A Mathematical Model of Glucose-Insulin Interaction with Time Delay

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Abstract

This paper extends the work by incorporating a time delay in the interaction term of the glucose and insulin. The new model is analyzed and the stability properties have been concluded. Numerical simulations are carried out to confirm the theoretical results.

Keywords: Glucose-Insulin; Stability; Diabetes; Equilibrium point; Asymptotically

Introduction

Body cells need energy to function, and glucose is the main source of energy for the cells in the living organisms [1]. Glucose comes from carbohydrate food after a meal intake. Therefore, the glucose level increases after food intake [2-4]. Here comes the role of the Insulin, which is a hormone that is produced by the Pancreas for the purpose of regulating the Glucose concentration in the blood [5-8]. The main function of the Insulin is to help the body cells to absorb the free blood glucose [9-11]. After its release by the Pancreas, the Insulin attaches with the cells and signals them to absorb the sugar from the blood stream [12-14].

Generally, two types of failure in controlling the level of sugar in the blood can occur. The first failure type happens when the Pancreas fails to produce enough amount of insulin [15]. This failure type is referred to as juvenile diabetes. The second failure type happens when the body cells fail to respond to insulin in a proper way [16]. The diabetes mellitus includes diabetes of types 1 and 2 [16,17].

According to the International Diabetes Federation (IDF), 387 million people have diabetes in 2014, and this number will rise to 592 million by 2035. The number of people with diabetes of type 2 is increasing in every country of the world. Moreover, Diabetes caused 4.9 million deaths in 2014; every seven seconds a person dies from diabetes (http://www.idf.org/diabetesatlas/update-2014).

The modeling of the glucose-insulin system has become an interesting topic, where, many mathematical models were developed to understand and predict the Glucose-Insulin dynamics [1,9,11,12,17]. Among the most widely used models to study diabetes dynamics, is the minimal model which is used in the interpretation of the intravenous glucose tolerance test [5]. In this paper, we extend the work [9] by considering a time delay. We propose the following general model for the interaction of glucose and insulin

\[
\begin{align*}
\frac{dx}{dt} &= -a_1x(t) - a_2x(t-\tau)y(t-\tau) + a_3, \quad t \in [0, T], \\
\frac{dy}{dt} &= b_1x(t) - b_2y(t), \quad t \in [0, T].
\end{align*}
\]

(1)

with initial data:

\[
\begin{align*}
x(\theta) &= \phi(\theta), \theta < 0, \\
y(\theta) &= \psi(\theta), \theta < 0
\end{align*}
\]

(2)

where x ≥ 0, y ≥ 0, x represents glucose concentration, y represents insulin concentration, a1 is the rate constant which represents insulin-independent glucose disappearance, a2 is the rate constant which represents insulin-dependent glucose disappearance, a3, b1, and b2 are the glucose infusion rate and the rate constant which represents insulin production due to glucose stimulation, b3 is the rate constant which represents insulin degradation and the time delay τ represents the time taken by Pancreas to respond to the feedback of the glucose level.

The rest of this paper is organized as follows. We discuss the qualitative analysis of the model behavior. The description of the numerical method for solving the proposed model is introduced. Then, we show the behavior of the model for different values of the time delay. Finally, these conclusions are discussed.

Stability Analysis

In this we discuss the qualitative analysis of the model behavior. We shall now investigate the dynamics of the delay system eqn. (1). The steady state or equilibrium (fixed point) of the system eqn. (1) is one of which

\[
x(t)=x(t-\tau)=x(0)=x^*, \quad y(t)=y(t-\tau)=y(0)=y^* t
\]

and as a consequence all the time derivatives vanish identically. Hence substituting

\[
x(t) = x(t-\tau) \quad \text{and} \quad \dot{x} = 0,
\]

\[
y(t) = y(t-\tau) \quad \text{and} \quad \dot{y} = 0,
\]

in eqn. (1), it is easy to see that eqn. (1) has the equilibrium point \((x^*, y^*)\) as the following:

\[
x^* = -\frac{a_1b_2}{2a_3} + \frac{a_2b_1}{2a_3}, \quad y^* = -\frac{a_2b_3}{2a_3} + \frac{a_2b_1}{2a_3}.
\]

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The interior-equilibrium point \( E_1(x',y') \) exists unconditionally as \( x' \) and \( y' \) are always positive as all the parameters are considered positive.

To linearize the model about the equilibrium point \( E_1 \) of model eqn. (1), let \( u(t)\equiv x(t)-x' \), \( v(t)\equiv y(t)-y' \). Then we obtain the linearized model

\[
\frac{du}{dt} = -a_1u(t) - a_2y'u(t - \tau) - a_3x'v(t - \tau) - a_4u(t - \tau) - a_5x'x' + a_6v,
\]

\[dy \over dt \] \[= b_1u(t) - b_2v(t).
\]

(4)

After removing nonlinear terms, we obtain the linear vibrational system, by using equilibria conditions as

\[
\frac{du}{dt} = -a_1u(t) - a_2y'u(t - \tau) - a_3x'v(t - \tau),
\]

\[dy \over dt \] \[= b_1u(t) - b_2v(t).
\]

(5)

From the linearized model we obtain the characteristic equation

\[
\Delta(\lambda, \tau) = \lambda^2 + a_1\lambda + b_1 = 0,
\]

where \( a=a_1+b_1, b=b_2, c=a_2b_1x' + a_3b_2y' \) and \( d=a_4b_2c \).

For \( \tau=0 \), the characteristic eqn. (6) becomes

\[
\Delta(\lambda, 0) = \lambda^2 + (a+b)\lambda + c + d = 0.
\]

(7)

Now, sum of the roots=\(-(a+b)\) and product of the roots=\(c+d\). Thus, we can say that both the roots of eqn. (7) are real and negative or complex conjugate with negative real parts if and only if

\[a+b>0 \text{ and } c+d=0. \]

(8)

Hence, in the absence of time delay, the equilibrium point \( E_1 \) is locally asymptotically stable if and only if both conditions \( a+b>0 \) and \( c+d=0 \) hold simultaneously.

Following [9]: The interior-equilibrium point \( E_1 \) is locally asymptotically stable when \( \tau \) is small if

\[
(b_1, a_2x')^2 = 4\lambda_1(a_1, a_2y').
\]

(9)

Now for \( \tau \neq 0 \), the interior-equilibrium point \( E_1 \) is locally asymptotically stable if and only if both conditions \( a+b>0 \) and \( c+d=0 \) hold simultaneously.

Following [9]: The interior-equilibrium point \( E_1 \) is locally asymptotically stable when \( \tau \) is small if

\[
\omega = \sqrt{\left(a^2 - b^2 - 2d\right)^2 + 4\left(d^2 - c^2\right)^2} / \sqrt{2}.
\]

From the eqn.(11), it follows that if

\[a^2 - b^2 - 2d>0 \text{ and } d^2 - c^2>0
\]

then eqn. (10) does not have any real solutions. From eqn. (11), we see that there is a unique positive solution \( w_0 \) if

\[c^2 - d^2<0
\]

(13)

If \(d^2-c^2<0, b^2-a^2<2d>0, \) and

\[(b^2-a^2)^2=4(d^2-c^2)
\]

(14)

hold, then there are two positive solutions \( \omega_0 \).

To find the necessary and sufficient conditions for nonexistence of delay induced instability, we now use the following theorem.

Theorem 1.1

A set of necessary and sufficient conditions for \( E_1 \) to be asymptotically stable for all \( \tau \geq 0 \) is the following.

1. The real parts of all the roots of \( \Delta(\lambda, \tau) = 0 \) are negative [10].

For all real \( m \) and \( \tau \geq 0 \), \( \Delta(i\omega, \tau) 
eq 0 \), \( i = \sqrt{-1} \).

Theorem 1.2

If conditions eqn.(8) and eqn.(12) and Theorem 1.1 are satisfied, then the equilibrium \( E_1 \) is asymptotically stable for all \( \tau \geq \tau_0 \), and unstable for \( \tau>\tau_0 \). Furthermore, as \( \tau \) increases through \( \tau_0 \), \( E_1 \) bifurcates into small amplitude periodic solutions, where \( \tau_0=\tau_0^* \) as \( n=0 \).

Proof: For \( \tau=0 \), \( E_1 \) is asymptotically stable if condition eqn. (8) holds. Hence, by Butler’s lemma, \( E_1 \) remains stable for \( \tau<\tau_0 \). We now have to show that

\[
\frac{d\left(\text{Re}\lambda\right)}{d\tau}<0
\]

Thus this will signify that there exists at least one eigenvalue with positive real part for \( \tau>\tau_0 \). Moreover, the conditions of Hopf bifurcation [8] are then satisfied yielding the required periodic solution.

Substituting \( w_0 \) into eqn.(9) and solving for \( \tau \), we get

\[
\tau_0 = \frac{1}{w_0^2} \arctan \left( \frac{w_0(2a_0 - 2bd + b_0w_0^2)}{2\pi n(w_0^2 - w_0^2 + b_0w_0^2)} \right), \quad n = 0, 1, 2,...
\]

(15)

Substituting \( \omega_n \) into eqn.(9) and solving for \( \tau \), we obtain

\[
\tau_0^* = \frac{1}{w_0^2} \arctan \left( \frac{w_0(2a_0 - 2bd + b_0w_0^2)}{2\pi n(w_0^2 - w_0^2 + b_0w_0^2)} \right), \quad k = 0, 1, 2,...
\]

(16)

Differentiating eqn. (6) w.r.t. \( \tau \), we obtain

\[
\frac{d\lambda}{d\tau} = \frac{2\lambda + a + be^{-2\tau} - \tau(bh + c)e^{-\tau}}{-\lambda + a\lambda + d} + \frac{b}{\lambda(h + c)} - \frac{\tau}{\lambda}
\]

(17)

and by using

\[
e^{-\tau} = -\left( \frac{\lambda + a\lambda + d}{\lambda(h + c)} \right)
\]

we obtain

\[
\frac{d\left(\text{Re}\lambda\right)}{d\tau} = \text{sign}(\text{Re}\lambda) \left[ \frac{2\lambda + a + be^{-2\tau} - \tau(bh + c)e^{-\tau}}{\left(-\lambda + a\lambda + d\right) - \lambda^2} \right]
\]

(19)

It follows that

\[
\frac{d\left(\text{Re}\lambda\right)}{d\tau}|_{\tau=\tau_0^*, \omega_{n=0}} > 0
\]

Thus, the transversality condition holds, and hence, Hopf bifurcation occurs at \( \tau=\tau_0^* \). This completes the proof.

Theorem 1.3

Let \( \lambda \) be defined in eqn.(16). If in eqns. (8) and (14) are satisfied, then the equilibrium point \( E_1 \) is stable when and unstable when, for some
positive integer $m$. Thus, there are bifurcations at the equilibrium point $E_k$ when $\tau = \tau_k$, $k=0,1,2,...$

Proof: If conditions in eqn. (8) and (14) are satisfied, then to prove the theorem we need only to verify the transversality conditions [6,7].

Thus, the transferability conditions are satisfied.

**Description of the Numerical Method**

In this we follow the work [3,4] to design a positivity preserving numerical method for solving model.

Let $N$ be any positive integer, and $h=T/N$. Let $t_j = j \cdot h$, for $j=0,...,N$. Let $x^i = x(t_i)$ and $y^j = y(t_j)$. Then, model eqn. (1) can be approximated by the non-standard finite difference formula:

$$x^{i+1} - x^i = -a_1 x^i + a_3 y_{i+1}^j, \quad j = 0,...,N-1$$

$$y^{j+1} - y^j = b_1 x^i - b_3 y_{j+1}^j, \quad j = 0,...,N-1$$

from which we obtain the two relations:

$$x^{i+1} = x^i + h \left( a_1 - a_3 y_{i+1}^j \right) / (1 + a_1), \quad j = 0,...,N-1$$

(20)

$$y^{j+1} = y^j + h \left( b_1 - b_3 y_{j+1}^j \right) / (1 + b_1), \quad j = 0,...,N-1$$

(21)

where, $x^i$ approximates $x(t_i)$ and $y^j$ approximates $y(t_j)$.

Let

$$S = \frac{\tau}{h},$$

then $\tau = Sh$. Now, $x(t_i - \tau)$ can be approximated by $x(t_{i-j})$ for $j \geq S$ and $y(t_i - \tau)$ can be approximated by $y(t_{i-j})$ for $j \geq S$.

Then,

$$y(t_i - \tau) = \begin{cases} \phi(t_i - \tau) & j < S \\ y^{j+1} & j \geq S \end{cases}$$

(23)

**Numerical Simulations**

In this, section we give some numerical simulations supporting our theoretical predictions. This present some examples and numerical simulations to verify our theoretical results proved in the previous using matlab program.

We consider the system:

$$\frac{dx}{dt} = -0.1135x(t) - 1.4x(t-\tau)y(t-\tau) + 1.0023$$

$$\frac{dy}{dt} = 0.22x(t) - 0.2972y(t).$$

There is a positive equilibrium $(x^*,y^*)=(0.9302,0.6886)$.

**Case I:** For $\tau=0$. In this case, the numerical simulation (Figures 1 and 2) shows that both the glucose and insulin converge in finite time to their equilibrium values $x^*0.9302$ and $y^*0.6886$, respectively.

![Figure 1: Glucose-Insulin dynamics for $\tau=1.0738$.](image1)

![Figure 2: Phase plain of the Glucose-Insulin dynamics for $\tau=1.0738$.](image2)
Also in eqn. (9), i.e., the interior-equilibrium point \( E_1 \) is locally asymptotically stable when \( \tau = 0 \) since
\[
(b_1, a_1 x^*) - 0.9022 < 4b_2 (a_2, a_2 y^*) = 1.4259.
\]

Case II: For \( \tau \neq 0 \), by Theorem 1.2, there is a critical value \( \tau_c = 1.2738 \). The computer simulations (Figures 1-4), show that \( E_1 \) is asymptotically stable when \( \tau = 1.0738 < \tau_c = 1.2738 \). When \( \tau \) passes through the critical value \( \tau_c = 1.2738 \), \( E_1 \) loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from \( E_1 \). When \( \tau > \tau_c = 1.2738 \), \( E_1 \) is unstable (Figures 5 and 6).

**Conclusion**

Delay differential equations are an interesting form of differential equations, with many different applications, particularly in the biological and medical worlds. In this paper, a mathematical model has been proposed and analyzed to study the dynamics of glucose and insulin in the human body. Numerical simulations are carried out to demonstrate the results obtained. Appropriately determining the range of the delay based on physiology and clinical data is important in theoretical study. All the numerical results and graphs presented in the project were in agreement with those presented in the relevant corresponding papers. Our results reveal the conditions on the parameters so that the periodic solution exist surrounding the interior equilibrium. It shows that \( \tau_c \) is a critical value for the parameter \( \tau \). Furthermore, the direction of Hopf bifurcation and the stability of bifurcated periodic solutions are investigated. From the above results, we conclude that the model is physiologically consistent and may be a useful tool for further research on diabetes.

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