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A New Approximation to Standard Normal Distribution Function

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Abstract

This paper, presents three news-improved approximations to the Cumulative Distribution Function (C.D.F.). The first approximation improves the accuracy of approximation given by Polya (1945). In this first new approximation, we reduce the maximum absolute error (MAE) from0.000314 to 0.00103. For this first new approximation, Aludaat and Alodat were reduce the (MAE) from 0.000314 to 0.001972. The second new approximation improve Tocher's approximation, we reduce the (MAE) from, 0.166 to 0.00577. For the third new approximation, we combined the two previous approximations. Hence, this combined approximation is more accurate and its inverse is hard to calculate. This third approximation reduces the (MAE) to be less than 2.232e-004. The two improved previous approximations are less accurate, but his inverse is easy to calculate. Finally, we give an application to the third approximation for pricing a European Call using Black-Scholes Model.

Keywords: Cumulative distribution function; Normal distribution; Maximum absolute error

Introduction

The cumulative distribution function (CDF) plays an important role in financial mathematics and especially in pricing options with Black-Scholes Model. The European option pricing call given by Black-Scholes Model is

$$C = SN(d) - Ke^{-rT}N\left(d - \sigma\sqrt{T}\right) \tag{1}$$

Where
$$d = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{2}$$

S, the current price, K the exercise price, r interest rate, T time option and σ volatility [1-8]. The cumulative distribution function (CDF) is

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{Z} e^{-\frac{t^2}{2}}dt \tag{3}$$

The (CDF) has not a closed form. His evaluation is an expensive task. For evaluate the (CDF) at a point z we need compute the integral under the probability density function (PDF) given by $\varphi(t) = e^{-0.5t^2} / \sqrt{2\pi}$.

In much research, we find approximations, with closed forms, for the area under the standard normal curve. Otherwise, we need consulting Tables of cumulative standard normal probabilities. Hence, in the literature, we find several approximations to this function from Polya [7] to Yerukala et al. [9]. For this raison, we use some approximations to this CDF. (Polya's approximation and Tocher's approximation) [10-11].

Improving Polya's Approximation

We consider the case of $z \geq 0$. (For $z \geq 0$, $N(z) = 1 - N(-z)$).

The original Polya's approximation given by:

$$N_{polya}(z) \approx \frac{1}{2}\left\{1 + \sqrt{1 - e^{-az^2}}\right\}, \text{ where } a = \frac{2}{\pi}. \tag{2.1}$$

The Maximum absolute Error (MAE)

$$\text{M.A.E.}_{Polya} = \max_z \left|\text{N}_{Polya}(z) - N(z)\right| = 0.003138181653387 \ . \tag{2.2}$$

Aludaat K.M and Alodat M.T [1] proposed the same formula with $a = \sqrt{\frac{\pi}{8}}$ instead of $a = \frac{2}{\pi}$. They have

$$\text{M.A.E.}_{Aludaat} = \max_z \left|\text{N}_{Aludaat}(z) - N(z)\right| = 0.001971820656170 \ .$$

In this paper, we write the formula (2.1) and (2.2) as

$$N_{Malki}(z) \approx \text{a} + \text{b}\sqrt{1 - e^{-cz^2}} \tag{2.3}$$

Hence, we $N_{Malki}(z) \approx a + b\sqrt{1 - e^{-cz^2}}$ search the parameters a,b and c that

$$\text{M.A.E.}_{Malki} = \max_z \left|\text{N}_{Malki}(z) - N(z)\right| \tag{2.4}$$

Was the smallest possible using the following algorithm?

$h = 0.00001; H = 20h; Er = 0.00314;$

$a_0 = 0.5, b_0 = 0.5, c_0 = \frac{2}{\pi},$

$for\ a = a_0 - H : h : a_0 + H\ for\ b = b_0 - H : h : b_0 + H$

$for\ c = c_0 - H : h : c_0 + H;\ \ M = a + b\sqrt{1 - e^{-cz^2}}$,

$e = \max_z \left|\text{M} - N(z)\right|;\ if\,(Er > e)\, Er = e; A = a; B = b; C = c; end;$

$a_0 = A, b_0 = B, c_0 = C.$

Repeat 3) to 6) until convergence

Using our algorithm, we find the best parameters

$$\text{a}^* = 0.50103; \text{b}^* = 0.49794; \text{c}^* = 0.62632 \tag{2.4}$$

Hence the best formula is

$$\boldsymbol{N}_{\boldsymbol{Malki}1}(z) = 0.50103 + 0.49794\sqrt{1 - \boldsymbol{e}^{-0.62632z^2}}, \tag{2.5}$$

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Note that the absolute error as function of z variable noted by ($|z| \geq 5$, $\max_z E(z) \leq 4.46e-009$),

shows the graph of Absolute Error for Polya, Aludaat and Malki as function of -5 ≤ z ≤ -5, ($|z| \geq 5$, $\max_z E(z) \leq 4.46e-009$) (Figure 1)

Improving Tocher's Approximation

The Original Tocher approximation is

$$N_{Tocher}(z) = 1/\left(1+e^{-\sqrt{\frac{2}{\pi}}z}\right) \text{ with}$$

$$\max_z \left|N_{Tocher}(z) - \mathrm{N}(z)\right| = 0.165811983691380 \approx 0.166. \quad (3.1)$$

These approximations have the form (3.2)

Hence, we search the parameters a, b and c $N_{Malki2}(z) = \frac{\mathrm{a}}{\mathrm{b}+e^{-cz}}$ that

$$\mathrm{M.A.E.}_{Malki2} = \max_z \left|\mathrm{N}_{Malki2}(z) - N(z)\right| \quad (3.3)$$

Was the smallest possible using the following algorithm?

$h = 0.00001; H = 20h;\ Er = 0.166;$

$a_0 = 1, b_0 = 1, c_0 = \sqrt{\frac{2}{\pi}}$

$for\, a = a_0 - H : h : a_0 + H\ for\, b = b_0 - H : h : b_0 + H$

$for\, c = c_0 - H : h : c_0 + H; M = \frac{a}{b+e^{-cz}},$

$e = \max_z \left|\mathrm{M} - N(z)\right|; if\,(Er > e)\, Er = e; A = a; B = b; C = c; end;$

$a_0 = A, b_0 = B, c_0 = C$

Repeat (3) to(6) until convergence

Using our algorithm, we find the best parameters

a^*=0.97186; b^*=0.97186; c^*=1.69075 (3.4)

Hence the best-improved formula for Tocher's approximation is

$$N_{Malki2}(z) = \frac{0.97186}{0.96628 + e^{-1.69075z}} \quad (3.5)$$

$$\max_z \left|N_{Malki2}(z) - N(z)\right| = 0.005774676414954 \quad (3.6)$$

Comparison of Absolute Error for Original Tocher, Modified Tocher and Malki 2 as function of z variable (-5 ≤ z ≤ 5) (Figure 2).

Combined Formula

As the third new approximation formula, we consider the two previous formula

$$N_{Malki1}(z) = 0.50103 + 0.49794\sqrt{1 - e^{-0.62632z^2}} \quad \text{and}$$

$$N_{Malkii2}(\mathrm{z}) = \frac{0.97186}{0.96628 + e^{-1.69075z^*}}$$

Hence, we consider the third new formula as

$$N_{Malki3}(z) = \omega N_{Malki1}(z) + (1-\omega) N_{Malki2}(z), for, (0 \leq \omega \leq 1) \quad (4.1)$$

We search the optimum parameter ω that the

$$\mathrm{M.A.E.}_{Malki3} = \max_z \left|\mathrm{N}_{Malki3}(z) - N(z)\right|$$

Was the smallest possible. We find optimum parameter $\omega^* = 0.16$

The new third approximation is

$$N_{Malki}(\mathrm{z}) = 0.16 N_{Malki}(\mathrm{z}) + 0.84\, \mathrm{N}_{Malki2}(\mathrm{z}) \quad (4.2)$$

The adjusted formula is

$$N_{Malki3}(z) = \frac{0.1544976}{0.96568 + e^{-1.68975z}} + 0.4212652 + 0.4189696\sqrt{1 - e^{-0.62642z^2}} \quad (4.3)$$

For, this approximation we have:

$$\max_z \left|\mathrm{N}_{Malki3}(z) - N(z)\right| = 2.231943559627414\mathrm{e}-004 \quad (4.4)$$

(Figure 3).

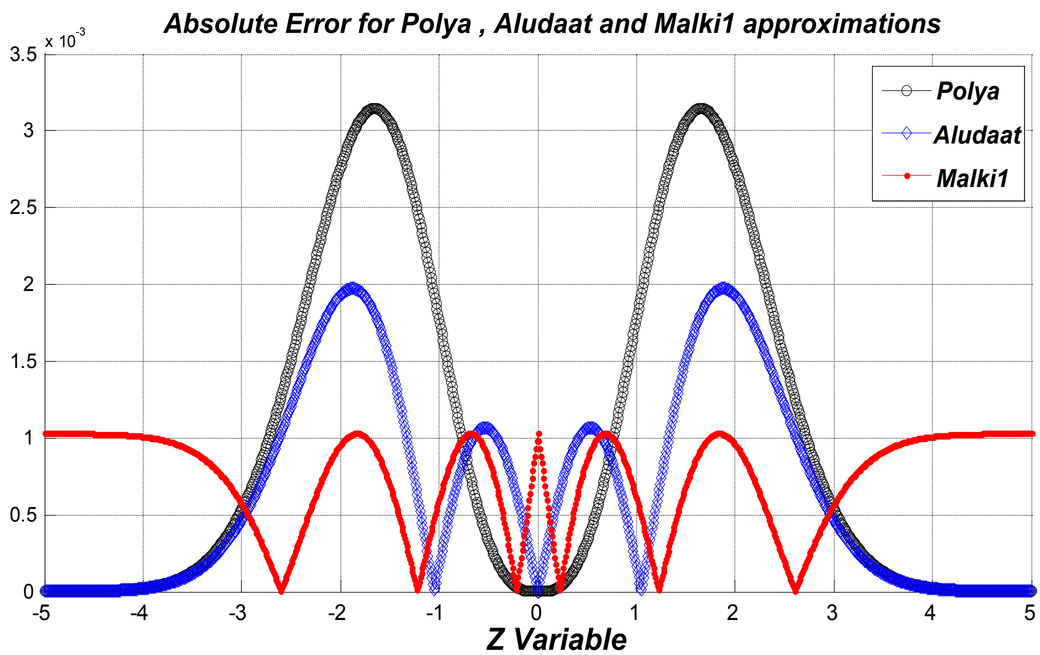

Figure 1: Comparison of absolute error for Polya, Aludaat and Malki1.

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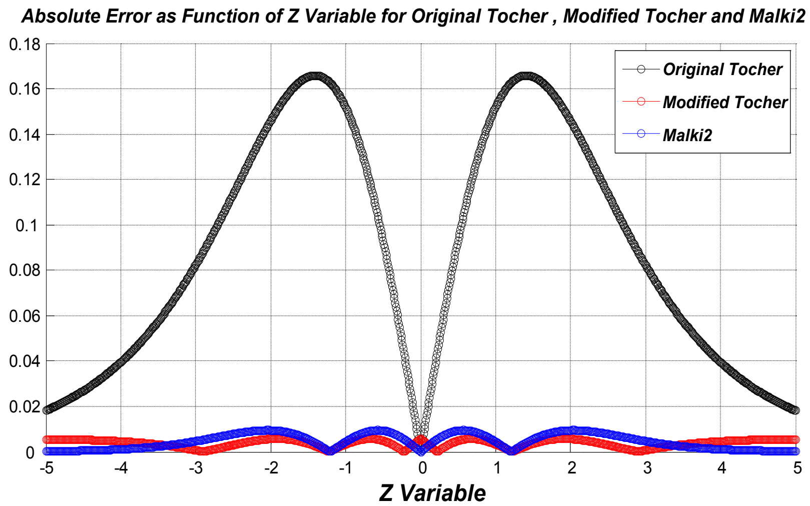

Figure 2: Gives the curves of original absolute error and the new absolute error.

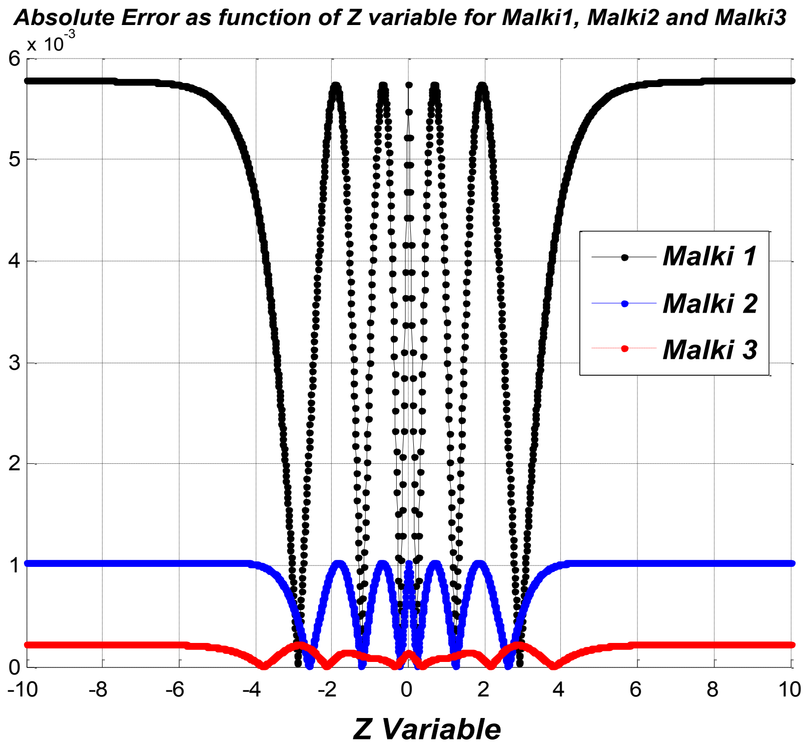

Figure 3: Absolute error as function of Z variable for Malki1, Malki2 and Malki3.

Application with Black-Scholes Model

For

$$S = 35; K = 30; r = 0.065; T = 1.2; \sigma = 0.35; \quad (5.1)$$

To calculate a Call European option we compute

$$d = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = 0.797198914562755 \quad (5.2)$$

And $d - \sigma\sqrt{T} = 0.413793124309139$ (5.3)

Hence

$$C = SN(d) - Ke^{-rT}N\left(d - \sigma\sqrt{T}\right) = 9.228813813962439 \quad (5.4)$$

Using, N_{Malki3} we have

$$C_3 = SN_{Malki3}(d) - Ke^{-rT}N_{Malki3}\left(d - \sigma\sqrt{T}\right) = 9.231095739041432 \quad (5.5)$$

The absolute error is $|C - C_3| \le 0.0023$ (5.6)

Conclusion

We have proposed three approximations to the cumulative distribution function of the standard normal distribution. The first approximation improves the Polya's formula in accuracy. The second new approximation improve the accuracy of Tocher's formula. The third formula is a combination of the two previous formula. The MAE for the first approximation is 0.00103. The MAE for the second approximation is 0.00577. For the third approximation the MAE is less than 2.232e−004. Finally, we insert an application to option pricing of a Call European option based on Black-Scholes formula.

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