
 End If

 If Near0(Cb) = 0 And Near0(Ca) <> 0 Then
  R.X = Sgn(b.Z) * a.Z * a.X / Ca
  R.Y = Sgn(b.Z) * a.Z * a.Y / Ca
  R.Z = -Sgn(b.Z) * Ca
  GoTo To_End
 End If

  R.X = a.Z / b.Z
  R.Y = 0
  R.Z = 0

 To_End:
 Calc_Vector_Notation R 'reference to 3d sub code
 DivA_3d = R
 Exit Function
 Set_To_Zero:
  R.X = 0
```

```
 R.Y = 0
 R.Z = 0
 R.R = 0
 R.Alfa = 0
 R.Beta = 0
 DivA_3d = R
End Function
```

Appendix C: 4d core visual basic source code.

```
Option Compare Database
Option Explicit
'----------------------------------
' CORE 4d ALGEBRA
' V2.9 OPTIMIZED
'----------------------------------
'Public Const Pi = 3.14159265358979
'--------------------------------------------
'AVOID THE USE OF SMALL NUMBER IN SIMULATION (OR VERY BIG NUMBERS)
'THE PRECISION IS LIMITED, THE MANTISSA HAVE 15 DIGIT
'Public Const MaxDigit = 12, AsZero = 10 ^ -12
'We can round the results of calculus or not
Private Const Round_Results = True
'--------------------------------------------

'The definition of the Complex4d type
Type Complex4d
 X As Double
 Y As Double
 Z As Double
 T As Double
 R As Double
 Alfa As Double
 Beta As Double
 Gamma As Double
End Type

'The initialization number in cartesian notation
Function Init_Algebric_4d(X As Double, Y As Double, Z As Double, T As Double) As Complex4d
Dim R As Complex4d
R.X = X
R.Y = Y
R.Z = Z
R.T = T
Calc_Vector_Notation R
Init_Algebric_4d = R
End Function

'The initialization number in vector notation
Function Init_Vector_4d(R As Double, Alfa As Double, Beta As Double, Gamma As Double) As Complex4d
Dim S As Complex4d
S.R = R
S.Alfa = Alfa
S.Beta = Beta
S.Gamma = Gamma
To_Algebric_Notation S
Init_Vector_4d = S
End Function

'The Sum A+B
Function Sum_4d(a As Complex4d, b As Complex4d) As Complex4d
Dim R As Complex4d
'Standard sum
R.X = a.X + b.X
R.Y = a.Y + b.Y
R.Z = a.Z + b.Z
R.T = a.T + b.T
Calc_Vector_Notation R
Sum_4d = R
End Function

'The Difference A-B
Function Diff_4d(a As Complex4d, b As Complex4d) As Complex4d
Dim R As Complex4d
'Standard diff
R.X = a.X - b.X
R.Y = a.Y - b.Y
R.Z = a.Z - b.Z
```

```
R.T = a.T - b.T
Calc_Vector_Notation R
Diff_4d = R
End Function

'The Product A*B
Function Mul_4d(a As Complex4d, b As Complex4d) As Complex4d
Dim R As Complex4d
R.R = a.R * b.R
R.Alfa = Modulus(a.Alfa + b.Alfa, 2 * Pi)
R.Beta = Modulus(a.Beta + b.Beta, 2 * Pi)
R.Gamma = Modulus(a.Gamma + b.Gamma, 2 * Pi)
To_Algebric_Notation R
Mul_4d = R
End Function

'The algebric product
Function MulA_4d(a As Complex4d, b As Complex4d) As Complex4d
Dim Ca As Double, Cb As Double, R As Complex4d
Dim Ra1 As Double, Rb1 As Double, R1 As Double, Kx As Double

If Near0(a.R) = 0 Or Near0(b.R) = 0 Then GoTo Set_To_Zero

Ra1 = Sqr(a.X ^ 2 + a.Y ^ 2 + a.Z ^ 2)
Rb1 = Sqr(b.X ^ 2 + b.Y ^ 2 + b.Z ^ 2)

If Near0(Ra1) = 0 And Near0(Rb1) = 0 Then 'x=y=z=0
 R.X = Abs(b.T * a.T)
 R.Y = 0
 R.Z = 0
 R.T = 0
 GoTo To_End
End If

R1 = Abs(Ra1 * Rb1 - a.T * b.T)

Ca = Sqr(a.X ^ 2 + a.Y ^ 2)
Cb = Sqr(b.X ^ 2 + b.Y ^ 2)

If Near0(Ra1 * Rb1) = 0 Then
 If Near0(Ra1) = 0 Then
 Kx = R1 / Rb1
 R.X = Kx * b.X
 R.Y = Kx * b.Y
 R.Z = Kx * b.Z
 R.T = Ra1 * b.T + Rb1 * a.T
 GoTo To_End
 Else
 Kx = R1 / Ra1
 R.X = Kx * a.X
 R.Y = Kx * a.Y
 R.Z = Kx * a.Z
 R.T = Ra1 * b.T + Rb1 * a.T
 GoTo To_End
 End If
End If

If Near0(Ca) <> 0 And Near0(Cb) <> 0 Then
 'generic case
 Kx = R1 / (Ra1 * Rb1)
 R.X = Kx * (Ca * Cb - a.Z * b.Z) * (a.X * b.X - a.Y * b.Y) / (Ca * Cb)
 R.Y = Kx * (Ca * Cb - a.Z * b.Z) * (a.X * b.Y + a.Y * b.X) / (Ca * Cb)
 R.Z = Kx * (Ca * b.Z + Cb * a.Z)
 R.T = Ra1 * b.T + Rb1 * a.T
 GoTo To_End
End If

If Near0(Ca) = 0 And Near0(Cb) <> 0 Then
 Kx = R1 / (Ra1 * Rb1)
 R.X = -Kx * a.Z * b.Z * b.X / Cb
 R.Y = -Kx * a.Z * b.Z * b.Y / Cb
 R.Z = Kx * a.Z * Cb
 R.T = Ra1 * b.T + Rb1 * a.T
 GoTo To_End
End If

If Near0(Cb) = 0 And Near0(Ca) <> 0 Then
 Kx = R1 / (Ra1 * Rb1)
 R.X = -Kx * a.Z * b.Z * a.X / Ca
```

```
 R.Y = -Kx * a.Z * b.Z * a.Y / Ca
 R.Z = Kx * b.Z * Ca
 R.T = Ra1 * b.T + Rb1 * a.T
 GoTo To_End
End If

If Near0(Cb) = 0 And Near0(Ca) = 0 And Near0(Ra1 * Rb1) <> 0 Then
 R.X = -R1 * Sgn(a.Z * b.Z)
 R.Y = 0
 R.Z = 0
 R.T = Ra1 * b.T + Rb1 * a.T
 GoTo To_End
End If

To_End:
Calc_Vector_Notation R
MulA_4d = R
Exit Function
Set_To_Zero:
 R.X = 0
 R.Y = 0
 R.Z = 0
 R.T = 0
 R.R = 0
 R.Alfa = 0
 R.Beta = 0
 R.Gamma = 0
 MulA_4d = R
End Function

'The Division A/B
Function Div_4d(a As Complex4d, b As Complex4d) As Complex4d
Dim R As Complex4d
R.R = a.R / b.R
R.Alfa = Modulus(a.Alfa - b.Alfa, 2 * Pi)
R.Beta = Modulus(a.Beta - b.Beta, 2 * Pi)
R.Gamma = Modulus(a.Gamma - b.Gamma, 2 * Pi)
To_Algebric_Notation R
Div_4d = R
End Function

'The algebric division
Function DivA_4d(a As Complex4d, b As Complex4d) As Complex4d
Dim Ca As Double, Cb As Double, R As Complex4d
Dim Ra1 As Double, Rb1 As Double, R1 As Double, Rb As Double, Kx As Double

If Near0(a.R) = 0 Or Near0(b.R) = 0 Then GoTo Set_To_Zero

Rb = 1 / b.R
Ra1 = Sqr(a.X ^ 2 + a.Y ^ 2 + a.Z ^ 2)
Rb1 = Sqr(b.X ^ 2 + b.Y ^ 2 + b.Z ^ 2)

If Near0(Ra1) = 0 And Near0(Rb1) = 0 Then 'x=y=z=0
 R.X = Abs(a.T / b.T)
 R.Y = 0
 R.Z = 0
 R.T = 0
 GoTo To_End
End If

R1 = Abs(Ra1 * Rb1 + a.T * b.T)

Ca = Sqr(a.X ^ 2 + a.Y ^ 2)
Cb = Sqr(b.X ^ 2 + b.Y ^ 2)

If Near0(Ra1 * Rb1) = 0 Then
 If Near0(Ra1) = 0 Then
 Kx = Rb ^ 2 * R1 / Rb1
 R.X = Kx * b.X
 R.Y = -Kx * b.Y
 R.Z = -Kx * b.Z
 R.T = Rb ^ 2 * (-Ra1 * b.T + Rb1 * a.T)
 GoTo To_End
 Else
 Kx = Rb ^ 2 * R1 / Ra1
 R.X = Kx * a.X
 R.Y = Kx * a.Y
 R.Z = Kx * a.Z
```

Citation: Sonaglioni L (2016) A New Number Theory-Algebra Analysis II. J Appl Computat Math 5: 289. doi:10.4172/2168-9679.1000289


```
 R.T = Rb ^ 2 * (-Ra1 * b.T + Rb1 * a.T)
 GoTo To_End
 End If
End If

If Near0(Ca) <> 0 And Near0(Cb) <> 0 Then
 'generic case
 Kx = Rb ^ 2 * R1 / (Ra1 * Rb1)
 R.X = Kx * (Ca * Cb + a.Z * b.Z) * (a.X * b.X + a.Y * b.Y) / (Ca * Cb)
 R.Y = Kx * (Ca * Cb + a.Z * b.Z) * (-a.X * b.Y + a.Y * b.X) / (Ca * Cb)
 R.Z = Kx * (-Ca * b.Z + Cb * a.Z)
 R.T = Rb ^ 2 * (-Ra1 * b.T + Rb1 * a.T)
 GoTo To_End
End If

If Near0(Ca) = 0 And Near0(Cb) <> 0 Then
 Kx = Rb ^ 2 * R1 / (Ra1 * Rb1)
 R.X = Kx * a.Z * b.Z * b.X / Cb
 R.Y = -Kx * a.Z * b.Z * b.Y / Cb
 R.Z = Kx * a.Z * Cb
 R.T = Rb ^ 2 * (-Ra1 * b.T + Rb1 * a.T)
 GoTo To_End
End If

If Near0(Cb) = 0 And Near0(Ca) <> 0 Then
 Kx = Rb ^ 2 * R1 / Ra1
 R.X = Kx * a.Z * Sgn(b.Z) * a.X / Ca
 R.Y = Kx * a.Z * Sgn(b.Z) * a.Y / Ca
 R.Z = -Kx * Sgn(b.Z) * Ca
 R.T = Rb ^ 2 * (-Ra1 * b.T + Rb1 * a.T)
 GoTo To_End
End If

If Near0(Cb) = 0 And Near0(Ca) = 0 And Near0(Ra1 * Rb1) <> 0 Then
 R.X = Rb ^ 2 * R1 * Sgn(a.Z * b.Z)
 R.Y = 0
 R.Z = 0
 R.T = Rb ^ 2 * (-Ra1 * b.T + Rb1 * a.T)
 GoTo To_End
End If

To_End:
Calc_Vector_Notation R
DivA_4d = R
Exit Function
Set_To_Zero:
 R.X = 0
 R.Y = 0
 R.Z = 0
 R.T = 0
 R.R = 0
 R.Alfa = 0
 R.Beta = 0
 R.Gamma = 0
 DivA_4d = R
End Function

'The 1/S
Function Inverse_4d(S As Complex4d) As Complex4d
Dim R As Complex4d
R.R = 1 / S.R
R.Alfa = Modulus(-S.Alfa, 2 * Pi)
R.Beta = Modulus(-S.Beta, 2 * Pi)
R.Gamma = Modulus(-S.Gamma, 2 * Pi)
To_Algebric_Notation R
Inverse_4d = R
End Function

'S^X; X Real
Function S_elev_X_4d(S As Complex4d, X As Double) As Complex4d
Dim R As Complex4d
R.R = S.R ^ X
R.Alfa = Modulus(S.Alfa * X, 2 * Pi)
R.Beta = Modulus(S.Beta * X, 2 * Pi)
R.Gamma = Modulus(S.Gamma * X, 2 * Pi)
To_Algebric_Notation R
S_elev_X_4d = R
End Function
```

```
'Square Root of S
Function Sqr_4d(S As Complex4d) As Complex4d
Dim R As Complex4d
R.R = Sqr(S.R)
R.Alfa = Modulus(S.Alfa / 2, 2 * Pi)
R.Beta = Modulus(S.Beta / 2, 2 * Pi)
R.Gamma = Modulus(S.Gamma / 2, 2 * Pi)
To_Algebric_Notation R
Sqr_4d = R
End Function

'Rotation and Elongation
Function Rotation_4d(S As Complex4d, dAlfa As Double, dBeta As Double, dGamma As Double, Optional dr As Double = 0) As Complex4d
Dim R As Complex4d
R = S
If Near0(R.R) = 0 And Near0(dr) = 0 Then
 Rotation_4d = R
 Exit Function
End If
R.R = R.R + dr
R.Alfa = Modulus(S.Alfa + dAlfa, 2 * Pi)
R.Beta = Modulus(S.Beta + dBeta, 2 * Pi)
R.Gamma = Modulus(S.Gamma + dGamma, 2 * Pi)
To_Algebric_Notation R
Rotation_4d = R
End Function

'Creates ds from a vector S and dAlfa,dBeta and dr
Function Differentiate_Vector_4d(S As Complex4d, dAlfa As Double, dBeta As Double, dGamma As Double, dr As Double) As Complex4d
Dim dx As Double, dy As Double, dz As Double, dt As Double, ds As Complex4d
Dim dr1 As Double, R1 As Double
R1 = Sqr(S.X ^ 2 + S.Y ^ 2 + S.Z ^ 2)
dr1 = dr * Cos(S.Gamma) - S.R * Sin(S.Gamma) * dGamma
dt = dr * Sin(S.Gamma) + S.R * Cos(S.Gamma) * dGamma
dz = dr1 * Sin(S.Beta) + S.R * Cos(S.Beta) * dBeta
dy = dr1 * Cos(S.Beta) * Sin(S.Alfa) - S.R * Sin(S.Beta) * Sin(S.Alfa) * dBeta + S.R * Cos(S.Beta) * Cos(S.Alfa) * dAlfa
dx = dr1 * Cos(S.Beta) * Cos(S.Alfa) - S.R * Sin(S.Beta) * Cos(S.Alfa) * dBeta - S.R * Cos(S.Beta) * Sin(S.Alfa) * dAlfa
ds = Init_Algebric_4d(dx, dy, dz, dt)
Differentiate_Vector_4d = ds
End Function

'Scalar product
Function A_V_B_4d(a As Complex4d, b As Complex4d) As Double
A_V_B_4d = a.X * b.X + a.Y * b.Y + a.Z * b.Z + a.T * b.T
End Function

'Versor of S
Function Versor_4d(S As Complex4d) As Complex4d
Dim R As Complex4d, R0 As Double
If Near0(S.R) = 0 Then GoTo Set_To_Zero
R = S
R0 = R.R
R.R = 1
R.X = R.X / R0
R.Y = R.Y / R0
R.Z = R.Z / R0
R.T = R.T / R0
Versor_4d = R
Exit Function
Set_To_Zero:
 R.X = 0
 R.Y = 0
 R.Z = 0
 R.T = 0
 R.R = 0
 R.Alfa = 0
 R.Beta = 0
 R.Gamma = 0
 Versor_4d = R
End Function

'Return vector A along components on B axes; B new real axes
Function Project_A_on_B_4d(a As Complex4d, b As Complex4d) As Complex4d
```

```
Dim Wx As Complex4d, Wy As Complex4d, Wz As Complex4d, Wt As Complex4d, R As Complex4d, R0 As Double
Dim X As Double, Y As Double, Z As Double, T As Double
Dim BVx As Double, BVy As Double, BVz As Double, BVt As Double

If Near0(b.R) = 0 Then GoTo Set_To_Zero
If Near0(a.R) = 0 Then GoTo Set_To_Zero

'Versors Wx, Wy and Wz the new base
Wx = Versor_4d(b)
'----------------------------------------------------
' Optimization
'Wy = Init_Algebric_4d(-Wy.Y, Wy.X, 0,0)
Wy.X = -Wx.Y
Wy.Y = Wx.X
Wy.Z = 0
Wy.T = 0
'Wy = Versor_4d(Wy)
R0 = Sqr(Wy.X ^ 2 + Wy.Y ^ 2 + Wy.Z ^ 2 + Wy.T ^ 2)
If Near0(R0) = 0 Then GoTo Set_To_Zero 'New quatern is undetermine
Wy.X = Wy.X / R0
Wy.Y = Wy.Y / R0
'Wy.Z = Wy.Z / R0
'Wy.T = Wy.T / R0
'----------------------------------------------------
'
'----------------------------------------------------
'consider Wz as
'Wz = A_X_B_3d(Wx, Wy)+ T=0
Wz.X = Wx.Y * Wy.Z - Wx.Z * Wy.Y
Wz.Y = Wx.Z * Wy.X - Wx.X * Wy.Z
Wz.Z = Wx.X * Wy.Y - Wx.Y * Wy.X
Wz.T = 0
'----------------------------------------------------
'
'Wt: Take Wx and make it ortogonal respect to T
If Near0(Wx.T) = 0 Then
 Wt.X = 0
 Wt.Y = 0
 Wt.Z = 0
 Wt.T = 0
Else
 Wt.X = Wx.X
 Wt.Y = Wx.Y
 Wt.Z = Wx.Z
 Wt.T = -(Wx.X ^ 2 + Wx.Y ^ 2 + Wx.Z ^ 2) / Wx.T
End If

'Wt = Versor_4d(Wt)
R0 = Sqr(Wt.X ^ 2 + Wt.Y ^ 2 + Wt.Z ^ 2 + Wt.T ^ 2)
If Near0(R0) = 0 Then R0 = 1 'this do not stop the calculus
Wt.X = Wt.X / R0
Wt.Y = Wt.Y / R0
Wt.Z = Wt.Z / R0
Wt.T = Wt.T / R0

'Project A on Wx, Wy, Wz, Wt
BVx = A_V_B_4d(a, Wx)
BVy = A_V_B_4d(a, Wy)
BVz = A_V_B_4d(a, Wz)
BVt = A_V_B_4d(a, Wt)

R = Init_Algebric_4d(BVx, BVy, BVz, BVt)
Project_A_on_B_4d = R
Exit Function
Set_To_Zero:
 R.X = 0
 R.Y = 0
 R.Z = 0
 R.T = 0
 R.R = 0
 R.Alfa = 0
 R.Beta = 0
 R.Gamma = 0
 Project_A_on_B_4d = R
End Function

'THE TRASFORMATION FROM CARTESIAN TO VECTOR NOTATION
```

Citation: Sonaglioni L (2016) A New Number Theory-Algebra Analysis II. J Appl Computat Math 5: 289. doi:10.4172/2168-9679.1000289


```
Private Sub Calc_Vector_Notation(S As Complex4d)
Dim SinGamma As Double, CosGamma As Double, R1 As Double
Dim SinBeta As Double, CosBeta As Double, SinAlfa As Double, CosAlfa As Double

Check_Algebric_Zero_4d S

'Calc r
S.R = Sqr(S.X ^ 2 + S.Y ^ 2 + S.Z ^ 2 + S.T ^ 2)
If Near0(S.R) = 0 Then GoTo Set_To_Zero

R1 = Sqr(S.X ^ 2 + S.Y ^ 2 + S.Z ^ 2)

'Solve Gamma....
SinGamma = S.T / S.R
'SinGamma can be <=0

CosGamma = R1 / S.R
'CosGamma >=0 always
If Round(CosGamma, MaxDigit) = 0 Then '->R1=0; considerate T
 If Round(SinGamma, MaxDigit) = 0 Then GoTo Set_To_Zero 'i.e. T=0, and R1=0
 S.Gamma = Pi / 2 * Sgn(S.T)
End If

S.Gamma = ArcSin(SinGamma)

If Near0(R1) = 0 Then 'pure T vector
 S.X = 0
 S.Y = 0
 S.Z = 0
 S.Alfa = 0
 S.Beta = 0
 S.Gamma = Pi / 2 * Sgn(S.T)
 Exit Sub
End If

'Solve Beta....
SinBeta = S.Z / R1
CosBeta = Sqr(S.X ^ 2 + S.Y ^ 2) / R1

If Round(CosBeta, MaxDigit) = 0 Then
 S.Beta = Pi / 2 * Sgn(S.Z)
 S.Alfa = 0
 Exit Sub
End If

S.Beta = ArcSin(SinBeta)

'Solve Alfa....
SinAlfa = S.Y / (R1 * CosBeta)
CosAlfa = S.X / (R1 * CosBeta)

If Round(CosAlfa, MaxDigit) = 0 Then
 If Round(SinAlfa, MaxDigit) = 0 Then
 S.Alfa = 0
 Else
 S.Alfa = Pi / 2 * Sgn(S.Y)
 End If
Else
 S.Alfa = ArcSin(SinAlfa)
 If CosAlfa < 0 Then
 'If CosAlfa<0 ... -> Quadrant 2 o quadrant 4
 If Near0(S.Alfa) <> 0 Then
 S.Alfa = (Pi - Abs(S.Alfa)) * Sgn(S.Y)
 Else
 S.Alfa = Pi
 End If
 End If
End If

Exit Sub
Set_To_Zero:
 S.X = 0
 S.Y = 0
 S.Z = 0
 S.R = 0
 S.Alfa = 0
 S.Beta = 0
 S.Gamma = 0
End Sub
```

```
'THE TRASFORMATION FROM VECTOR TO CARTESIAN NOTATION
Private Sub To_Algebric_Notation(S As Complex4d)
Dim R1 As Double, CosBeta As Double, CosGamma As Double
If Near0(S.R) = 0 Then GoTo Set_To_Zero

'Solve X,Y,Z, T
S.T = S.R * Sin(S.Gamma)
CosGamma = Cos(S.Gamma)

If Near0(CosGamma) = 0 Then
 'The Vector is a pure T vector, so
 S.Z = 0
 S.Y = 0
 S.X = 0
 'Alfa and Beta irrelevant, set to 0
 S.Beta = 0
 S.Alfa = 0
 S.Gamma = Pi / 2 * Sgn(S.T)
 Exit Sub
End If

R1 = S.R * Abs(CosGamma)

'Solve Alfa, Beta
S.Z = R1 * Sin(S.Beta)
CosBeta = Cos(S.Beta)

If Near0(CosBeta) = 0 Then
 S.Y = 0
 S.X = 0
 S.Alfa = 0
Else
 S.Y = R1 * CosBeta * Sin(S.Alfa)
 S.X = R1 * CosBeta * Cos(S.Alfa)
End If

Calc_Vector_Notation S
Exit Sub
Set_To_Zero:
 S.X = 0
 S.Y = 0
 S.Z = 0
 S.T = 0
 S.R = 0
 S.Alfa = 0
 S.Beta = 0
 S.Gamma = 0
End Sub

Private Sub Check_Algebric_Zero_4d(S As Complex4d)
S.Z = Near0(S.Z)
S.Y = Near0(S.Y)
S.X = Near0(S.X)
S.T = Near0(S.T)
End Sub
'-----------------------------------
'END CORE 4d ALGEBRA
'-----------------------------------
```

References

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