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A New Asymptotic Study to the 3-Dimensional Radial Schrodinger Equation under Modified Quark-antiquark Interaction Potential

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Abstract

In present work, the modified 3-dimensional modified radial Schrödinger equation is analytically solved for the non-relativistic interactions under the modified quark-antiquark interaction potential in the symmetries of Non-Commutative Quantum Mechanics (NCQM), using the generalized Bopp's shift method in the case of perturbed Non-Relativistic Quantum Chromodynamics (pNRQCD). The new energy eigenvalues (ground state, first excited state and nth excited state) and the corresponding Hamiltonian operator are calculated in 3-dimensional Non-Commutative Real Space Phase (NC: 3D-RSP) symmetries instead of solving modified Schrödinger Weyl Moyal star product. The present results, in (NC: 3D-RSP), are applied on the charmonium $c\bar{c}$, bottomonium $b\bar{b}$ and $b\bar{c}$ mesons. It is found that the perturbative solutions of discrete spectrum and quarkoniums mass can be expressed by the the parabolic cylinder functions function, Gamma function, the discreet atomic quantum numbers (j,l,s,m) of the $Q\bar{Q}$ state and the potential parameters (*a*,*b*,*c*), in addition to non-commutativity parameters (Θ and $\bar{\theta}$). The total complete degeneracy of new energy levels of the modified quark-antiquark interaction potential changed to become equals to the value $3n^2$ instead the values n^2 in ordinary Commutative Quantum Mechanics (CQM). Our obtained results are in good agreement with the already existing literatures in NCQM.

Keywords: Schrodinger equation; The heavy quarkonium system; The quark-antiquark interaction potential; Non-commutative space phase; Bopp's shift method

Introduction

It is well known that, the quark-antiquark interaction potential or the quarkonium potential, which obtained from Cornell potential by adding the harmonic term, is one of confining potentials. It was one of the most popular model for study the interactions in the Non-Relativistic Quantum Chromodynamics NRQCD such systems as quarkonium (heavy quarkonia) consisting of heavy quark and antiquark (charmonium $c\bar{c}$, bottomonium $b\bar{b}$ and $b\bar{c}$ mesons). The study of this potential, in NRQCD, is a particular interest for detecting characterizing the electromagnetic characteristics of mesons and the mass spectra for coupled states [1-8]. It is consist of three terms. One of the terms is responsible for the Coulomb interaction of quarks and the second corresponds to the string interaction, which provides confinement, while the third is the harmonic term, known to researchers and plays a very important role in various fields of physics and chemistry. Furthermore, this potential plays a vital role in different branches of physics such as atomic and molecular physics, particle physics, plasma physics and solid-state physics [7]. The main objective is to develop the research article [1] and expanding it to the hug symmetry known by Non-Commutative Quantum Mechanics (NCQM) in order to achieve a more accurate physical vision so that this study becomes valid in the field of nanotechnology. On the other hand to explore the possibility of creating new applications and more profound interpretations in the sub-atomics and nano scales using new version the modified quark-antiquark interaction potential, which has the following form:

$$V_{qp}(r)=ar^2+br-\frac{c}{r}\rightarrow V_{qp}(\hat{r})=V_{qp}(r)+\left\{\frac{c}{2r^3}-\frac{b}{2r}-a\right\}\vec{L}\vec{\Theta} \tag{1}$$

We refer to this term $\vec{L}\vec{\Theta}$ in the materials and methods section. It should be noted that, the non-commutativity was introduced firstly by W. Heisenberg in 1930 [9] and then by H. Syndre in 1947 [10]. The new structure of NCQM based to new canonical commutations relations in both Schrödinger SP and Heisenberg HP, respectively, as follows (Throughout this paper, the natural units $c=\hbar=1$ will be used) [11-15]:

$$\begin{aligned}
&\left[\hat{x}_\mu \overset{*}{,} \hat{p}_\nu\right]=\left[\hat{x}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)\right]=i\delta_{\mu\nu}\hbar\Rightarrow\Delta\hat{x}_\mu\Delta\hat{p}_\nu\geq\frac{\delta_{\mu\nu}}{2}\\
&\left[\hat{x}_\mu \overset{*}{,} \hat{x}_\nu\right]=\left[\hat{x}_\mu(t) \overset{*}{,} \hat{x}_\nu(t)\right]=i\theta_{\mu\nu}\Rightarrow\Delta\hat{x}_\mu\Delta\hat{x}_\nu\geq\frac{\theta_{\mu\nu}}{2}\\
&\left[\hat{p}_\mu \overset{*}{,} \hat{p}_\nu\right]=\left[\hat{p}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)\right]=i\bar{\theta}_{\mu\nu}\Rightarrow\Delta\hat{p}_\mu\Delta\hat{p}_\nu\geq\frac{\bar{\theta}_{\mu\nu}}{2}
\end{aligned} \tag{2}$$

Where the indices $\mu,\nu\equiv\overline{1,3}$. This means that the principle of uncertainty for Heisenberg generalized to include another two new uncertainties related to the positions $(\hat{x}_\mu,\hat{x}_\nu)$ and the momenta's $(\hat{p}_\mu,\hat{p}_\nu)$, in addition to the ordinary uncertainty of $(\hat{x}_\mu,\hat{p}_\nu)$ in CQM. The very small two parameters $\theta^{\mu\nu}$ and $\bar{\theta}^{\mu\nu}$ (compared to the energy) are elements of two antisymmetric real matrixes, parameters of non-commutativity and $(*)$ denote to the Weyl Moyal star product, which is generalized between two arbitrary functions $(f,g)(x,p)\rightarrow(\hat{f},\hat{g})(\hat{x},\hat{p})$ to the new form $\hat{f}(\hat{x},\hat{p})\hat{g}(\hat{x},\hat{p})\equiv(f*g)(x,p)$ in (NC: 3D-RSP) symmetries [16-23]:

$$(f,g)\ (x,p)\rightarrow(f*g)(x,p)=\left(fg-\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x f\,\partial_\nu^x g-\frac{i}{2}\bar{\theta}^{\mu\nu}\partial_\mu^p f\,\partial_\nu^p g\right)(x,p) \tag{3}$$

The second and the third terms in the above equation are present the effects of (space-space) and (phase-phase) non-commutativity properties. However, the new operators $\hat{\xi}(t)=(\hat{x}_\mu\vee\hat{p}_\mu)(t)$ in HP are

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depending to the corresponding new operators $\hat{\xi} = \hat{x}_\mu \vee \hat{p}_\nu$ in SP from the following projections relations:

$$\xi(t) = \exp(i\hat{H}_{qp}(t-t_0))\xi\exp(-i\hat{H}_{qp}(t-t_0)) \Rightarrow$$
$$\hat{\xi}(t) = \exp(i\hat{H}_{nc-qp}(t-t_0)) * \hat{\xi} * \exp(-i\hat{H}_{nc-qp}(t-t_0)) \quad (4)$$

Here $\xi = (x_\mu \vee p_\nu)$ and $\xi(t) = (x_\mu \vee p_\nu)(t)$, while the dynamics of new systems $\frac{d\xi(t)}{dt}$ are described from the following motion equations in NCQM:

$$\frac{d\xi(t)}{dt} = \left[\xi(t), \hat{H}_{qp}\right] \Rightarrow \frac{d\hat{\xi}(t)}{dt} = \left[\hat{\xi}(t) \overset{*}{,} \hat{H}_{nc-qp}\right] \quad (5)$$

The two operators $\hat{H}_{qp}$ and $\hat{H}_{nc-qp}$ are presents the quantum Hamiltonian operators for the quark-antiquark interaction potential and modified quark-antiquark interaction potential, in the CQM and its extension NCQM, respectively. This paper consists of five sections and the organization scheme is given as follows: In next section, the theory part, we briefly review the SE with the quark-antiquark interaction potential [1-2]. The Section 3 is devoted to studying the MSE by applying the generalized Bopp's shift method and obtained the modified quark-antiquark interaction potential and the modified spin-orbital operator. Then, we apply the standard perturbation theory to find the quantum spectrum of (ground state, first excited state and the n^{th} excited state) which produced automatically by the effects of modified spin-orbital and modified Zeeman interactions. After that, in the fourth section, a discussion of the main results is presented in addition to determine the new formula of mass spectra of the of quarkonium system (the charmonium $c\bar{c}$, bottomonium $b\bar{b}$ and $b\bar{c}$ mesons) in (NC: 3D-RSP) symmetries. Finally, in the last section, summary and conclusions are presented.

Theory

Overview of the Eigen-functions and the energy Eigen-values for the quark-antiquark interaction potential in CQM

We shall recall briefly in this section, the time independent Schrödinger equation SE for the quark- antiquark interaction potential (quarkonium potential) [1-2]:

$$V(r) = ar^2 + br - \frac{c}{r} \quad (6)$$

The relative spatial coordinate between the two quarks is r, $a\rangle 0$, b and are c purely phenomenological constants of the model. If we insert this potential into SE, the radial part function $U(r) = \frac{R(r)}{r}$ is given as [1]:

$$\frac{d^2U(r)}{dr^2} + \frac{2}{r}\frac{dU(r)}{dr} + 2\mu\left[E - ar^2 - br + \frac{c}{r} - \frac{l(l+1)}{2\mu r^2}\right]U(r) = 0$$
$$\rightarrow \frac{d^2R(r)}{dr^2} + 2\mu\left[E - ar^2 - br + \frac{c}{r} - \frac{l(l+1)}{2\mu r^2}\right]R(r) = 0 \quad (7)$$

Here $\mu = \frac{m_q m_{\bar{q}}}{m_q + m_{\bar{q}}}$ the reduced mass for the quarkonium particle for example $c\bar{c}$, $b\bar{b}$ and $c\bar{b}$. The complete wave function $\Psi(r,\theta,\phi) = \frac{R(r)}{r}Y_l^m(\theta,\phi)$ is given by [1]:

$$\Psi(r,\theta,\phi) = C_{nl}(-2)^n \frac{\Gamma(2\eta+1+n)}{\Gamma(2\eta+n)} r^{\eta-\frac{1}{2}} \exp(-\alpha r^2 - \beta r)\,{}_1F_1(-n, 2\eta+1; 2\beta r) Y_l^m(\theta,\phi) \quad (8)$$

In addition, the energy E_{nl} of the potential in Eq. (8) [1]:

$$E_{nl} = \sqrt{\frac{a}{2\mu}}(2n+2l+3) + \frac{(4n-2l-2)b^2}{8(l+1)a} \quad (9)$$

where $\alpha = \sqrt{\frac{\mu a}{2}}$, $\beta = b\sqrt{\frac{\mu}{2a}}$, $\eta = l + 1/2$ and C_{nl} is a normalizing constant. While n is a natural number accounting for the radial excitation while l is a non-negative integer number which represents the orbital angular momentum. On account of the relation between the confluent hypergeometric function of the first kind and the generalized Laguerre polynomial.

${}_1F_1(-n, 2\eta+1; 2\beta r) = \frac{n!\Gamma(2\eta+1)}{\Gamma(2\eta+1+n)} L_n^{2\eta}(2\beta r)$, Eq. (8) is rewritten as:

$$\Psi(r,\theta,\varphi) = C_{nl}(-2)^n \frac{n!\Gamma(2\eta+1)}{\Gamma(2\eta+n)} r^{\eta-\frac{1}{2}} \exp(-\alpha r^2 - \beta r) L_n^{2\eta}(2\beta r) Y_l^m(\theta,\varphi) \quad (10)$$

Materials and Methods

Solution of MSE for modified quark-antiquark interaction potential in pNRQCD

In this section, we shall give an overview or a brief preliminary for modified quarkonium potential in (NC: 3D-RSP) symmetries. To perform this task the physical form of MSE, it is necessary to replace ordinary 3-dimensional Hamiltonian operators $\hat{H}_{qp}(x_\mu, p_\mu)$, complex wave function $\Psi(\vec{r})$ and energy E_{nl} by new Hamiltonian operators $\hat{H}_{nc-qp}(\hat{x}_\mu, \hat{p}_\mu)$, new complex wave function $\hat{\Psi}(\vec{\hat{r}})$ and new values E_{nc-cp}, respectively. In addition to replace the ordinary product by the Weyl Moyal star product, which allow us to constructing the MSE in (NC-3D: RSP) symmetries as [24-28]:

$$\hat{H}_{qp}(x_\mu, p_\mu)\Psi(\vec{r}) = E_{nl}\Psi(\vec{r}) \Rightarrow \hat{H}_{nc-qp}(\hat{x}_\mu, \hat{p}_\mu) * \Psi(\vec{\hat{r}}) = E_{nc-qp}\Psi(\vec{\hat{r}}) \quad (11)$$

The Bopp's shift method has been successfully applied to relativistic and non-relativistic (non-commutative quantum mechanical problems) using Modified Dirac Equation (MDE), Modified Klein-Gordon Equation (MKGE) and MSE. This method has produced very promising results for a number of situations having physical, chemical interest. The method reduces MDE, MKGE and MSE to the Dirac equation, Klein-Gordon and SE, respectively, under two-similtaniously translations in space and phase $x_\mu \to \hat{x}_\mu \equiv x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu$ and $\hat{p}_\mu = p_\mu + \frac{\bar{\theta}_{\mu\nu}}{2}x_\nu$. It based on the following new commutators [10-14, 28-31]:

$$\left[\hat{x}_\mu, \hat{x}_\nu\right] = \left[\hat{x}_\mu(t), \hat{x}_\nu(t)\right] = i\theta_{\mu\nu}$$
$$\left[\hat{p}_\mu, \hat{p}_\nu\right] = \left[\hat{p}_\mu(t), \hat{p}_\nu(t)\right] = i\bar{\theta}_{\mu\nu} \quad (12)$$

The new generalized positions and momentum coordinates $(\hat{x}_\mu, \hat{p}_\nu)$ in (NC: 3D-RSP) are defined in terms of the commutative counterparts (x_μ, p_ν) in CQM *via*, respectively [12-16, 28-31]:

$$(x_\mu, p_\nu) \Rightarrow (\hat{x}_\mu, \hat{p}_\nu) = \left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu, p_\mu + \frac{\bar{\theta}_{\mu\nu}}{2}x_\nu\right) \quad (13)$$

The above equation allows us to obtain the two operators $(\hat{r}^2, \hat{p}^2)$ in (NC-3D: RSP) symmetries [26-31]:

$$(r^2, p^2) \Rightarrow (\hat{r}^2, \hat{p}^2) = \left(r^2 - \mathbf{L}\Theta, p^2 + \vec{\mathbf{L}}\vec{\bar{\theta}}\right) \quad (14)$$

The two couplings $\mathbf{L}\Theta$ and $\vec{\mathbf{L}}\vec{\bar{\theta}}$ are $(L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13})$ and

$\left(L_x\overline{\theta}_{12}+L_y\overline{\theta}_{23}+L_z\overline{\theta}_{13}\right)$, respectively, while ($L_x$, L_y and L_z) are the three components of angular momentum operator $\vec{L}$ and $\Theta_{\mu\nu}=\theta_{\mu\nu}/2$. Thus, the reduced Schrödinger equation (without star product) can be written as:

$$\hat{H}_{nc-qp}\left(\hat{x}_\mu,\hat{p}_\mu\right)*\Psi\left(\vec{\hat{r}}\right)=E_{nc-qp}\Psi\left(\vec{\hat{r}}\right)\Rightarrow H_{nc-qp}\left(\hat{x}_\mu,\hat{p}_\mu\right)\psi\left(\vec{r}\right)=E_{nc-qp}\psi\left(\vec{r}\right) \quad (15)$$

The new operator of Hamiltonian $H_{nc-cp}\left(\hat{x}_\mu,\hat{p}_\nu\right)$ can be expressed as:

$$H_{cp}\left(x_\mu,x_\mu\right)\Rightarrow H_{nc-cp}\left(\hat{x}_\mu,\hat{p}_\mu\right)\equiv H\left(\hat{x}_\mu=x_\mu-\frac{\theta_{\mu\nu}}{2}p_\nu,\hat{p}_\mu=p_\mu+\frac{\overline{\theta}_{\mu\nu}}{2}x_\nu\right)$$
$$=\frac{\hat{p}^2}{2\mu}+V_{cp}\left(\hat{r}=\sqrt{\left(x_\mu-\frac{\theta_{\mu\nu}}{2}p_\nu\right)\left(x_\mu-\frac{\theta_{\mu\nu}}{2}p_\nu\right)}\right) \quad (16)$$

Where $V_{qp}\left(\hat{r}\right)$ denote to the modified quarkonium potential in (NC: 3D-RSP) symmetries:

$$V_{qp}\left(r\right)\Rightarrow V_{qp}\left(\hat{r}\right)=a\hat{r}^2+b\hat{r}-\frac{c}{\hat{r}} \quad (17)$$

Again, applying Eq. (14) to find the three terms ($b\hat{r}$ and $\left(-\frac{c}{\hat{r}}\right)$ and $\left(a\hat{r}^2\right)$), which will be used to determine the modified quarkonium potential $V_{qp}\left(\hat{r}\right)$, as follows:

$$\begin{cases}\frac{c}{r}\rightarrow\frac{c}{\hat{r}}=\frac{c}{r}+\frac{c}{2r^3}\vec{\mathbf{L}}\vec{\Theta}+O\left(\Theta^2\right)\\ br\rightarrow b\hat{r}=br-\frac{b}{2r}\vec{\mathbf{L}}\vec{\Theta}+O\left(\Theta^2\right)\\ ar^2\rightarrow a\hat{r}^2=ar^2-a\vec{\mathbf{L}}\vec{\Theta}+O\left(\Theta^2\right)\end{cases} \quad (18)$$

Substituting, Eq. (18) into Eq. (17), gives the modified quark-antiquark interaction potential in (NC-3D: RSP) symmetries as follows:

$$V_{qp}\left(\hat{r}\right)=V_{qp}\left(r\right)+\left\{\frac{c}{2r^3}-\frac{b}{2r}-a\right\}\vec{\mathbf{L}}\vec{\Theta} \quad (19)$$

By making the substitution above equation into Eq. (16), we find the global our working new modified Hamiltonian operator $H_{qp}\left(x_\mu,p_\nu\right)$ satisfies the equation in (NC: 3D-RSP) symmetries:

$$H_{qp}\left(x_\mu,p_\nu\right)\Rightarrow H_{nc-qp}\left(\hat{r}\right)=H_{qp}\left(x_\mu,p_\nu\right)+H_{\text{per-}qp}\left(r,\Theta,\overline{\theta}\right) \quad (20)$$

where the operator $H_{qp}\left(x_\mu,p_\nu\right)$ is just the ordinary Hamiltonian operator for quarkonium potential in CQM

$$H_{qp}\left(x_\mu,p_\mu\right)=\frac{p^2}{2\mu}+ar^2+br-\frac{c}{r} \quad (21)$$

while the rest part $H_{\text{per-}qp}\left(r,\Theta,\overline{\theta}\right)$ (the perturbative Hamiltonian operator) is proportional with two infinitesimals parameters (Θ and $\overline{\theta}$):

$$H_{\text{per-}qp}\left(r,\Theta,\overline{\theta}\right)=\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\vec{\mathbf{L}}\vec{\Theta}+\frac{\vec{\mathbf{L}}\vec{\overline{\boldsymbol{\theta}}}}{2\mu} \quad (22)$$

Thus, we can consider $H_{\text{per-}qp}(r)$ as a perturbation terms compared with the principal Hamiltonian operator $H_{qp}\left(x_\mu,p_\mu\right)$ in (NC: 3D-RSP) symmetries. After profound calculation, one can show that, the new radial function $R_{nl}(r)$ satisfying the following differential equation for modified quarkonium potential:

$$\frac{d^2R_{nl}(r)}{dr^2}+2\mu\left[E_{nl}-ar^2-br+\frac{c}{r}-\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\vec{\mathbf{L}}\vec{\Theta}-\frac{\vec{\mathbf{L}}\vec{\overline{\theta}}}{2\mu}-\frac{l(l+1)}{2\mu r^2}\right]R_{nl}(r)=0 \quad (23)$$

The exact modified spin-orbital operator for heavy quarkonium systems under modified quarkonium potential in Pnrqcd

In this subsection, we will apply the same strategy, which we have seen exclusively in some of our published scientific works [26-32]. Under such particular choice, one can easily reproduce both couplings ($\vec{\mathbf{L}}\vec{\Theta}$ and $\vec{\mathbf{L}}\vec{\overline{\theta}}$) to the new physical forms ($\gamma\Theta\vec{L}\vec{S}$ and $\gamma\overline{\theta}\vec{L}\vec{S}$), respectively. Thus, the perturbative Hamiltonian operator $H_{\text{per-}qp}\left(r,\Theta,\overline{\theta}\right)$ for heavy quarkonium systems, will be transforms to modified spin-orbital operator $H_{\text{so-}qp}\left(r,\Theta,\overline{\theta}\right)$, under modified quarkonium potential as follows:

$$H_{\text{per-}qp}\left(r,\Theta,\overline{\theta}\right)\rightarrow H_{\text{so-}qp}\left(r,\Theta,\overline{\theta}\right)\equiv\gamma\left\{\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)+\frac{\overline{\theta}}{2\mu}\right\}\vec{L}\vec{S} \quad (24)$$

Here $\Theta=\sqrt{\Theta_{12}{}^2+\Theta_{23}{}^2+\Theta_{13}{}^2}$, $\overline{\theta}=\sqrt{\overline{\theta}_{12}{}^2+\overline{\theta}_{23}{}^2+\overline{\theta}_{13}{}^2}$ and $\gamma\approx\frac{1}{137}$ is a new constant, which play the role of fine structure constant in the electromagnetic interaction or QED theory, we have chosen the two vectors $\vec{\Theta}$ and $\vec{\overline{\boldsymbol{\theta}}}$ parallel to the spin $\vec{S}$ of heavy quarkonium systems. Furthermore, the above perturbative terms $H_{\text{so-}qp}(r)$ can be rewritten to the following new form:

$$H_{so-qp}\left(r,\Theta,\overline{\theta}\right)=\frac{\gamma}{2}\left\{\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\Theta+\frac{\overline{\theta}}{2\mu}\right\}\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right) \quad (25)$$

Where $\vec{J}$ and $\vec{S}$ are defined the operators of the total angular momentum and spin of quarkonium systems. This operator, $H_{\text{so-}qp}\left(r,\Theta,\overline{\theta}\right)$, traduces the coupling between spin $\vec{S}$ and orbital momentum $\vec{L}$. The set ($H_{so-qp}\left(r,\Theta,\overline{\theta}\right)$, J^2, L^2, S^2 and J_z forms a complete of conserved physics quantities. For $\vec{S}=\vec{1}$, the eigenvalues of the spin orbital coupling operator are $k(l)\equiv\frac{1}{2}\{j(j+1)-l(l+1)-2\}$ corresponding $j=l+1$ (spin great), $j=l$ (spin middle) and $j=l-1$ (spin little), respectively, then, one can form a diagonal (3×3) matrix for modified quarkonium potential in (NC: 3D-RSP) symmetries, with diagonal elements $\left(H_{so-qp}\right)_{11}$, $\left(H_{so-qp}\right)_{22}$ and $\left(H_{so-qp}\right)_{33}$ are given by:

$$\begin{aligned}\left(H_{so-qp}\right)_{11}&=\gamma k_1(l)\left(\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{if } j=l+1\\ \left(H_{so-qp}\right)_{22}&=\gamma k_2(l)\left(\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{if } j=l\\ \left(H_{so-qp}\right)_{33}&=\gamma k_3(l)\left(\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{if } j=l-1\end{aligned} \quad (26)$$

Here $\left(k_1(l),k_2(l),k_3(l)\right)\equiv\frac{1}{2}(l,-2,-2l-2)$ and $|l-1|\leq j\leq|l+1|$ is the total quantum number. The non-null diagonal elements $\left(H_{so-qp}\right)_{11}$ $\left(H_{so-qp}\right)_{22}$ and $\left(H_{so-qp}\right)_{33}$ of modified Hamiltonian operator '$H_{nc-qp}\left(\hat{r}\right)$ will change the energy values E_{nl} by creating three new values:

$$\begin{cases}E_{\text{g-}qp}=\left\langle\Psi\left(r,\theta,\phi\right)\right|\left(H_{so-qp}\right)_{11}\left|\Psi\left(r,\theta,\phi\right)\right\rangle\\ E_{\text{m-}qp}=\left\langle\Psi\left(r,\theta,\phi\right)\right|\left(H_{so-qp}\right)_{22}\left|\Psi\left(r,\theta,\phi\right)\right\rangle\\ E_{\text{l-}qp}=\left\langle\Psi\left(r,\theta,\phi\right)\right|\left(H_{so-qp}\right)_{33}\left|\Psi\left(r,\theta,\phi\right)\right\rangle\end{cases} \quad (27)$$

We will see them in detail in the next subsection. Through our observation of the expression of $H_{\text{so-}qp}(r)$, which appear in the equation (24), we see it as proportionate to two infinitesimals parameters (Θ and Θ). Thus, in what follows, we proceed to solve the modified radial part of the MSE that is, equation (23) by applying standard perturbation theory to find acceptable solutions at first order of two parameters Θ and $\bar{\theta}$. The proposed solutions for MSE under modified the quark-antiquark interaction potential includes energy corrections, which produced automatically from two principal physical phoneme's, the first one is the effect of modified spin-orbital interaction and the second is the modified Zeeman effect while the stark effect can be appear in the linear part of the modified quark-antiquark interaction potential.

The exact modified spin-orbital spectrum for heavy quarkonium systems under the modified quark-antiquark interaction potential in pNRQCD

The purpose here is to give a complete prescription for determine the energy level of ground state, first excited state and n^{th} excited state, of heavy quarkonium systems. We first find the corrections ($E_{so-gqp}\left(k_1(l),a,b,c,n\right)$, $E_{so-mqp}\left(k_2(l),a,b,c,n\right)$ and $E_{so-lqp}\left(k_3(l),a,b,c,n\right)$) for heavy quarkonium system such as (the charmonium $c\bar{c}$, bottomonium $b\bar{c}$ and $b\bar{c}$) mesons that have the quark and antiquark flavor under modified the quark-antiquark interaction potential, which have three polarity $j=l+1$ (spin great), $j=l$ (spin middle) and $j=l+1$ (spin little), respectively, at first order of two parameters ($\bar{\theta}$ and θ).

Moreover, by applying the perturbative theory, in the case of pNRQCD, we obtained the following results:

$$E_{so-gqp}=\gamma C_{nl}{}^{2}(-2)^{2n}\left(\frac{n!\Gamma(2\eta+1)}{\Gamma(2\eta+n)}\right)^{2}k_1(l)\int_0^{+\infty}r^{2\eta+1}\exp\left(-2\alpha r^2-2\beta r\right)\left[L_n^{2\eta}(2\beta r)\right]^2\left(\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\Theta+\frac{\bar{\theta}}{2\mu}\right)dr$$
$$E_{so-mqp}=\gamma C_{nl}{}^{2}(-2)^{2n}\left(\frac{n!\Gamma(2\eta+1)}{\Gamma(2\eta+n)}\right)^{2}k_2(l)\int_0^{+\infty}r^{2\eta+1}\exp\left(-2\alpha r^2-2\beta r\right)\left[L_n^{2\eta}(2\beta r)\right]^2\left(\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\Theta+\frac{\bar{\theta}}{2\mu}\right)dr$$
$$E_{so-lqp}=\gamma C_{nl}{}^{2}(-2)^{2n}\left(\frac{n!\Gamma(2\eta+1)}{\Gamma(2\eta+n)}\right)^{2}k_3(l)\int_0^{+\infty}r^{2\eta+1}\exp\left(-2\alpha r^2-2\beta r\right)\left[L_n^{2\eta}(2\beta r)\right]^2\left(\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\Theta+\frac{\bar{\theta}}{2\mu}\right)dr \tag{28}$$

We have used the orthogonality property of the spherical harmonics $\int Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)\sin(\theta)\,d\theta d\phi=\delta_{ll'}\delta_{mm'}$. Now, we can re-write the above equations to the simplified new form:

$$E_{so-gqp}(k_1,a,b,c,n,l,\mu)=\gamma C_{h}{}^{2}(-2)^{2n}\left(\frac{n!\Gamma(2l+3)}{\Gamma(2l+2+n)}\right)^{2}k_1(l)\left\{\Theta\left[T_1(n,c,l,\mu)+T_2(n,b,l,\mu)+T_3(n,a,l,\mu)\right]-\frac{\bar{\theta}}{2\mu}T_4(n,l,\mu)\right\}$$
$$E_{so-mqp}(k_2,a,b,c,n,l,\mu)=\gamma C_{h}{}^{2}(-2)^{2n}\left(\frac{n!\Gamma(2l+3)}{\Gamma(2l+2+n)}\right)^{2}k_2(l)\left\{\Theta\left[T_1(n,c,l,\mu)+T_2(n,b,l,\mu)+T_3(n,a,l,\mu)\right]-\frac{\bar{\theta}}{2\mu}T_4(n,l,\mu)\right\}$$
$$E_{so-lqp}(k_3,a,b,c,n,l,\mu)=\gamma C_{h}{}^{2}(-2)^{2n}\left(\frac{n!\Gamma(2l+3)}{\Gamma(2l+2+n)}\right)^{2}k_3(l)\left\{\Theta\left[T_1(n,c,l,\mu)+T_2(n,b,l,\mu)+T_3(n,a,l,\mu)\right]-\frac{\bar{\theta}}{2\mu}T_4(n,l,\mu)\right\} \tag{29}$$

Moreover, the expressions of the 4-factors $T_i\left(i=\overline{1,4}\right)$ are given by:

$$T_1(c,n,l,\mu)=\frac{c}{2}\int_0^{+\infty}r^{2\eta-2}\exp\left(-2\alpha r^2-2\beta r\right)\left[L_n^{2\eta}(2\beta r)\right]^2dr$$
$$T_2(b,n,l,\mu)=-\frac{b}{2}\int_0^{+\infty}r^{2\eta}\exp\left(-2\alpha r^2-2\beta r\right)\left[L_n^{2\eta}(2\beta r)\right]^2dr$$
$$T_3(a,n,l,\mu)=-a\int_0^{+\infty}r^{2\eta+1}\exp\left(-2\alpha r^2-2\beta r\right)\left[L_n^{2\eta}(2\beta r)\right]^2dr$$
$$T_4(l,n,\mu)=\int_0^{+\infty}r^{2\eta+1}\exp\left(-2\alpha r^2-2\beta r\right)\left[L_n^{2\eta}(2\beta r)\right]^2dr \tag{30}$$

For the ground state, we have $L_{n=0}^{2\eta}(2\beta r)=1$, thus, the expressions of the 4-factors $T_i\left(i=\overline{1,4}\right)$ will be simplified to the following form:

$$T_1(c,n=0,l,\mu)=\frac{c}{2}\int_0^{+\infty}r^{2\eta-1-1}\exp\left(-2\alpha r^2-2\beta r\right)dr$$
$$T_2(b,n=0,l,\mu)=-\frac{b}{2}\int_0^{+\infty}r^{2\eta+1-1}\exp\left(-2\alpha r^2-2\beta r\right)dr$$
$$T_3(a,n=0,l,\mu)=-a\int_0^{+\infty}r^{2\eta+2-1}\exp\left(-2\alpha r^2-2\beta r\right)dr$$
$$T_4(n=0,l,\mu)=\int_0^{+\infty}r^{2\eta+2-1}\exp\left(-2\alpha r^2-2\beta r\right)dr \tag{31}$$

It is convenient to apply the following special integral [33]:

$$\int_0^{+\infty}x^{\nu-1}\exp\left(-\varepsilon x^2-\gamma x\right)dx=(2\varepsilon)^{-\frac{\nu}{2}}\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\varepsilon}\right)D_{-\nu}\left(\frac{\gamma}{\sqrt{2\varepsilon}}\right) \tag{32}$$

Where $D_{-\nu}\left(\frac{\gamma}{\sqrt{2\varepsilon}}\right)$ denote to the parabolic cylinder functions function, $\Gamma(\nu)$ Gamma function $\mathrm{Re}1(\varepsilon)\rangle 0$ and $\mathrm{Re}1(\nu)\rangle 0$. After straightforward calculations, we can obtain the explicitly results:

$$T_1(c,n=0,l,\mu)=\frac{c}{2}(4\alpha)^{-\frac{2l+1}{2}}\Gamma(2l+1)\exp\left(\frac{\beta^2}{4\alpha}\right)D_{-(2l+1)}\left(\frac{\beta}{\sqrt{\alpha}}\right)$$
$$T_2(b,n=0,l,\mu)=-\frac{b}{2}(4\alpha)^{-\frac{2l+3}{2}}\Gamma(2l+3)\exp\left(\frac{\beta^2}{4\alpha}\right)D_{-(2l+3)}\left(\frac{\beta}{\sqrt{\alpha}}\right)$$
$$T_3(a,n=0,l,\mu)=-a(4\alpha)^{-l-2}\Gamma(2l+4)\exp\left(\frac{\beta^2}{4\alpha}\right)D_{-(2l+4)}\left(\frac{\beta}{\sqrt{\alpha}}\right)$$
$$T_4(n=0,l,\mu)=(4\alpha)^{-l-2}\Gamma(2l+4)\exp\left(\frac{\beta^2}{4\alpha}\right)D_{-(2l+4)}\left(\frac{\beta}{\sqrt{\alpha}}\right) \tag{33}$$

Allow us the two to obtain the exact modifications $E_{so-gqp}(k_1,a,b,c,n=0,l)$, $E_{so-mqp}(k_2,a,b,c,n=0,l)$ and $E_{so-lqp}(k_3,a,b,c,n=0,l)$ of the ground state as:

$$E_{so-gqp}(k_1,a,b,c,n=0,l,\mu)=2\gamma C_{00}{}^{2}k_1(l=0)\left\{\Theta T_{10}(a,b,c,n=0,l,\mu)+\frac{\bar{\theta}}{2\mu}T_4(n=0,l,\mu)\right\}$$
$$E_{so-mqp}(k_2,a,b,c,n=0,l,\mu)=2\gamma C_{00}{}^{2}k_2(l=0)\left\{\Theta T_{10}(a,b,c,n=0,l,\mu)+\frac{\bar{\theta}}{2\mu}T_4(n=0,l,\mu)\right\}$$
$$E_{so-lqp}(k_3,a,b,c,n=0,l,\mu)=2\gamma C_{00}{}^{2}k_3(l=0)\left\{\Theta T_{10}(a,b,c,n=0,l,\mu)+\frac{\bar{\theta}}{2\mu}T_4(n=0,l,\mu)\right\} \tag{34}$$

With

$$T_{10}(a,b,c,n=0,l=0,\mu)=T_1(c,n=0,l=0,\mu)+$$
$$+T_2(b,n=0,l=0,\mu)+T_3(a,n=0,l=0,\mu)$$

The obtained results, which presented in Eq. (34) are created by the effect of modified spin-orbital Hamiltonian operator $H_{so-qp}(r)$, thus the ground state energy the energy E_{00} in CQM will be generated into three values $E_{00}+E_{so-gqp}(k_1,a,b,c,n=0,l,\mu)$, $E_{00}+E_{so-gqp}(k_2,a,b,c,n=0,l,\mu)$ and $E_{00}+E_{so-gqp}(k_3,a,b,c,n=0,l,\mu)$. For the first excited state $n=1$, we have $L_{n=1}^{2\eta}(2\beta r)=-2\beta r+2\eta+1$, after a straightforward calculations, the expressions of the 4-factors $T_i\left(i=\overline{1,4}\right)$ for $n=1$ in Eq. (30) are simplified to the form:

$$T_1(c,n=1,l,\mu)=\frac{c}{2}\int_0^{+\infty}\exp\left(-2\alpha r^2-2\beta r\right)\left(f_1r^{2\eta+1-1}+f_2r^{2\eta-1}+f_3r^{2\eta-1-1}\right)dr$$
$$T_2(b,n=1,l,\mu)=-\frac{b}{2}\int_0^{+\infty}\exp\left(-2\alpha r^2-2\beta r\right)\left(f_1r^{2\eta+3-1}+f_2r^{2\eta+2-1}+f_3r^{2\eta+1-1}\right)dr$$
$$T_3(a,n=1,l,\mu)=-a\int_0^{+\infty}\exp\left(-2\alpha r^2-2\beta r\right)\left(f_1r^{2\eta+4-1}+f_2r^{2\eta+3-1}+f_3r^{2\eta+2-1}\right)dr$$
$$T_4(l,n=1,\mu)=\int_0^{+\infty}r\exp\left(-2\alpha r^2-2\beta r\right)\left(f_1r^{2\eta+4-1}+f_2r^{2\eta+3-1}+f_3r^{2\eta+2-1}\right)dr \tag{35}$$

Where $f_1 = 4\beta^2$, $f_2 = -4\beta(2\eta+1)$ and $f_3 = 4\eta^2 + 4\eta + 1$ Evaluating the integral in Eq. (35), applies the special integration, which given by Eq. (32), we obtain the results:

$$\begin{aligned}
T_1(c,n=1,l,\mu) &= \frac{c}{2}\exp\left(\frac{\beta^2}{4\alpha}\right)(f_1(4\alpha)^{-\frac{2l+3}{2}}\Gamma(2l+3)D_{-(2l+3)}\left(\frac{\beta}{\sqrt{\alpha}}\right)+f_2(4\alpha)^{-l-1}\Gamma(2l+2)D_{-(2l+2)}\left(\frac{\beta}{\sqrt{\alpha}}\right)\\
&+f_3(4\alpha)^{-\frac{2l+1}{2}}\Gamma(2l+1)D_{-(2l+1)}\left(\frac{\beta}{\sqrt{\alpha}}\right))\\
T_2(b,n=1,l,\mu) &= -\frac{b}{2}\exp\left(\frac{\beta^2}{4\alpha}\right)(f_1(4\alpha)^{-\frac{2l+5}{2}}\Gamma(2l+5)D_{-(2l+5)}\left(\frac{\beta}{\sqrt{\alpha}}\right)+f_2(4\alpha)^{-\frac{2l+4}{2}}\Gamma(2l+4)D_{-(2l+4)}\left(\frac{\beta}{\sqrt{\alpha}}\right)\\
&+f_3(4\alpha)^{-\frac{2l+3}{2}}\Gamma(2l+3)D_{-(2l+3)}\left(\frac{\beta}{\sqrt{\alpha}}\right))\\
T_3(a,n=1,l,\mu) &= -a\exp\left(\frac{\beta^2}{4\alpha}\right)(f_1(4\alpha)^{-\frac{2l+6}{2}}\Gamma(2l+6)D_{-(2l+6)}\left(\frac{\beta}{\sqrt{\alpha}}\right)+f_2(4\alpha)^{-\frac{2l+5}{2}}\Gamma(2l+5)D_{-(2l+5)}\left(\frac{\beta}{\sqrt{\alpha}}\right)\\
&+f_3(4\alpha)^{-\frac{2l+4}{2}}\Gamma(2l+4)D_{-(2l+4)}\left(\frac{\beta}{\sqrt{\alpha}}\right)) = -aT_4(l,n=1,\mu)
\end{aligned} \tag{36}$$

Allow us the two to obtain the exact modifications $E_{so-gqp}(k_1,a,b,c,n=1,l)$, $E_{so-mqp}(k_2,a,b,c,n=1,l)$ and $E_{so-lqp}(k_3,a,b,c,n=1,l)$ of the first excited state under the modified quark-antiquark interaction potential:

$$\begin{aligned}
E_{so-gqp}(k_1,a,b,c,n=1,l,\mu) &= 4\gamma C_{1l}^{\,2}k_1(l)\left\{\Theta T_{11}(a,b,c,n=1,l,\mu)+\frac{\bar{\theta}}{2\mu}T_4(n=1,l,\mu)\right\}\\
E_{so-mqp}(k_2,a,b,c,n=1,l,\mu) &= 4\gamma C_{1l}^{\,2}k_2(l)\left\{\Theta T_{11}(a,b,c,n=1,l,\mu)+\frac{\bar{\theta}}{2\mu}T_4(n=1,l,\mu)\right\}\\
E_{so-lqp}(k_3,a,b,c,n=1,l,\mu) &= 4\gamma C_{1l}^{\,2}k_3(l)\left\{\Theta T_{11}(a,b,c,n=1,l,\mu)+\frac{\bar{\theta}}{2\mu}T_4(n=1,l,\mu)\right\}
\end{aligned} \tag{37}$$

With $T_{11}(a,b,c,n=1,l,\mu) = T_1(c,n=1,l,\mu)+T_2(b,n=1,l,\mu)+T_3(a,n=1,l,\mu)$. In addition, and in the same way we find the exact modifications $E_{so-gqp}(k_1,a,b,c,n,l)$, $E_{so-mqp}(k_2,a,b,c,n,l)$ and $E_{so-lqp}(k_3,a,b,c,n,l)$ for n^{th} excited states of heavy quarkonium systems under modified quark-antiquark interaction potential in global quantum group symmetry (NC: 3D-RSP):

$$\begin{aligned}
E_{so-gqp}(k_1,a,b,c,n,l,\mu) &= \gamma C_{nl}^{\,2}C_{nl}^{\,2}(-2)^{2n}\left(\frac{n!\Gamma(2l+3)}{\Gamma(2l+2+n)}\right)^2 k_1(l)\left\{\Theta T_{1n}(a,b,c,n,l,\mu)+\frac{\bar{\theta}}{2\mu}T_4(n,l,\mu)\right\}\\
E_{so-mqp}(k_2,a,b,c,n,l,\mu) &= \gamma C_{nl}^{\,2}C_{nl}^{\,2}(-2)^{2n}\left(\frac{n!\Gamma(2l+3)}{\Gamma(2l+2+n)}\right)^2 k_2(l)\left\{\Theta T_{1n}(a,b,c,n,l,\mu)+\frac{\bar{\theta}}{2\mu}T_4(n,l,\mu)\right\}\\
E_{so-lqp}(k_3,a,b,c,n,l,\mu) &= \gamma C_{nl}^{\,2}C_{nl}^{\,2}(-2)^{2n}\left(\frac{n!\Gamma(2l+3)}{\Gamma(2l+2+n)}\right)^2 k_3(l)\left\{\Theta T_{1n}(a,b,c,n,l,\mu)+\frac{\bar{\theta}}{2\mu}T_4(n,l,\mu)\right\}
\end{aligned} \tag{38}$$

With $T_{1n}(a,b,c,n,l,\mu) = T_1(c,n,l,\mu)+T_2(b,n,l,\mu)+T_3(a,n,l,\mu)$

The exact modified magnetic spectrum for heavy quarkonium systems under modified quark- antiquark interaction potential in pNRQCD

Further to the important previously obtained results, now, we consider another important physically meaningful phenomena produced by the effect of the modified quark-antiquark interaction potential in perturbative NRQCD related to the influence of an external uniform magnetic field $\vec{B}$. To avoid the repetition in the theoretical calculations, it is sufficient to apply the following replacements:

$$\begin{cases}\vec{\Theta}\to\chi\vec{B}\\ \vec{\bar{\theta}}\to\bar{\sigma}\vec{B}\end{cases} \Rightarrow \left(\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\vec{\Theta}\vec{L}+\frac{\vec{\bar{\theta}}\vec{L}}{2\mu}\right) \text{ will -be- replace- by: } \left(\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\chi+\frac{\bar{\sigma}}{2\mu}\right)\vec{B}\vec{L} \tag{39}$$

Here χ and $\bar{\sigma}$ are two infinitesimal real proportional constants, and we choose the arbitrary uniform external magnetic field $\vec{B}$ parallel to the (Oz) axis, which allow us to introduce the new modified magnetic Hamiltonian $H_{m-qp}(r,\chi,\bar{\sigma})$ in (NC: 3D-RSP) symmetries as:

$$H_{so\text{-}qp}(r,\Theta,\bar{\theta}) \to H_{m-qp}(r,\chi,\bar{\sigma}) = \left(\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\chi+\frac{\bar{\sigma}}{2\mu}\right)\left\{\vec{B}\vec{J}-\aleph_z\right\} \tag{40}$$

Here $\aleph_z \equiv -\vec{S}\vec{B}$ denote to Zeeman effect in CQM, while $\aleph_{\text{mod}-z} \equiv \vec{B}\vec{J}-\aleph_z$ is the new Zeeman effect in NCQM. To obtain the exact NC magnetic modifications of energy for ground state, first excited state and n^{th} excited states of heavy quarkonium systems $E_{mag-qp}(m=0,a,b,c,n=0,l,\mu)$, $E_{mag-qp}(m=0,\pm1,a,b,c,n=1,l,\mu)$ and $E_{mag-cp}(m=\overline{-l,+l},a,b,c,n,l,\mu)$, we just replace $k_1(l)$ and Θ in the Eqs. (34), (37) and (38) by the following parameters m and χ, respectively:

$$\begin{aligned}
&E_{mag-qp}(m=0,a,b,c,n=0,l,\mu)=0\\
&E_{mag-qp}(m=0,\pm1,a,b,c,n=1,l,\mu)=4\gamma C_{1l}^{\,2}\left\{\chi T_{11}(n=1,,b,c,n,l,\mu)+\frac{\bar{\sigma}}{2\mu}T_4(n,l,\mu)\right\}Bm\\
&E_{mag-qp}(m=\overline{-l,+l},a,b,c,n,l,\mu)=\gamma C_{nl}^{\,2}(-2)^{2n}\left(\frac{n!\Gamma(2l+3)}{\Gamma(2l+2+n)}\right)^2\left\{\chi T_{1n}(n,a,b,c,n,l,\mu)+\frac{\bar{\sigma}}{2\mu}T_4(n,l,\mu)\right\}Bm
\end{aligned} \tag{41}$$

We have $-l \le m \le +l$, which allow us to fixing $(2l+1)$ values for discreet number m. It should be noted that the results obtained in Eq. (41) could find it by direct calculation $E_{mag-qp} = \langle\Psi(r,\theta,\phi)|H_{m-qp}(r,\chi,\bar{\sigma})|\Psi(r,\theta,\phi)\rangle$ that takes the following explicit relation:

$$\begin{aligned}
E_{mag-qp} &= \gamma C_{nl}^{\,2}(-2)^{2n}\left(\frac{n!\Gamma(2\eta+1)}{\Gamma(2\eta+n)}\right)^2 mB\\
&\int_0^{+\infty} r^{2\eta-1}\exp\left(-2\alpha r^2-2\beta r\right)\left(L_n^{2\eta}(2\beta r)\right)^2\left(\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\chi+\frac{\bar{\sigma}}{2\mu}\right)\left[Y_l^m(\theta,\phi)\right]^2 d\tau
\end{aligned} \tag{42}$$

Where $d\tau = r^2\sin(\theta)d\theta d\phi$, we have used the wave function that is in Eq. (10) and the perturbed magnetic Hamiltonian operator, Eq. (41) and the the orthogonality property of the spherical harmonics $\int Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)d\Omega = \delta_{ll'}\delta_{mm'}$ (with $d\Omega \equiv \sin(\theta)d\theta d\phi$), It is clear that the Eq. (42) can be rewritten as follows:

$$\begin{aligned}
E_{so-gqp}(k_1,a,b,c,n,l,\mu) &= \gamma C_{nl}^{\,2}(-2)^{2n}\left(\frac{n!\Gamma(2\eta+1)}{\Gamma(2\eta+n)}\right)^2\\
&\left\{\chi\left[T_1(n,c,l,\mu)+T_2(n,b,l,\mu)+T_3(n,a,l,\mu)\right]-\frac{\bar{\sigma}}{2\mu}T_4(n,l,\mu)\right\}mB
\end{aligned} \tag{43}$$

the expressions of the 4-factors $T_i(i=\overline{1,4})$ are given by Eq. (30). Then we find the magnetic specters of energy produced by the operator $H_{m-qp}(r,\chi,\bar{\sigma})$ for ground state and first excited states repeating the same calculations in the previous subsection. It should be noted that, the magnetic effects are very important in the case of elementary particles such as electrons [34].

Results

In the previous sub-sections, we obtained the solution of the modified Schrödinger equation for the modified quark-antiquark interaction potential, which is given in Eq. (22) by using the generalized Bopp's shift method and standard perturbation theory in pNRQCD. The energy eigenvalue is calculated in the 3D space-phase. The modified eigenenergies $(E_{\text{nc -gqp}}, E_{\text{nc -mqp}}, E_{\text{nc -lqp}})(n=0,m=0,a,b,c,l,\mu)$,

$\left(E_{\text{nc -}gqp}, E_{\text{nc -m}qp}, E_{\text{nc -l}qp}\right)\left(n=1,(m=0,\pm 1),a,b,c,l,\mu\right)$ and with spin $\vec{S}=\vec{1}$ for MSE for heavy quarkonium systems (the charmonium $c\bar{c}$, bottomonium $b\bar{b}$ and $b\bar{c}$ mesons) under modified quark-antiquark interaction potential are obtained in this paper on based to our original results presented on the Eqs. (34), (37), (38) and (41), in addition to the ordinary energy E_{nl} for the modified quark-antiquark interaction potential which presented in the Eq.(9):

$$E_{\text{nc -}gqp}\left(n=0,m=0,a,b,c,l,\mu\right)=E_{00}$$
$$E_{\text{nc -}mqp}\left(n=0,m=0,a,b,c,l,\mu\right)=E_{00}+2\gamma C_{00}{}^{2}\left\{\Theta T_{10}\left(n=0,a,b,c,l=0,\mu\right)+\frac{\bar{\theta}}{2\mu}T_{4}\left(n=0,l=0,\mu\right)\right\}$$
$$E_{\text{nc -}lqp}\left(n=0,m=0,a,b,c,l,\mu\right)=E_{00}+2\gamma C_{00}{}^{2}\left\{\Theta T_{10}\left(n=0,a,b,c,l=0,\mu\right)+\frac{\bar{\theta}}{2\mu}T_{4}\left(n=0,l=0,\mu\right)\right\} \tag{44}$$

$$E_{\text{nc -gqp}}\left(n=1,k_1,(m=0,\pm1),a,b,c,l,\mu\right)=E_{1l}+4\gamma C_{1l}{}^{2}\left\{\left(k_1(l=1)\Theta+\chi Bm\right)T_{11}\left(n=1,a,b,c,l,\mu\right)+\left(\frac{\bar{\theta}}{2\mu}k_1(l=1)+\frac{\bar{\sigma}}{2\mu}Bm\right)T_4\left(n=1,l,\mu\right)\right\}$$
$$E_{\text{nc -mqp}}\left(n=1,k_2,(m=0,\pm1),a,b,c,l,\mu\right)=E_{1l}+4\gamma C_{1l}{}^{2}\left\{\left(k_2(l=1)\Theta+\chi Bm\right)T_{11}\left(n=1,a,b,c,l,\mu\right)+\left(\frac{\bar{\theta}}{2\mu}k_2(l=1)+\frac{\bar{\sigma}}{2\mu}Bm\right)T_4\left(n=1,l,\mu\right)\right\}$$
$$E_{\text{nc -lqp}}\left(n=1,k3,(m=0,\pm1),a,b,c,l,\mu\right)=E_{1l}+4\gamma C_{1l}{}^{2}\left\{\left(k_3(l=1)\Theta+\chi Bm\right)T_{11}\left(n=1,a,b,c,l,\mu\right)+\left(\frac{\bar{\theta}}{2\mu}k_3(l=1)+\frac{\bar{\sigma}}{2\mu}Bm\right)T_4\left(n=1,l,\mu\right)\right\} \tag{45}$$

$$E_{\text{nc -gcqp}}\left(n,k_1,m=\overline{-l,+l},a,b,c,l,\mu\right)=E_{nl}+\gamma C_{nl}{}^{2}(-2)^{2n}\left(\frac{n!\Gamma(2l+3)}{\Gamma(2l+2+n)}\right)^{2}\left\{\left(k_1(l)\Theta+\chi Bm\right)T_{1n}\left(n,a,b,c,l,\mu\right)+\left(\frac{\bar{\theta}}{2\mu}k_1(l)+\frac{\bar{\sigma}}{2\mu}Bm\right)T_4\left(n,l,\mu\right)\right\}$$
$$E_{\text{nc -mqp}}\left(n,k_2,m=\overline{-l,+l},a,b,c,l,\mu\right)=E_{nl}+\gamma C_{nl}{}^{2}(-2)^{2n}\left(\frac{n!\Gamma(2l+3)}{\Gamma(2l+2+n)}\right)^{2}\left\{\left(k_2(l)\Theta+\chi Bm\right)T_{1n}\left(n,ba,b,c,l,\mu\right)+\left(\frac{\bar{\theta}}{2\mu}k_2(l)+\frac{\bar{\sigma}}{2\mu}Bm\right)T_4\left(n,l,\mu\right)\right\}$$
$$E_{\text{nc -lqp}}\left(n,k_3,m=\overline{-l,+l},a,b,c,l,\mu\right)=E_{nl}+\gamma C_{nl}{}^{2}(-2)^{2n}\left(\frac{n!\Gamma(2l+3)}{\Gamma(2l+2+n)}\right)^{2}\left\{\left(k_3(l)\Theta+\chi Bm\right)T_{1n}\left(n,a,b,c,l,\mu\right)+\left(\frac{\bar{\theta}}{2\mu}k_3(l)+\frac{\bar{\sigma}}{2\mu}Bm\right)T_4\left(n,l,\mu\right)\right\} \tag{46}$$

Where E_{00} and E_{1l} are the energy of ground state and first excited state of heavy quarkonium systems in the symmetries of quantum mechanics under the quark-antiquark interaction potential:

$$E_{00}=3\sqrt{\frac{a}{2\mu}}-\frac{2b^2}{8a} \quad \text{and} \quad E_{1l}=\sqrt{\frac{a}{2\mu}}(2l+5)+\frac{(2-2l)b^2}{8(l+1)a} \tag{47}$$

This is one of the main objectives of our research and by noting that, the obtained eigenvalues of energies are real's and

$$H_{qp}\left(x_\mu,p_\mu\right)\rightarrow H_{nc-qp}\left(x_\mu,p_\mu\right)\equiv\begin{pmatrix}\left(H_{nc-qp}\right)_{11} & 0 & 0\\ 0 & \left(H_{nc-qp}\right)_{22} & 0\\ 0 & 0 & \left(H_{nc-qp}\right)_{33}\end{pmatrix} \tag{48}$$

Where $\left(H_{nc-qp}\right)_{11}=-\frac{\Delta_{nc}}{2\mu}+H_{\text{int}-gqp}$,, $\left(H_{nc-qp}\right)_{22}=-\frac{\Delta_{nc}}{2\mu}+H_{\text{int}-mq}$ and $\left(H_{nc-qp}\right)_{33}=-\frac{\Delta_{nc}}{2\mu}+H_{\text{int}-lqp}$ with $\frac{\Delta_{nc}}{2\mu}=\frac{\Delta-\vec{\bar{\theta}}\vec{L}-\vec{\bar{\sigma}}\vec{L}}{2\mu}$, and the three modified interactions elements $\left(H_{\text{int}-gqp},H_{\text{int}-mcp},H_{\text{int}-lqp}\right)$ are given by:

$$V_{qp}(r)\rightarrow\begin{cases}H_{\text{int}-gqp}=ar^2+br-\frac{c}{r}+\gamma\left(k_1(l)\Theta+\chi\aleph_{\text{mod}-z}\right)\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\\ H_{\text{int}-mqp}=ar^2+br-\frac{c}{r}+\gamma\left(k_2(l)\Theta+\chi\aleph_{\text{mod}-z}\right)\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\\ H_{\text{int}-lqp}=ar^2+br-\frac{c}{r}+\gamma\left(k_3(l)\Theta+\chi\aleph_{\text{mod}-z}\right)\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\end{cases} \tag{49}$$

Thus, the ordinary kinetic term for the quark-antiquark interaction potential ($-\frac{\Delta}{2\mu}$) and ordinary interaction ($ar^2+br-\frac{c}{r}$) are replaced by new modified form of kinetic term $\frac{\Delta_{nc}}{2\mu}$ and new modified interactions ($H_{\text{int}-gqp}$, $H_{\text{int}-mqp}$ and $H_{\text{int}-lqp}$) respectively, in (NC-3D: RSP) symmetries. On the other hand, it is evident to consider the quantum number m takes $(2l+1)$ values and we have also three values for $(j=l\pm1,l)$, thus every state in usually three-dimensional space of energy for heavy quarkonium systems under modified quark-antiquark interaction potential will be $3\left((2l+1)\right)$ sub-states. To obtain the total complete degeneracy of energy level of the modified quark-antiquark interaction potential in (NC-3D: RSP) symmetries, we need to sum for all allowed values of l. Total degeneracy is thus,

$$\sum_{i=0}^{n-1}(2l+1)=n^2\rightarrow 3\left(\sum_{i=0}^{n-1}(2l+1)\right)\equiv 3n^2 \tag{50}$$

Note that the obtained new energy eigenvalues $\left(E_{\text{nc -}gqp},E_{\text{nc -m}qp},E_{\text{nc -l}qp}\right)\left(n,\left(m=\overline{-l,+l}\right),a,b,c,l,\mu\right)$ now depend to new discrete atomic quantum numbers (n,j,l,s) and m in addition to the parameters (a,b,c) of the modified quark-antiquark interaction potential. It is pertinent to note that when the atoms have $\vec{S}=\vec{0}$, the total operator can be obtains from the interval $|l-s|\leq j\leq|l+s|$, which allow us to obtaining the eigenvalues of the operator $\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right)$ as $k(j,l,s)\equiv 0$ and then the non-relativistic energy spectrum $\left(E_{\text{nc -}gqp},E_{\text{nc -m}qp},E_{\text{nc -l}qp}\right)\left(n,\left(m=\overline{-l,+l}\right),a,b,c,l,\mu\right)$ reads:

$$\left(E_{\text{nc -gcq}},E_{\text{nc -mcq}},E_{\text{nc -lcq}}\right)\left(n,\left(m=\overline{-l,+l}\right),a,b,c,\mu\right)=E_{nl}-\gamma C_{nl}{}^{2}\left\{\chi T_{1n}\left(n,l,\mu,m,a,b,c\right)-\frac{\bar{\sigma}}{2}T_4\left(n,l,\mu\right)\right\}Bm \tag{51}$$

One of the most important applications, in the extended model of pNRQCD, is to calculate the modified mass spectra of the heavy quarkonium systems (the mass of the quarkonium bound state), such as charmonium and bottomonium mesons, that have the quark and antiquark flavor in the symmetries of NCQM under modified quark-antiquark interaction potential. In order to achieve this goal, we generalize the traditional formula $M=2m_q+E_{nl}$ to the new form

$$M=2m_q+E_{nl}\rightarrow M_{nc-cp}=2m+\frac{1}{3}\left(E_{\text{nc -gqp}}+E_{\text{nc -mqp}}+E_{\text{nc -lqp}}\right)\left(n,\left(m=\overline{-l,+l}\right),a,b,c,l,\mu\right) \tag{52}$$

Here m_q is bare mass of quarkonium or twice the reduced mass of the system. Moreover, $\frac{1}{3}\left(E_{\text{nc -gqp}}+E_{\text{nc -mqp}}+E_{\text{nc -lqp}}\right)\left(n,\left(m=\overline{-l,+l}\right),a,b,c,l,\mu\right)$ is the non-polarized energies (energy independent of spin), which can determine from Eqs. (46) and (51). Thus, the modified mass of quarkonium system M_{nc-qp} can be according to the following new result:

$$M_{nc-cp}\left(n,a,b,c,l,m,\mu\right)=M\left(n,a,b,c,l,\mu\right)+\gamma C_{nl}{}^{2}(-2)^{2n}\left(\frac{n!\Gamma(2l+3)}{\Gamma(2l+2+n)}\right)^{2}\begin{cases}M_1 & \text{for} \quad \vec{S}=\vec{1}\\ M_2 & \text{for} \quad \vec{S}=\vec{0}\end{cases} \tag{53}$$

With $M_2\left(\vec{S}=\vec{0}\right)$ and $M_2\left(\vec{S}=\vec{0}\right)$ are given by:

$$M_1\left(\vec{S}=\vec{1}\right)=\left(\chi Bm+\frac{l+4}{6}\Theta+\right)T_{1n}\left(n,a,b,c,l,\mu\right)-\left(\frac{\bar{\sigma}}{2\mu}Bm+(l+4)\frac{\bar{\theta}}{12\mu}\right)T_4\left(n,l,\mu\right)$$
$$M_2\left(\vec{S}=\vec{0}\right)=\left\{\chi T_{1n}\left(n,a,b,c,l,\mu\right)+\frac{\bar{\sigma}}{2}T_4\left(n,l,\mu\right)\right\}Bm \tag{54}$$

Here $M(n,a,b,c,l,\mu)$ is the heavy quarkonium systems under the quark-antiquark interaction potential in CQM, which defined in [1]. The obtained modified mass of quarkonium system M_{nc-qp} equal the sum of corresponding value M in CQM and two perturbative terms proportional with two parameters ((Θ or χ) and ($\bar{\theta}$ or $\bar{\sigma}$)). If we consider $(\Theta \text{ or } \chi, \bar{\theta} \text{ or } \bar{\sigma}) \to (0,0)$, we recover the results of commutative space of ref. [1] obtained for the quark-antiquark interaction potential, which means that our calculations are correct.

Conclusion

In the present work, the 3-dimensional modified Schrodinger equation is analytically solved using the generalized Bopp's shift method and time independent standard perturbation theory. The quark-antiquark interaction potential is extended to include effect of non-commutativity space phase; we resume the main obtained results:

- Ordinary quark-antiquark interaction potential ($ar^2 + br - \frac{c}{r}$) were replaced by new modified interactions ($H_{\text{int}-gqp}$, $H_{\text{int}-mqp}$ and $H_{\text{int}-lqp}$) for heavy quarkonium systems,

- The ordinary kinetic term $-\frac{\Delta}{2\mu}$ modified to the new form $\frac{\Delta}{2\mu} \to \frac{\Delta_{nc}}{2\mu} = \frac{\Delta - \vec{\bar{\theta}}\vec{L} - \vec{\bar{\sigma}}\vec{L}}{2\mu}$ for heavy quarkonium systems under influence of the modified quark-antiquark interaction potential,

- We obtained the perturbative corrections (($E_{\text{nc -}gqp}, E_{\text{nc -m}qp}, E_{\text{nc -l}qp})(n=0, m=0, a, b, c, l, \mu)$, $(E_{\text{nc -}gqp}, E_{\text{nc -m}qp}, E_{\text{nc -l}qp})(n=1, (m=0, \pm 1), a, b, c, l, \mu)$ and) for ground state, first excited state and n^{th} excited state with (spin $\vec{S} = \vec{1}$ and $\vec{S} = \vec{0}$) for heavy quarkonium systems under influence of the modified quark-antiquark interaction potential are obtained.

- We have obtained the modified mass of quarkonium system $M_{nc-cp}(n, a, b, c, l, m, \mu)$ which equal the sum of corresponding value M in CQM and two perturbative terms proportional with two parameters ((Θ or $\bar{\theta}$) and ($\bar{\theta}$ or $\bar{\sigma}$)).

Through the of high value results, which we have achieved in present work, we hope to extend our recently work physics for further investigations of particles physics and other characteristics of quarkonium among others in the context of NRQCD theory. Finally, we can say that we have established our new theoretical model to describe the charmonium $c\bar{c}$, bottomonium $b\bar{b}$ and $b\bar{c}$ mesons in the symmetries of NCQM under modified quark-antiquark interaction potential, this is the main objective of this work.

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