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A New Conjugate Gradient Method for the Solution of Linear Ill-Posed Problem

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Abstract

A new conjugate gradient method is proposed for solving the linear ill-posed problem and the application to the identification of the multi-source dynamic loads on a surface of simply supported plate. The algorithm considered here is detailedly given and proved that the computational costs for the present method are nearly the same as the common conjugate gradient method, but the number of iteration steps is even less. Finally, the performances of numerical simulations are given, and verify the favorable theoretical properties of the present method.

Keywords: Ill-posed problems; Conjugate gradient method; Global convergence

Introduction

Many works have been done for regularization of linear ill-posed problems [1-5]. We are concerned with the problem of determining solutions $x^{\dagger}$ for the linear ill-posed problems

$$Ax = y, \quad (y \in R(A)) \tag{1.1}$$

where A is a bounded non-negative, self-adjoint and injective operator on a Hilbert space X and $y \in R(A)$, the range of A This problem is in general ill-posed in the sense that even if a unique solution for (1.1) exists, the solution may not depend continuously on the data y. This situation occurs if $R(A)$ is not closed. For each $\delta > 0$, let $y^{\delta} \in X$ be such that

$$\| y - y^{\delta} \| \le \delta \tag{1.2}$$

and known noise level δ.

In general, the problem of solving (1.1) is ill-posed. By ill-posedness, we always mean that the solution do not depend continuously on the data. In the case of multiple solutions, this is understandable in the sense of multivalued mappings. So, it is necessary to develop some inverse analysis techniques for coping with this kind of ill-posedness. Recently, in mathematical theory, these technology problems attract lots of attention in the ill-posedness and regularization methods [6-9]. An augmented Galerkin method was suggested to solve the first kind Fredholm integral equations problem which is often ill-posed [10]. Many researchers solve these ill-posed problems using wavelet basis method [11-13]. However, for solving the first kind Fredholm integral equations problem by the conjugate gradient method, as we know, very few papers can be found and very limited. In fact, these inverse problems mentioned by most of papers above are ill-posed. For an ill-posed problem, the linear system of the first kind Fredholm integral equation is severely ill-conditioned.

In fact, Tikhonov regularization and iterative method are usual methods for the linear ill-posed problems. However, the former will cost lots of time to choose regularization parameter, and the convergence rate of the latter is very slow. In order to avoid these problems, in this paper, we establish a new conjugate gradient method (MCG) for this problem based on the ideas of [14], and investigate the minimum of this minimization problem. This paper is organized as follows. In Section 2 we establish a new conjugate gradient method. In Section 3 we prove that this method can obtain global convergent property. In Section 4, we compare the solution to the inverse problem via a forward solver using MCG method versus the Landweber method and common conjugate gradient method. We conclude this paper in Section 5.

The Establishment of New Conjugate Gradient Method

The conjugate gradient methods are very efficient tools to solve the optimization problems [15,16]. In this section, we will consider the following n variables unstrained optimization problem

$$\min_{x \in R_n} f(x), \tag{2.1}$$

where $f: R_n \to R$ is smooth and its gradient $g(x)$ is available. The new nonlinear conjugate gradient method for (2.1) is defined by the iterative form

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 1, 2, \cdots, \tag{2.2}$$

where x_k is the k th iterative point, $\alpha_k > 0$ is a steplength, and d_k is the search direction defined by

$$d_k = \begin{cases} -g_k + \beta_k d_{k-1}, & k \ge 2 \\ -g_1, & k = 1, \end{cases} \tag{2.3}$$

where $\beta_k \in R^+$ is a scalar which determines the different conjugate gradient methods [17], and g_k is the gradient of $f(x)$ at the point of x_k. Many efforts have been exerted to the global convergence analysis of the conjugate gradient methods based on the different formula β_k [18-24]. In the already existing convergence analysis and implementa-

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tions of the conjugate gradient method, it is normally required that the strong Wolfe conditions holds, namely,

$$f(x_k+\alpha_k d_k)-f(x_k)\le \delta\alpha_k g_k^T d_k, \tag{2.4}$$

$$|g(x_k+\alpha_k d_k)^T d_k| \le -\sigma g_k^T d_k, \tag{2.5}$$

where $0<\delta<\sigma<1$.

In this paper, we will establish a new valid nonlinear conjugate gradient method under the following modified condition:

$$g(x_k+\alpha_k d_k)^T d_k > \sigma \| d_k \|^2 g_k^T d_k, \tag{2.6}$$

which is weaker than the usual one.

In fact, it is crucial to design a descent direction for implementing a conjugate gradient method. Let the current search direction d_k is a descent direction. Now we should find a β_k such that the search direction d_{k+1} is a descent direction, i.e.

$$d_{k+1}^T g_{k+1} = -\| g_{k+1}\|^2 + \beta_k g_{k+1}^T d_k < 0. \tag{2.7}$$

Let $c_k = \| g_{k+1}\|^2 \beta_k$, then (2.7) can be substituted by

$$c_k > g_{k+1}^T d_k.$$

Setting

$$c_k = (g_{k+1} - \| d_k\|^2 g_k)^T d_k,$$

then we can obtain our new formula

$$\beta_k = \| g_{k+1}\|^2 (g_{k+1} - \| d_k\|^2 g_k)^T d_k. \tag{2.8}$$

Noting that the inequality $d_k(g_{k+1} - \| d_k\|^2 g_k)^T > 0$ holds due to the line search condition (2.6), so the formula (2.8) is validly defined. It means that this formula (2.8) correspondingly generates a conjugate gradient method. Now we can define the new conjugate gradient method:

Algorithm (MCG method)

Step 0: Given $x_1 \in R^n$, set $d_1=-g_1, k=1$. If $g_1=0$, then stop.

Step 1: Find a $\alpha_k>0$ satisfying (2.4) and (2.6).

Step 2: Let $x_{k+1}=x_k+\alpha_k d_k$ and $g_{k+1}=g(x_{k+1})$. If $g_{k+1}=0$, then stop.

Step 3: Compute β_{k+1} by the formula (2.8) and generate d_{k+1} by (2.3).

Step 4: Set $k:=k+1$, go to Step 1.

Using the equality (2.3) and (2.8), we obtain

$$\begin{aligned} g_{k+1}^T d_{k+1} &= -\| g_{k+1}\|^2 + g_{k+1}^T d_k \| g_{k+1}\|^2 (g_{k+1} - \| d_k\|^2 g_k)^T d_k \\ &= \| d_k\|^2 g_k^T d_k \beta_k. \end{aligned} \tag{2.4}$$

General Convergence Results

In the following, we will investigate the convergence behavior of Algorithm 2.1 under the following two assumptions, which are often used in the literature to study the global convergence of conjugate gradient methods with inexact line search.

Assumption H1

f is bounded below in R^n. Moreover, f is continuously differentiable in a neighborhood of level set

$$\Psi = \{x \in R^n : f(x) \le f(x_1)\}.$$

Assumption H2

There exists a constant L such that for any $x, y \in \Psi, \gamma \in R^+$,

$$\| g(x) - \| y\|^2 g(x+\gamma y) \| \le L \| \gamma y \|.$$

Lemma

Suppose that Assumption H1 and H2 hold. Consider any iterative method of the form (2.2) and (2.3), and α_k satisfies the conditions (2.4) and (2.6). Then

$$\sum_{k=1}^{\infty} (g_k^T d_k)^2 \| d_k \|^2 < \infty. (3.1)$$

Proof

Noting the inequality (2.6), we have

$$(g_{k+1} - \| d_k\|^2 g_k)^T d_k \ge (\sigma-1) \| d_k\|^2 g_k^T d_k,$$

which together with the result of Assumption H2, we can obtain

$$\alpha_k \ge \sigma - 1 L g_k^T d_k \| d_k \|^2.$$

Due to (2.4), we have

$$f_{k+1} - f_k \le \delta(\sigma-1) L (g_k^T d_k)^2 \| d_k\|^2. \tag{3.2}$$

Summing the above inequality, and due to the bounded below property of f we immediately obtain the assertion.

Theorem

Suppose that $\{x_k, k=1,2,\cdots\}$ be generated by Algorithm 2.1. Assume further that Assumption H1 and H2 hold. In [25], there exists some constant $c>0$ such that for all $k\in N$,

$$g_k^T d_k \le -c \| g_k \|^2. \tag{3.3}$$

Then, the Algorithm either terminates a stationary point or converges in the sense that

$$\lim_{k\to\infty} \inf \| g_k \| = 0. \tag{3.4}$$

Proof. In fact, we can prove that all search directions are descent, namely

$$g_k^T d_k < 0, \quad k \ge 1 \tag{3.5}$$

It is easy to check that the inequality (3.5) holds for $k=1$.

Now we let $g_k^T d_k < 0$, and prove that $g_{k+1}^T d_{k+1} < 0$ is true, too. Using line search conditions we can obtain

$$(g_{k+1} - \| d_k\|^2 g_k)^T d_k \ge (\sigma-1) \| d_k\|^2 g_k^T d_k > 0 \tag{3.6}$$

Noting the equality (2.9) and (3.6), then we have that (3.5) holds for

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$k+1$. This implies that (3.5) is true for all $k \ge 1$.

Exploiting the equality (2.3), we can obtain

$$\begin{aligned} \|d_{k+1}\|^2 (g_{k+1}^T d_{k+1})^2 &= 1\|d_k\|^2 (g_k^T d_k)^2 - 2g_{k+1}^T d_{k+1} - \|g_{k+1}\|^2 (g_{k+1}^T d_{k+1})^2 \\ &\le 1\|g_{k+1}\|^2 + 1\|d_k\|^2 (g_k^T d_k)^2 \end{aligned} \quad (3.7)$$

Therefore, by using (3.4), we can derive

$$\|d_{k+1}\|^2 (g_{k+1}^T d_{k+1})^2 \le 1\|g_{k+1}\|^2 + 1c^4 \|g_k\|^6 \quad (3.8)$$

Assume that the result of Theorem is not true, then there exists a constant $\varepsilon > 0$ such that

$$\|g_k\| \ge \varepsilon, \quad k \ge 1 \quad (3.9)$$

By the above inequality, we have

$$(g_{k+1}^T d_{k+1})^2 \|d_{k+1}\|^2 \ge c^4 \varepsilon^6 1 + c^4 \varepsilon^4$$

Then, we have

$$\sum_{k=1}^{\infty} (g_k^T d_k)^2 \|d_k\|^2 = \infty,$$

which obviously contradicts the inequality (3.1). Then the proof is complete.

Numerical Examples and Discussion

Benchmark Test

In this section, we will valid the numerical consequences of the new conjugate gradient method. We first consider the first kind of Fredholm integral equation

$$\int_0^1 e^{ts} x(s) ds = e^{t+1} - 1t + 1, \quad t \in [0,1] \quad (4.1)$$

It is easy to check that the true solution of Eq.(4.1) is $x(s) = e^s$. In general terms, we consider the perturbed equation

$$\int_0^1 e^{ts} x(s) ds = y^\delta(t), \quad t \in [0,1] \quad (4.2)$$

Discretizing Eq. (4.2), we can obtain

$$1N \sum_{j=1}^{N} e^{t_i s_j} x(s_j) = y_i^\delta, \quad i, j = 1, 2, \cdots, N \quad (4.3)$$

where

$$t_i = i - 1N, s_j = j - 1N, y_i^\delta = y(t_i) + \theta_i \delta,$$

θ_i is a random number and satisfies $|\theta_i| \le 1$.

To analyze the convergence performances of the present method, we denote $N = 50$ as the number of grid and choose different noisy level δ. The termination condition is $\|g(x_k)\| \le 0.0001$. Applying PC-MATLAB environment, we obtain the following results.

The comparison of the true solution with the numerical results by the MCG method, the CG method in which the formula for β_k is $\beta_k = \|g_{k+1}\|^2 \|g_k\|^2$, and Landweber method is illustrated in Figures 1-3. It is clearly shown that the computational results of the new conjugate gradient method are better than those of Landweber method and

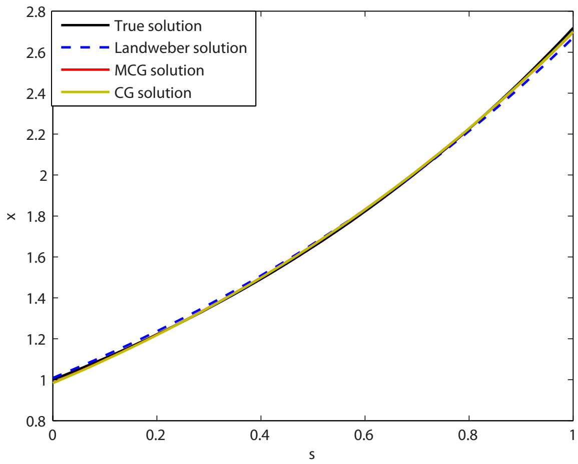

Figure 1: $N_{Landweber} = 376, N_{MCG} = 2, N_{CG} = 7, \delta = 0.001$

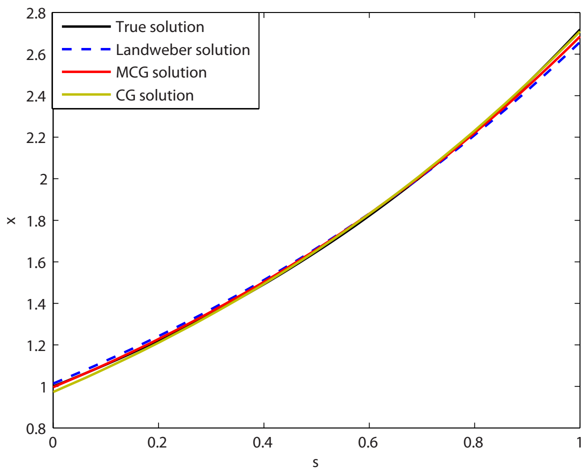

Figure 2:. $N_{Landweber} = 400, N_{MCG} = 7, N_{CG} = 11, \delta = 0.01$.

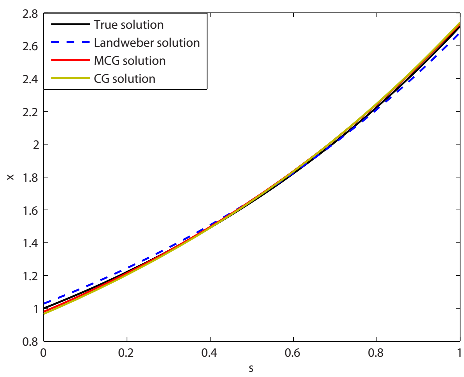

Figure 3: $N_{Landweber} = 426, N_{MCG} = 12, N_{CG} = 16, \delta = 0.1$

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	MCG method		Landweber method		CG method	
Noisy level	Iterative number	Average Error (%)	Iterative number	Average Error (%)	Iterative number	Average Error (%)
0.001	2	0.2404	376	0.5342	7	0. 2492
0.01	7	0.2682	400	0.7019	11	0. 3822
0.1	12	0.3141	426	0.7843	16	0. 5373

Table 1: Numerical results of equation (4.1).

			CG method		MCG method	
	Time point	Real force	Identified force	Error (%)	Identified force	Error (%)
Sine	0.001	1000	987. 03	1.30	976.79	2.32
Triangle	0.0006	480	458.02	2.75	486.23	0.78
Sine	0.003	-1000	-951.62	4.84	-966.67	3.33
Triangle	0.001	800	760.68	4.91	699.69	12.54
Sine	0.0045	707.11	680.35	2.68	699.6	0.75
Triangle	0.0016	320	325.86	0.73	306.4	1.70
Sine	0.0063	-453.99	-447.06	0.69	-453.9	0.01
Triangle	0.0033	-560	-567.8	0.97	-567.42	0.93
Sine	0.0073	-891.01	-906.74	1.57	-886.61	0.44
Triangle	0.0038	-160	-156.7	0.41	-130.11	3.74
Error (%)			Maximum	Average	Maximum	Average
Sine			8.71	2.89	8.55	1.38
Triangle			7.93	2.53	12.54	1.43

Table 2: The identified force at five time points at noise level .

CG method. In fact, more detailed results about them are indicated in Table 1. From this table, we can know that at the noisy level the iterative number of CG method and Landweber are 7 and 376, respectively, both larger than the present method. In addition, the iteration error of the MCG method is smaller than that of CG method and Landweber method. When the noisy level becomes larger and larger, the iterative number of these methods mentioned above increases. From the performances of Table 1, we can assure that the proposed method is more precise and effective than CG method and Landweber method. In a word, the present method is superior to the Landweber method and CG method. Meanwhile, the numerically optimal convergence rate of the regularized solutions roughly coincides with the theoretical analysis.

Application

To illustrate the present methodology for use in determining the unknown time-dependent multi-source dynamic loads acting on simply supported plate, we need to know the following knowledge for a linear elastic structure.

Here we consider the multi-source dynamic load identification problem for a linear and time-invariant dynamic system. The response at an arbitrary receiving point in a structure can be expressed as a convolution integral of the forcing time-history and the corresponding Green's kernel in time domain [26,27] :

$$y(t)=\int_0^t G(t-\tau)p(\tau)d\tau \tag{4.4}$$

where $y(t)$ is the response which can be displacement, velocity, acceleration, strain, etc. $G(t)$ is the corresponding Green's function, which is the kernel of impulse response. $p(t)$ is the desired unknown dynamic load acting on the structure.

By discretizing this convolution integral, the whole concerned time period is separated into equally spaced intervals, and the equation (4.4) is transformed into the following system of algebraic equation:

$$Y(t)=G(t)P(t)$$

or equivalently,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} g_1 & & & \\ g_2 & g_1 & & \\ \vdots & \vdots & \ddots & \\ g_m & g_{m-1} & \cdots & g_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix} \Delta t,$$

where y_i, g_i, and p_i are response, Green's function matrix and input force at time $t=i\Delta t$, respectively. Δt is the discrete time interval. Since the structure without applied force is static before force is applied, y_0 and g_0 are equal to zero. All the elements in the upper triangular part of G are zeros and are not shown. The special form of the Green's function matrix reflects the characteristic of the convolution integral.

To recover the time history $P(t)$, the knowledge of $y(t)$ and $G(t)$ are required. In fact, the response at a receiving point and the numerical Green's function of a structure can be obtained by finite element method. However, the problem of identifying the dynamic load $P(t)$ by $y(t)$ and $G(t)$ is usually ill-posed, and cannot be solved by inverse matrix method. In the following, our method will be suggested to solve this problem.

A practical engineering problem is to determine the vertical forces acting on simply supported plate as shown in Figure 4. Its material properties are as: $\rho = 7.8\times10^{-9} kg/m^3, E = 2.0\times10^5 MPa, \nu = 0.3$.

The vertical concentrated load is applied to the outside surface and the measured response is the vertical displacement. Three straight

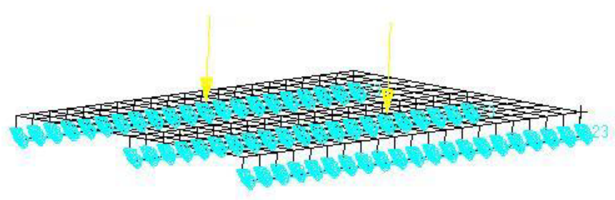

Figure 4: The finite element model of simply supported plate.

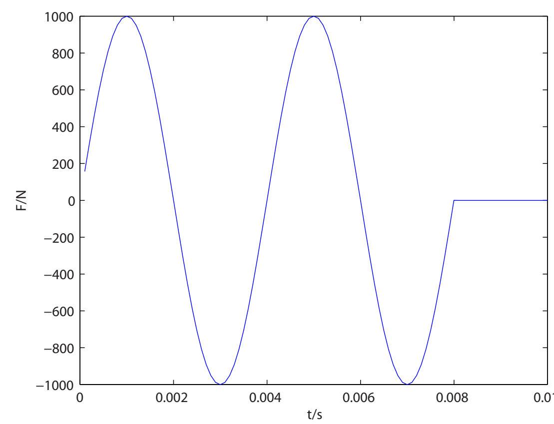

Figure 5: The vertical concentrated sine load acting on the outside surface.

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members of simply supported plate are fixed, and the others are free. We establish its finite element model as shown in Figure 4. The arrow in Figure 4 denotes the action point of dynamic force.

The concentrated loads are defined as follows:

$$F_1(t) = \begin{cases} q_1 \sin(2\pi t t_d), & 0 \le t \le 2t_d \\ 0, & t<0 \quad and \quad t>2t_d \end{cases}$$

$$F_2(t) = \begin{cases} 4q_2 t/t_d, & 0 \le t \le t_d/4 \\ 2q_2 - 4q_2 t/t_d, & t_d/4 < t \le 3t_d/4 \\ 4q_2 t/t_d - 4q_2, & 3t_d/4 < t \le t_d \\ 0, & t > t_d \end{cases}$$

where t_d is the time cycle of sine force, and $q_i(i=1,2)$ is a constant amplitude of the force. When $t_d = 0.004s, q_1 = 1000N$, and $q_2 = 800N$, the sine force and triangle force are shown in Figure 5 and Figure 6.

Herein, the experimental data of response is simulated by the com-

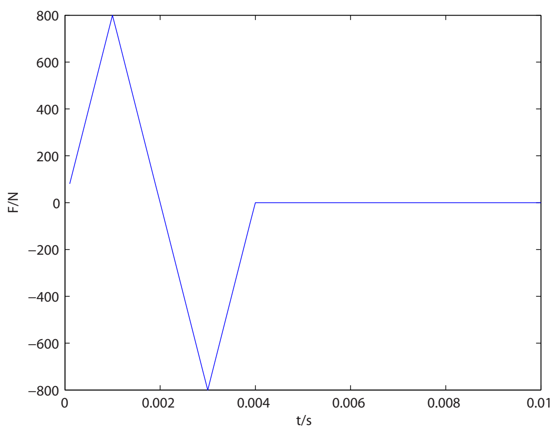

Figure 6: The vertical concentrated triangle load acting on the outside surface.

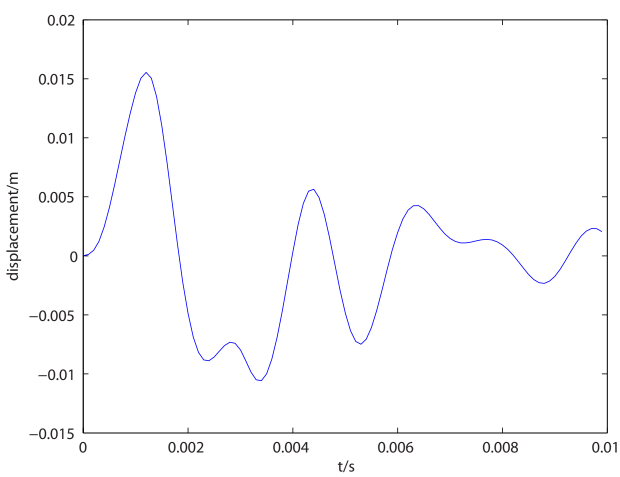

Figure 7: The corresponding vertical displacement response at one point.

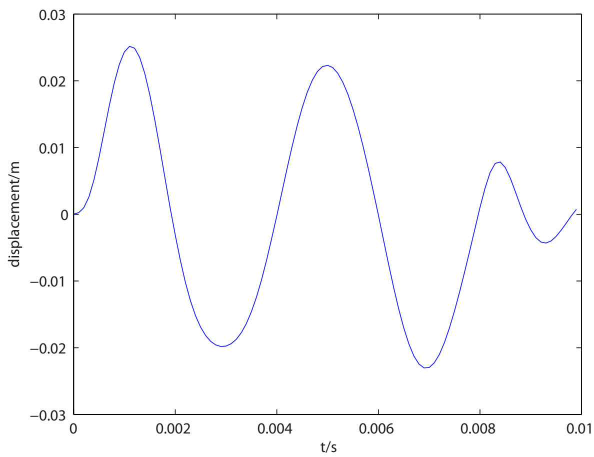

Figure 8: The corresponding vertical displacement response at the other point.

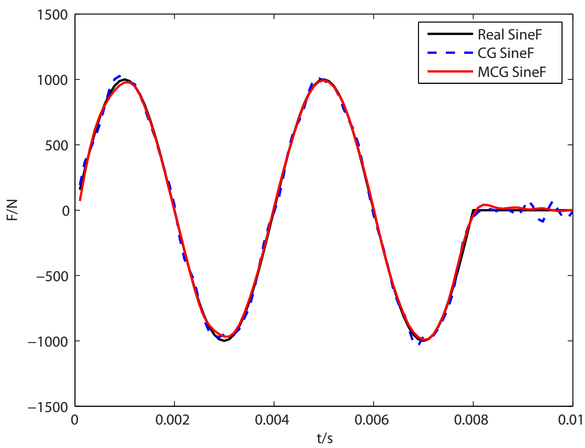

Figure 9: The identified sine force at noise level 5%.

puted numerical solution, and the corresponding vertical displacement response can be obtained by finite element method as shown in Figure 7and Figure 8. Furthermore, a noise is directly added to the computer-generated response to simulate the noise-contaminated measurement, and the noisy response is defined as follows:

$$Y_{err} = Y_{cal} + l_{noise} \cdot std(Y_{cal}) \cdot rand(-1,1),$$

where Y_{cal} is the computer-generated response; $std(Y_{cal})$ is the standard deviation of Y_{cal}; $rand(-1,1)$ denotes the random number between -1 and $+1$; l_{noise} is a parameter which controls the level of the noise contamination.

In order to investigate the effect of measurement error on the accuracy of the estimated values, we consider the case of noise level namely 5%, and the present method is adopted to determine the dynamic forces. By using a similar argument in Benchmark test, so the optimal solution obtained by the present method will be compared to those by CG method. The comparison will be made quantitatively by way of the relative estimation error:

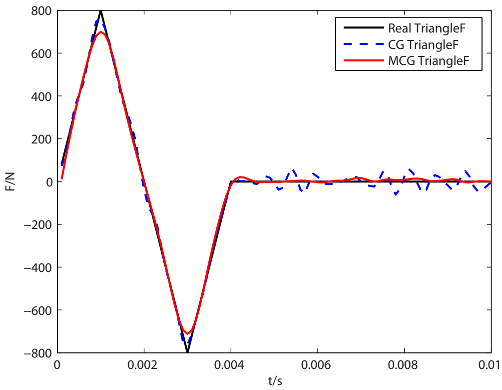

Figure 10: The identified triangle force at noise level 5%.

$$\tilde{F} = | F_{Real} - F_{Identified} F_{Real} | \tag{4.6}$$

To evaluate the effectiveness of these methods mentioned above, five time points are selected, and for each point the identified force will be compared with the corresponding actual force.

The results of numerical simulations are as follows:

From Figure 9 and Figure 10, it can be shown that CG method and MCG method can both stably and effectively identify the multi-source dynamic loads by the measured noisy responses. Moreover, the more detailed results by them at five time points are listed in Table 2. It can be found that at these five time points for noise level $\pm 5\%$, the most deviations of the identified loads by the present method are smaller than CG method, which dues to better efficient identification. It can be also found that the most deviations by CG method and the present method concentrate in the range of $9\%, 13\%$, respectively. In addition, for the identification of sine force, the maximal deviation and average deviation by the present method are $8.55\%, 1.38\%$, respectively, obviously smaller than the former. Furthermore, we can find that the maximal deviation and average deviation of the identification of triangle force by the present method are $12.54\%, 1.43\%$, respectively, which shows that MCG is better than CG method. Meanwhile, the number of iterations by the present method is 16, smaller than the CG method. In a word, the present algorithm achieves an excellent estimation, and also gives satisfactory results when recovering the loading time function.

Conclusion

In this paper, a new conjugate gradient method is presented and considered as an alternative to approximate the true solution of the ill-posed problem of Fredholm integral equations of the first kind. Finally the present method is applied to the identification of the multi-source dynamic loads on simply supported plate. It has been found that we can establish the global convergence and linear convergence rate for convex functions. Meanwhile, numerical simulations have shown that the present method reduces the number of iterations and quickens the speed of convergence of the regularized solution, and demonstrate that the present method is stable and effective.

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