A New Asymptotic Study to the 3-Dimensional Radial Schrödinger Equation under Modified Quark-antiquark Interaction Potential

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Abstract

In present work, the modified 3-dimensional modified radial Schrödinger equation is analytically solved for the non-relativistic interactions under the modified quark-antiquark interaction potential in the symmetries of Non-Commutative Quantum Chromodynamics (pNRQCD). The new energy eigenvalues (ground state, first excited state and nth excited state) and the corresponding Hamiltonian operator are calculated in 3-dimensional Non-Commutative Real Space Phase (NC: 3D-RSP) symmetries instead of solving modified Schrödinger Weyl Moyal star product. The present results, in (NC: 3D-RSP), are applied on the charmonium $c\bar{c}$, bottomonium $b\bar{b}$ and $c\bar{c}$ mesons. It is found that the perturbative solutions of discrete spectrum and quarkoniums mass can be expressed by the parabolic cylinder functions, Gamma function, the discreet atomic quantum numbers $(j, l, s, m)$ of the $\Omega^-$ state and the potential parameters $(a, b, c)$, in addition to non-commutativity parameters $(\Theta, \mu)$. The total complete degeneracy of new energy levels of the modified quark-antiquark interaction potential changed to become equals to the value $3\pi^2$ instead the values $\pi^2$ in ordinary Commutative Quantum Mechanics (CQM). Our obtained results are in good agreement with the already existed literatures in NCQM.

Keywords: Schrödinger equation; The heavy quarkonium system; The quark-antiquark interaction potential; Non-commutative space phase; Bopp’s shift method

Introduction

It is well known that, the quark-antiquark interaction potential or the quarkonium potential, obtained from Cornell potential by adding the harmonic term, is one of confining potentials. It was one of the most popular model for study the interactions in the Non-Relativistic Quantum Chromodynamics NRQCD such systems as quarkonium (heavy quarkonia) consisting of heavy quark and antiquark (charmonium $c\bar{c}$, bottomonium $b\bar{b}$ and $c\bar{c}$ mesons). The study of this potential, in NRQCD, is a particular interest for detecting characterizing the electromagnetic characteristics of mesons and the mass spectra for coupled states [1-8]. It is consist of three terms. One of the terms is responsible for the Coulomb interaction of quarks and the second corresponds to the string interaction, which provides confinement, while the third is the harmonic term, known to researchers and plays a very important role in various fields of physics and chemistry. Furthermore, this potential plays a vital role in different branches of physics such as atomic and molecular physics, particle physics, plasma physics and solid-state physics [7]. The main objective is to develop the research article [1] and expanding it to the hug symmetry known by physics and solid-state physics [7]. The main objective is to develop the research article [1] and expanding it to the hug symmetry known by physics and solid-state physics [7].

In 1930 [9] and then by H. Syndre in 1947 [10]. The new structure of NCQM based to new canonical commutations relations in both Schrödinger SP and Heisenberg HP, respectively, as follows (Throughout this paper, the natural units $\hbar = c = 1$ will be used) [11-15]:

$$V_{g'}(r) = ar^2 + br + c \rightarrow V_{g'}(\hat{r}) = V_{g'}(r) + \left[\frac{i}{\tau} \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \theta} \right]$$

We refer to this term $\vec{\mathbf{L}} \vec{\mathbf{O}}$ in the materials and methods section. It should be noted that, the non-commutativity was introduced firstly by W. Heisenberg in 1930 [9] and then by H. Syndre in 1947 [10]. The new structure of NCQM based to new canonical commutations relations in both Schrödinger SP and Heisenberg HP, respectively, as follows (Throughout this paper, the natural units $\hbar = c = 1$ will be used) [11-15]:

$$\left[ \hat{\mathbf{r}} \mathbf{p} \right] = \left[ \hat{\mathbf{p}} \mathbf{r} \right] = i \hbar \Rightarrow \Delta \hat{\mathbf{r}} \Delta \hat{\mathbf{p}} \geq \frac{\hbar}{2}$$

Where the indices $\mu, \nu = \{1, 2, 3\}$. This means that the principle of uncertainty for Heisenberg generalized to include another two new uncertainties related to the positions $(\hat{x}_\mu, \hat{x}_\nu)$ and the momentans $(\hat{p}_\mu, \hat{p}_\nu)$, in addition to the ordinary uncertainty of $(\hat{x}, \hat{p})$ in CQM. The very small two parameters $\vartheta^1$ and $\vartheta^2$ compared to the energy) are elements of two antisymmetric real matrices, parameters of non-commutativity and $(\ast)$ denote to the Weyl Moyal star product, which is generalized between two arbitrary functions $(f, g)(x, p) \rightarrow \{f, g\} = (f \ast g)(x, p)$ in (NC: 3D-RSP) symmetries [16-23]:

$$(f, g)(x, p) \rightarrow (f \ast g)(x, p) = \left[ \frac{i}{\tau} \int \vartheta \mathbf{O} f \mathbf{p} + \mathbf{O} f \mathbf{p} + \int \frac{i}{\vartheta} \mathbf{O} f \mathbf{p} + \mathbf{O} f \mathbf{p} \right]$$

The second and the third terms in the above equation are present the effects of (space-space) and (phase-phase) non-commutativity properties. However, the new operators $\vartheta_\mu(x, p) = \left[ \hat{x}_\mu \right] \mathbf{p} \circ \hat{p}_\mu(x, p)$ in HP are

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depending to the corresponding new operators $\hat{\xi} = \hat{\xi}_\mu \vee \hat{p}_\nu$ in SP from the following projections relations:

$$\xi(t) = \exp(i\hat{H}_{\omega}(t-t_0))\exp(-i\hat{H}_{\omega}(t-t_0)) \quad \Rightarrow \quad \xi(t) = \exp(i\hat{H}_{\omega-w}(t-t_0))$$

Here $\xi = (x_\mu \vee p_\nu)$ and $\xi(t) = (x_\mu \vee p_\nu)(t)$, while the dynamics of new systems $d\xi(t)$ are described from the following motion equations in NCQM:

$$\frac{d\xi(t)}{dt} = \left[\xi(t), \hat{H}_{\omega-w}\right] \quad \Rightarrow \quad \frac{d\xi(t)}{dt} = \left[\xi(t), \hat{H}_{\omega-w}\right]$$

The two operators $\hat{H}_\omega$ and $\hat{H}_{\omega-w}$ are presents the quantum Hamiltonian operators for the quark-antiquark interaction potential and modified quark-antiquark interaction potential, in the CQM and its extension NCQM, respectively. This paper consists of five sections and the organization scheme is given as follows: In next section, the theory part, we briefly review the SE with the quark-antiquark interaction potential [1-2]. The Section 3 is devoted to studying the MSE by applying the generalized Bopp’s shift method and obtained the modified quark-antiquark interaction potential in (NC: 3D-RSP) symmetries. To in addition to determine the confluent hypergeometric function of the first kind and the generalized Laguerre polynomial.

Materials and Methods

Solution of MSE for modified quark-antiquark interaction potential in pNRQCD

In this section, we shall give an overview or a brief preliminary for modified quarkonium potential in (NC: 3D-RSP) symmetries. To perform this task the physical form of MSE, it is necessary to replace ordinary 3-dimensional Hamiltonian operators $\hat{H}_\omega(x_\mu, \rho_\nu)$, complex wave function $\psi$ and energy $E_{\omega}$ by new Hamiltonian operators $\hat{H}_{\omega-w}(\tilde{x}_\mu, \tilde{\rho}_\nu)$, new complex wave function $\Psi(\tilde{r})$ and new values $E_{\omega-w}$, respectively. In addition to replace the ordinary product by the Weyl Moyal star product, which allow us to constructing the MSE in (NC: 3D: RSP) symmetries as [24-28]:

$$\hat{H}_\omega(x_\mu, \rho_\nu)\xi(t) = H_{\omega-w}(\tilde{x}_\mu, \tilde{\rho}_\nu)\xi(t) = \hat{H}_{\omega-w}(\tilde{x}_\mu, \tilde{\rho}_\nu)\Psi(\tilde{r}) = E_{\omega-w}\Psi(\tilde{r})$$

The Bopp’s shift method has been successfully applied to relativistic and non-relativistic (non-commutative quantum mechanical problems) using Modified Dirac Equation (MDE), Modified Klein-Gordon Equation (MKGE) and MSE. This method has produced very promising results for a number of situations having physical, chemical interest. The method reduces MDE, MKGE and MSE to the Dirac equation, Klein-Gordon and SE, respectively, under two-simultaneously translations in space and phase $x_\mu \rightarrow x_\mu = x_\mu - \frac{\theta_{\omega}}{2}\rho_\nu$ and $\rho_\nu \rightarrow \rho_\nu = \rho_\nu + \frac{\theta_{\omega}}{2}x_\mu$. It based on the following new commutators [10-14, 28-31]:

$$[\hat{\xi}_\mu, \hat{\xi}_\nu] = [\hat{\xi}_\mu(t), \hat{\xi}_\nu(t)] = i\theta_{\mu\nu}$$

$$[\hat{\rho}_\mu, \hat{\rho}_\nu] = [\hat{\rho}_\mu(t), \hat{\rho}_\nu(t)] = i\theta_{\mu\nu}$$

The new generalized positions and momentum coordinates $\tilde{x}_\mu, \tilde{\rho}_\nu$ in (NC: 3D- RSP) are defined in terms of the commutative counterparts $(x_\mu, \rho_\nu)$ in CQM via, respectively [12-16, 28-31]:

$$\left(x_\mu, \rho_\nu\right) \Rightarrow (\tilde{x}_\mu, \tilde{\rho}_\nu) = \left(x_\mu - \frac{\theta_{\omega}}{2}\rho_\nu, \rho_\nu + \frac{\theta_{\omega}}{2}x_\mu\right)$$

The above equation allows us to obtain the two operators $(\tilde{r}^2, \tilde{p}^2)$ in (NC: 3D: RSP) symmetries [26-31]:

$$(\tilde{r}^2, \tilde{p}^2) \Rightarrow (\tilde{r}^2, \tilde{p}^2) = (r^2 - i\tilde{L}_\Omega, p^2 + \tilde{L}_\tilde{\Omega})$$

The two couplings $L_i\tilde{\Omega}$ and $\tilde{L}_\Omega$ are $(L_1\Theta_{12} + L_2\Theta_{13} + L_3\Theta_{12})$ and...
The exact modified spin-orbital operator for heavy quarkonium systems under modified quarkonium potential in $\text{P}_{\text{nrqcd}}$

In this subsection, we will apply the same strategy, which we have seen exclusively in some of our published scientific works [26-32]. Under such particular choice, one can easily reproduce both couplings ($\vec{\Theta}$ and $\vec{\Theta}$) to the new physical forms ($\vec{\Theta}$ and $\vec{\Theta}$), respectively. Thus, the perturbative Hamiltonian operator $H_{\text{pert}} (r, \Theta, \vec{\Theta})$ for heavy quarkonium systems, will be transformed to modified spin-orbital operator $H_{\text{mod}} (r, \Theta, \vec{\Theta})$, under modified quarkonium potential as follows:

$$H_{\text{mod}} (r, \Theta, \vec{\Theta}) = H_{\text{pert}} (r, \Theta, \vec{\Theta}) = \gamma \left[ \frac{c}{2\sqrt{1+2l}} \hat{L} \hat{S} + \frac{\vec{\Theta}}{2\mu} \right]$$

Here $\Theta = \frac{1}{2} (\Theta_{1} + \Theta_{2} + \Theta_{3})$, $\vec{\Theta} = \sqrt{\Theta_{1}^{2} + \Theta_{2}^{2} + \Theta_{3}^{2}}$, and $\gamma = \frac{1}{\sqrt{2\mu}}$ is a new constant, which play the role of fine structure constant in the electromagnetic interaction or QED theory, we have chosen the two vectors $\Theta$ and $\vec{\Theta}$ parallel to the spin $\hat{S}$ of heavy quarkonium systems. Furthermore, the above perturbative terms $H_{\text{pert}} (r)$ can be rewritten to the following new form:

$$H_{\text{mod}} (r, \Theta, \vec{\Theta}) = \frac{\gamma}{2} \left[ \frac{c}{2\sqrt{1+2l}} \hat{L} \hat{S} + \frac{\vec{\Theta}}{2\mu} \right]$$

where $\hat{L}$ and $\hat{S}$ are defined the operators of the total angular momentum and spin of quarkonium systems. This operator, $H_{\text{mod}} (r, \Theta, \vec{\Theta})$, traduces the coupling between spin $\hat{S}$ and orbital momentum $\hat{L}$. The set $\{H_{\text{mod}} (r, \Theta, \vec{\Theta})\}_1, \{H_{\text{mod}} (r, \Theta, \vec{\Theta})\}_2$, and $\{H_{\text{mod}} (r, \Theta, \vec{\Theta})\}_3$ forms a complete of conserved physics quantities. For $\hat{S} = 1$, the eigenvalues of the spin orbital coupling operator are $k(l) = \frac{1}{2} (j(j+1)-l(l+1)-2l)$ corresponding $j = l+1$ (spin great), $j = l$ (spin middle) and $j = l-1$ (spin little), respectively, then, one can form a diagonal (3x3) matrix for modified quarkonium potential in (NC: 3D-RSP) symmetries, with diagonal elements $(H_{\text{mod}} (r, \Theta, \vec{\Theta})_{11}, H_{\text{mod}} (r, \Theta, \vec{\Theta})_{22}$ and $(H_{\text{mod}} (r, \Theta, \vec{\Theta})_{33}$ are given by:

$$H_{\text{mod}} (r, \Theta, \vec{\Theta})_{11} = \gamma k_{l} (\frac{c}{2\sqrt{1+2l}} \hat{L} \hat{S} + \frac{\vec{\Theta}}{2\mu})_{j = l+1}$$

$$H_{\text{mod}} (r, \Theta, \vec{\Theta})_{22} = \gamma k_{l} (\frac{c}{2\sqrt{1+2l}} \hat{L} \hat{S} + \frac{\vec{\Theta}}{2\mu})_{j = l}$$

$$H_{\text{mod}} (r, \Theta, \vec{\Theta})_{33} = \gamma k_{l} (\frac{c}{2\sqrt{1+2l}} \hat{L} \hat{S} + \frac{\vec{\Theta}}{2\mu})_{j = l-1}$$

Here $(k_{l}(l), k_{l}(l), k_{l}(l)) = \frac{1}{2} (l-1, -2l-1)$ and $\| \vec{\Theta} \| = j \leq l$ is the total quantum number. The non-null diagonal elements $(H_{\text{mod}} (r, \Theta, \vec{\Theta})_{11}, H_{\text{mod}} (r, \Theta, \vec{\Theta})_{22}$ and $(H_{\text{mod}} (r, \Theta, \vec{\Theta})_{33})$ of modified Hamiltonian operator $H_{\text{mod}} (r)$ will change the energy values $E_{\text{mod}}$ by creating three new values:

$$E_{\text{mod}} (r) = \Psi (r, \Theta, \vec{\Theta})$$

where we will see them in detail in the next subsection. Through our observation of the expression of $H_{\text{mod}} (r)$, which appear in the equation (24), we see it as proportionate to two infinitesimals
parameters ($\Theta$ and $\Theta$). Thus, in what follows, we proceed to solve the modified radial part of the MSE that is, equation (23) by applying standard perturbation theory to find acceptable solutions at first order of two parameters $\Theta$ and $\Theta$. The proposed solutions for MSE under modified the quark-antiquark interaction potential includes energy corrections, which produced automatically from two principal physical phonemes, the first one is the effect of modified spin-orbital interaction and the second is the modified Zeeman effect while the stark effect can be appear in the linear part of the modified quark-antiquark interaction potential.

**The exact modified spin-orbital spectrum for heavy quarkonium systems under the modified quark-antiquark interaction potential in pNRQCD**

The purpose here is to give a complete prescription for determine the energy level of ground state, first excited state and $n$th excited state of heavy quarkonium systems. We first find the corrections ($E_{\text{qq},a,b,c,n}$, $E_{\text{qq},a,b,c,n}$) and $E_{\text{qq},a,b,c,n}$) for heavy quarkonium system such as (the charmonium $\Xi_c$, bottomonium $\Upsilon_c$ and $\Upsilon_b$) mesons that have the quark and antiquark flavor under modified the quark-antiquark interaction potential, which have three polarity $j = l+1$ (spin great), $j = l$ (spin middle) and $j = l+1$ (spin little), respectively, at first order of two parameters ($\Theta$ and $\Theta$).

Moreover, by applying the perturbative theory, in the case of pNRQCD, we have the following results:

\begin{align*}
E_{\text{qq},a,b,c,n} &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [L_0^2(2\beta r)]^2 dr \\
E_{\text{qq},a,b,c,n} &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [L_0^2(2\beta r)]^2 dr \\
E_{\text{qq},a,b,c,n} &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [L_0^2(2\beta r)]^2 dr
\end{align*}

We have used the orthogonality property of the spherical harmonics $Y_m^j(\Theta, \Phi) Y_m^j(\Theta, \Phi)^* \sin(\Theta) d\Theta d\Phi = \delta_{m'm} \delta_{jj'}$. Now, we can re-write the above equations to the simplified new form:

\begin{align*}
E_{\text{qq},a,b,c,n} &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [L_0^2(2\beta r)]^2 dr \\
E_{\text{qq},a,b,c,n} &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [L_0^2(2\beta r)]^2 dr \\
E_{\text{qq},a,b,c,n} &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [L_0^2(2\beta r)]^2 dr
\end{align*}

Moreover, the expressions of the 4-factors $T_i$ ($i = 1, 2, 3, 4$) are given by:

\begin{align*}
T_1(c,n,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [L_0^2(2\beta r)]^2 dr \\
T_2(b,n,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [L_0^2(2\beta r)]^2 dr \\
T_3(a,n,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [L_0^2(2\beta r)]^2 dr \\
T_4(l,n,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [L_0^2(2\beta r)]^2 dr
\end{align*}

For the ground state, we have $L_0^2(2\beta r) = 1$, thus, the expressions of the 4-factors $T_i$ ($i = 1, 2, 3, 4$) will be simplified to the following form:

\begin{align*}
T_1(c,n = 0,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) dr \\
T_2(b,n = 0,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) dr \\
T_3(a,n = 0,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) dr \\
T_4(l,n,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) dr
\end{align*}

It is convenient to apply the following special integral [33]:

\begin{align*}
\int x^{-\alpha} e^{(-a x^2 - b x^2) dx} &= (2e) \frac{\Gamma(\nu)}{\nu} \exp\left(\frac{\nu^2}{8\alpha} D_{\nu} \left(\frac{x}{2\alpha e}\right)\right)
\end{align*}

Where $\Gamma(\nu)$ denote to the parabolic cylinder functions, $\Gamma(\nu)$ Gamma function $\Gamma(\nu) > 0$ and $\Gamma(\nu) > 0$. After straightforward calculations, we can obtain the explicitly results:

\begin{align*}
T_1(c,n = 0,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) dr \\
T_2(b,n = 0,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) dr \\
T_3(a,n = 0,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) dr
\end{align*}

Allow us the two to obtain the exact modifications $E_{\text{qq},a,b,c,n} = 0, l, m$ and $E_{\text{qq},a,b,c,n} = 0, l, m$ of the ground state as:

\begin{align*}
E_{\text{qq},a,b,c,n} &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) dr \\
E_{\text{qq},a,b,c,n} &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) dr
\end{align*}

With

\begin{align*}
T_1(c,n = 0,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) dr \\
T_2(b,n = 0,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) dr
\end{align*}

The obtained results, which presented in Eq. (34) are created by the effect of modified spin-orbital Hamiltonian operator $H_{\text{spin}}(r)$, thus the ground state energy the energy $E_0$ in CQM will be generated into three values $E_0 = E_{\text{qq},a,b,c,n} = 0, l, m$ and $E_0 = E_{\text{qq},a,b,c,n} = 0, l, m$. For the first excited state $n = 1$, we have $E_{1}^{2\beta r} = -2\beta r + 2n + 1$. After straightforward calculations, the expressions of the 4-factors $T_i$ ($i = 1, 2, 3, 4$) for $n = 1$ in Eq. (30) are simplified to the form:

\begin{align*}
T_1(c,n = 1,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [f(x_{n+1} + f(x_{n+1})] dr \\
T_2(b,n = 1,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [f(x_{n+1} + f(x_{n+1})] dr \\
T_3(a,n = 1,l,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [f(x_{n+1} + f(x_{n+1})] dr \\
T_4(l,n = 1,m) &= \frac{C}{2} \int \exp(-2\alpha^2 - 2\beta r) [f(x_{n+1} + f(x_{n+1})] dr
\end{align*}
Where $f_1 = 4\beta^2$, $f_2 = -4\beta(2\eta + 1)$ and $f_3 = 4\eta^2 + 4\eta + 1$ Evaluating the integral in Eq. (35), applies the special integration, which given by Eq. (32), we obtain the results:

$$\Gamma_{E_n}(l,0,\eta,\mu) = \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)} + f_1(l,0,\eta,\mu) \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$$

(36)

$$\Gamma_{E_n}(l,0,\eta,\mu) = \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)} + f_1(l,0,\eta,\mu) \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$$

Allow us the two to obtain the exact modifications $E_{m+1/2}(k_i,a,b,c,n=1,\mu)$, $E_{m+1/2}(k_i,a,b,c,n=1,\mu)$ and $E_{m+1/2}(k_i,a,b,c,n=1,\mu)$ of the first excited state under the modified quark-antiquark interaction potential:

$$E_{m+1/2}(k_i,a,b,c,n=1,\mu) = 4\Gamma_{E_n}(l,0,\eta,\mu) \left( \theta_{0,1,0,1,0,0} \right) \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)} + \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$$

(37)

With $T_{m}(a,b,c,n=0,\mu) = T_{m}(a,b,c,n=0,\mu)$ and $T_{m}(a,b,c,n=0,\mu)$ for $n^1/2$ excited states of heavy quarkonium systems under modified quark-antiquark interaction potential in global quantum group symmetry (NC: 3-D RSP):

$$E_{m+1/2}(k_i,a,b,c,n=1,\mu) = 4\Gamma_{E_n}(l,0,\eta,\mu) \left( \theta_{0,1,0,1,0,0} \right) \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)} + \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$$

In addition, and in the same way we find the exact modifications $E_{m+1/2}(k_i,a,b,c,n=1,\mu)$, $E_{m+1/2}(k_i,a,b,c,n=1,\mu)$ and $E_{m+1/2}(k_i,a,b,c,n=1,\mu)$ for $m^1/2$ excited states under modified quark-antiquark interaction potential in global quantum group symmetry (NC: 3-D RSP):

$$E_{m+1/2}(k_i,a,b,c,n=1,\mu) = 4\Gamma_{E_n}(l,0,\eta,\mu) \left( \theta_{0,1,0,1,0,0} \right) \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)} + \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$$

(38)

With $T_{m}(a,b,c,n=0,\mu) = T_{m}(a,b,c,n=0,\mu)$, $T_{m}(a,b,c,n=0,\mu)$ and $T_{m}(a,b,c,n=0,\mu)$ for $m^1/2$ excited states of heavy quarkonium systems under modified quark-antiquark interaction potential in global quantum group symmetry (NC: 3-D RSP):

$$E_{m+1/2}(k_i,a,b,c,n=1,\mu) = 4\Gamma_{E_n}(l,0,\eta,\mu) \left( \theta_{0,1,0,1,0,0} \right) \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)} + \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$$

(39)

We choose the arbitrary uniform external magnetic field $B$ parallel to the $(Oz)$ axis, which allow us to introduce the new modified magnetic Hamiltonian $H_{m+1/2}$ in (NC: 3-D RSP) symmetries as:

$$H_{m+1/2}(r,\vartheta,\varphi) = \Gamma_{E_n}(l,0,\eta,\mu) \left( \frac{c}{2\gamma^2} \right) - \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$$

(40)

Here $N_{m+1/2} = -S\frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$ denote to Zeeman effect in CQM, while $N_{m+1/2} = -S\frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$ is the new Zeeman effect in NCQM. To obtain the exact NC magnetic modifications of energy for ground state, first excited state and $n^1/2$ excited states of heavy quarkonium systems $E_{m+1/2}(m=0,a,b,c,n=0,\mu)$, $E_{m+1/2}(m=0,\pm 1,a,b,c,n=0,\mu)$ and $E_{m+1/2}(m=0,\pm 1,a,b,c,n=0,\mu)$, we just replace $k_1(i)$ and $\Theta$ in the Eqs. (34), (37) and (38) by the following parameters $m$ and $\mu$, respectively:

$$E_{m+1/2}(m=0,a,b,c,n=0,\mu) = 4\Gamma_{E_n}(l,0,\eta,\mu) \left( \theta_{0,1,0,1,0,0} \right) \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)} + \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$$

(41)

We have $-\mu \leq m \leq +\mu$, which allow us to fixing $(2l + 1)$ values for discrete number $M$. It should be noted that the results obtained in Eq. (41) could find it by direct calculation $E_{m+1/2}(m=0,a,b,c,n=0,\mu)$, $E_{m+1/2}(m=0,\pm 1,a,b,c,n=0,\mu)$, $E_{m+1/2}(m=0,\pm 1,a,b,c,n=0,\mu)$ and $E_{m+1/2}(m=0,\pm 1,a,b,c,n=0,\mu)$ that takes the following explicit relation:

$$E_{m+1/2}(m=0,a,b,c,n=0,\mu) = \Gamma_{E_n}(l,0,\eta,\mu) \left( \theta_{0,1,0,1,0,0} \right) \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)} + \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$$

(42)

Where $d\tau = r^2 \sin(\theta) d\theta d\phi$, we have used the wave function that is in Eq. (10) and the perturbed magnetic Hamiltonian operator, Eq. (41) and the the orthogonality property of the spherical harmonics $Y^m_l(\Theta,\Phi) Y^{m'}_l(\Theta,\Phi) d^2\Sigma = \delta_{mm'} \delta_{ll'}$, (with $d\Omega = \sin(\theta) d\theta d\phi$). It is clear that the Eq. (42) can be rewritten as follows:

$$E_{m+1/2}(m=0,a,b,c,n=0,\mu) = \Gamma_{E_n}(l,0,\eta,\mu) \left( \theta_{0,1,0,1,0,0} \right) \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)} + \frac{\Gamma_{E_n}(l,0,\eta,\mu)}{\Gamma_{E_n}(l,0,\eta,\mu)}$$

(43)

the expressions of the 4-factors $T(\mu,1,1,\mu)$ are given by Eq. (30). Then we find the magnetic specters of energy produced by the operator $H_{m+1/2}$ for ground state and first excited states repeating the same calculations in the previous subsection. It should be noted that, the magnetic effects are very important in the case of elementary particles such as electrons [34].

Results

In the previous sub-sections, we obtained the solution of the modified Schrödinger equation for the modified quark-antiquark interaction potential, which is given in Eq. (22) by using the generalized Bopp’s shift method and standard perturbation theory in pNRQCD. The energy eigenvalue is calculated in the 3D space-phase. The modified eigenenergies $E_{m+1/2}(E_{m+1/2},E_{m+1/2},E_{m+1/2},E_{m+1/2})$.
for MSE for heavy quarkonium systems (the charmonium $c\bar{c}$, bottomonium $b\bar{b}$ and $b\bar{c}$ mesons) under modified quark-antiquark interaction potential are obtained in this paper based on our original results presented on the Eqs. (34), (37), (38) and (41), in addition to the ordinary energy $E_o$ for the modified quark-antiquark interaction potential which presented in the Eq. (9):

$$E_{\text{m}}(n=0,m=0,a,b,c,l,\mu) = E_o$$

$$E_{\text{n}}(n=0,m=0,a,b,c,l,\mu) = E_o + 2C_{n}\left(\Theta_{\text{m}}(n=0,a,b,c,l,\mu) + \frac{\rho}{2\mu}(n=0,l=0,\mu)\right)$$

$$E_{\text{l}}(n=0,m=0,a,b,c,l,\mu) = E_o + 2C_{n}\left(\Theta_{\text{m}}(n=0,a,b,c,l,\mu) + \frac{\rho}{2\mu}(n=0,l=0,\mu)\right)$$

Thus, the ordinary kinetic term for the quark-antiquark interaction potential is the non-polarized energies (energy independent of spin), which are given by:

$$H_{\text{m}}(n=0,m=0,a,b,c,l,\mu) = \frac{\Delta}{2\mu} + H_{\text{m-ep}} + H_{\text{m-ep}}$$

This is one of the main objectives of our research and by noting the obtained eigenvalues of energies are real and

$$E_{\text{m}}(x_p,r) \rightarrow H_{\text{m}}(x_p,r) = \left(\begin{array}{c}
H_{\text{m}-\text{ep}}(x_p,r) \\
0 \\
0
\end{array}\right)$$

Where $E_{\text{m}}$ and $E_o$ are the energy of ground state and first excited state of heavy quarkonium systems in the symmetries of quantum mechanics under the quark-antiquark interaction potential:

$$E_{\text{m}} \equiv \frac{3a}{2\mu} \frac{2b^2}{8a}$$

and $E_o = \frac{a}{2\mu} \sqrt{2(2+5) - \frac{(2-2)b^2}{8(1+1)a}}$.

One of the most important applications, in the extended model of pNRQCD, is to calculate the modified mass spectra of the heavy quarkonium systems (the mass of the quarkonium bound state), such as charmonium and bottomonium mesons, that have the quark and antiquark flavor in the symmetries of NQCM under modified quark-antiquark interaction potential. In order to achieve this goal, we generalize the traditional formula $M = 2m_q + E_{\text{al}}$ to the new form

$$M = 2m_q + E_{\text{o}} \rightarrow M_{\text{m}} = 2m_q + \frac{1}{3}\left(E_{\text{m}} + E_{\text{m}} + E_{\text{m}}\right)\left(n, m = \frac{-7+\mu}{\mu}, a,b,c,l,\mu\right)$$

Here $m_q$ is bare mass of quarkonium or twice the reduced mass of the system. Moreover, $\frac{1}{3}\left(E_{\text{m}} + E_{\text{m}} + E_{\text{m}}\right)\left(n, m = \frac{-7+\mu}{\mu}, a,b,c,l,\mu\right)$ is the non-polarized energies (energy independent of spin), which can be determined from Eqs. (46) and (51). Thus, the modified mass of quarkonium system $M_{\text{m}}$ can be according to the following new result:

$$M_{\text{m}}(n,a,b,c,l,m,\mu) = M(\frac{n}{m},a,b,c,l,\mu) + \gamma_{\text{c-eff}}\left(-\frac{\gamma}{2}(2+3)\right)\left(\frac{\gamma}{2(2+3)}\right)$$

$$M_{\text{m}}(n,a,b,c,l,\mu)$$

With $M_{\text{m}}(\tilde{S} = \tilde{0})$ and $M_{\text{m}}(\tilde{S} = \tilde{0})$ are given by:

$$M_{\text{m}}(\tilde{S} = \tilde{1}) = \frac{\gamma}{2}B_{\tilde{S}} + \frac{\gamma}{6}T_{\tilde{S}}(n,a,b,c,l,\mu) = \frac{\gamma}{2}B_{\tilde{S}} + \frac{\gamma}{12}T_{\tilde{S}}(n,a,b,c,l,\mu)$$

$$M_{\text{m}}(\tilde{S} = \tilde{0}) = \frac{\gamma}{2}T_{\tilde{S}}(n,a,b,c,l,\mu) = \frac{\gamma}{2}T_{\tilde{S}}(n,a,b,c,l,\mu)$$

Thus, the ordinary kinetic term for the quark-antiquark interaction potential ($\frac{\Delta}{2\mu}$) and ordinary interaction ($ar^2 + br^2 - \frac{c}{2}$) are replaced by new modified form of kinetic term $\frac{\Delta}{2\mu}$ and new modified interactions ($H_{\text{m}} + g_{\text{ep}} + H_{\text{m}} + H_{\text{m}}$) respectively, in (NC-3D: RSP) symmetries. On the other hand, it is evident to consider the quantum number $m$ takes $(2+1)$ values and we have also three values for $(j = \pm 1, \ell)$, thus every state in usually three-dimensional space of energy for heavy quarkonium systems under modified quark-antiquark interaction potential will be $3((2+1)+1)$ sub-states. To obtain the total complete degeneracy of energy level of the modified quark-antiquark interaction potential in (NC-3D: RSP) symmetries, we need to sum for all allowed values of $l$. Total degeneracy is thus, 

$$\sum_{l=0}^{(2+1)}(2l+1) = \sum_{l=0}^{(2+1)}(2l+1) = 3n^2$$

Note that the obtained new energy eigenvalues ($E_{\text{m}}$, $E_{\text{m}}$, $E_{\text{m}}$, $E_{\text{m}}$) now depend to new discrete atomic quantum numbers ($n, j, l, s$) and $m$ in addition to the ordinary kinetic term for the quark-antiquark interaction potential. It is pertinent to note that when the atoms have $\tilde{S} = 0$, the total operator can be obtained from the interval $\left[-\frac{1}{2} \leq j \leq \frac{1}{2} + \mu\right]$, which allow us to obtaining the eigenvalues of the operator $\left(J^2 - \frac{\gamma}{2(2+3)}\right)$ as $k(j, l, s) = 0$ and then the non-relativistic energy spectrum ($E_{\text{m}}$, $E_{\text{m}}$, $E_{\text{m}}$, $E_{\text{m}}$) reads:

$$E_{\text{m}}(n,a,b,c,l,\mu) = E_{\text{m}}(n,a,b,c,l,\mu) = E_{\text{m}}(n,a,b,c,l,\mu) = E_{\text{m}}(n,a,b,c,l,\mu)$$

$$\left(J^2 - \frac{\gamma}{2(2+3)}\right)$$

$$\sum_{l=0}^{(2+1)}(2l+1) = \sum_{l=0}^{(2+1)}(2l+1) = 3n^2$$

(50)
terms proportional with two parameters \( ((\Theta \, \mathcal{O} \, \chi)\text{ and } (\overrightarrow{\partial} \, \chi))\). If we consider \( (\Theta \, \mathcal{O} \, \chi, \overrightarrow{\partial} \, \chi, \overrightarrow{\sigma}) \to (0,0)\), we recover the results of commutative space of ref. [1] obtained for the quark-antiquark interaction potential, which means that our calculations are correct.

Conclusion

In the present work, the 3-dimensional modified Schrodinger equation is analytically solved using the generalized Bopp’s shift method and time independent standard perturbation theory. The quark-antiquark interaction potential is extended to include effect of non-commutativity space phase; we resume the main obtained results:

- Ordinary quark-antiquark interaction potential \((ar^2 + br - \tilde{z})\) were replaced by new modified interactions \((H_{\text{int, gpp}}, H_{\text{int, mpp}}\text{ and } H_{\text{int, lpp}})\) for heavy quarkonium systems,

- The ordinary kinetic term \(\frac{\Delta}{2\mu}\) modified to the new form \(\frac{\Delta_{\text{nc}}}{2\mu} = \frac{\Delta - \overrightarrow{\partial} \overrightarrow{L} - \overrightarrow{\sigma} \overrightarrow{L}}{2\mu}\) for heavy quarkonium systems under influence of the modified quark-antiquark interaction potential,

- We obtained the perturbative corrections \((E_{\text{nc, gpp}}, E_{\text{nc, mpp}}, E_{\text{nc, lpp}})\) \((n = 0, m = 0, a, b, c, l, \mu)\), \((E_{\text{nc, gpp}}, E_{\text{nc, mpp}}, E_{\text{nc, lpp}})\) \((n = 1, m = 0, \pm 1, a, b, c, l, \mu)\text{ and} \) for ground state, first excited state and \(n^\prime\) excited state with \((\text{spin } s = 1 \text{ and } s^\prime = 0)\) for heavy quarkonium systems under influence of the modified quark-antiquark interaction potential are obtained.

- We have obtained the modified mass of quarkonium system \(M_{\text{nc, gpp}}(n, a, b, c, l, m, \mu)\) which equal the sum of corresponding value \(M\) in CQM and two perturbative terms proportional with two parameters \((((\Theta \, \mathcal{O} \, \chi)\text{ and } (\overrightarrow{\partial} \, \chi)))\).

Through the of high value results, which we have achieved in present work, we hope to extend our recently work physics for further investigations of particles physics and other characteristics of quarkonium among others in the context of NRQCD theory. Finally, we can say that we have established our new theoretical model to describe the charmonium \(c\bar{c}\), bottomonium \(b\bar{b}\) and \(c\bar{c}\) mesons in the symmetries of NCQM under modified quark-antiquark interaction potential, this is the main objective of this work.

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