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A Generlization of the Eneström-Kakeya Theorem

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Abstract

In this paper we obtain a generalization of well-known result of Eneström -Kakeya concerning the bounds for the moduli of the zeros of polynomials with complex coefficients which improve upon some results due to A. Aziz and Q.G Mohammad and others.

Keywords: Polynomial; Zeros; Eneström–Kakeya Theorem

Introduction and Statement of Results

The following result known as Eneström - Kakeya theorem, is well known in the theory of distribution of zeros of polynomials was firstly proved by Eneström [1] and Kakeya [2].

Theorem 1.1: If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients such that

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0,$$

then all the zeros of $p(z)$ lie in $|z| \leq 1$. (1.1)

In literature [3-5] there exist several extensions of Eneström -Kakeya theorem. By using Schwartz lemma, Aziz and Mohammad [6] generalized Eneström -Kakeya theorem in a different way and proved:

Theorem 1.2: If $p(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real positive coefficients. If $t_1 > t_2 \geq 0$ can be found such that

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \quad r=1,2,\ldots,n+1; a_{-1} = a_{n+1} = 0,$$

then all the zeros of $p(z)$ lie in $|z| \leq t_1$. (1.2)

The following generalization of Theorem B is due to Rather et al [7].

Theorem 1.3: If $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}\alpha_j = \alpha_j$ and

$Im\alpha_j = \beta_j, j=0,1,2,\ldots\ldots,n$ If $t_1 > t_2 \geq 0$, can be found such that

$$t_1t_2\alpha_r + (t_1 - t_2)\alpha_{r-1} - \alpha_{r-2} \geq 0 \text{ for } r=1,2,3,\ldots..,k+1,$$

$$t_1t_2\alpha_r + (t_1 - t_2)\alpha_{r-1} - \alpha_{r-2} \leq 0 \text{ for } r=k+2,\ldots..,n+1,$$

and

$$t_1t_2\beta_r + (t_1 - t_2)\beta_{r-1} - \beta_{r-2} \geq 0 \text{ for } r=1,2,3,\ldots..,m+1,$$

$$t_1t_2\beta_r + (t_1 - t_2)\beta_{r-1} - \beta_{r-2} \leq 0 \text{ for } r=m+2,\ldots..,n+1,$$

$0 \leq k \leq n, 0 \leq m \leq n, \alpha_{-1} = \beta_{-1} = \alpha_{n+1} = \beta_{n+1} = 0, \alpha_n > 0$ then all the zeros of $P(z)$ lie in

$$|z| \leq \frac{t_1}{|a_n|}\left\{2t_1^{k-n}(\alpha_k + t_2\alpha_{k+1}) + 2t_1^{m-n}(\beta_m + t_2\beta_{m+1}) - (\alpha_n + \beta_n)\right\} \quad (1.3)$$

In this paper, as a generalization of Theorem (1.3) , We prove the following result.

Theorem1.4: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \geq 1$ with $\operatorname{Re}\alpha_j = \alpha_j$ and $Im\alpha_j = \beta_j, j=0,1,2,\ldots\ldots,n.$ If $t_1 \geq t_2 \geq 0, t_1 \neq 0$ can be found such that

$$t_1t_2\alpha_r + (t_1 - t_2)\alpha_{r-1} - \alpha_{r-2} \geq 0 \text{ for } r=1,2,3,\ldots..,k+1,$$

$$t_1t_2\alpha_r + (t_1 - t_2)\alpha_{r-1} - \alpha_{r-2} \leq 0 \text{ for } r=k+2,\ldots..,n+1$$

and

$$t_1t_2\beta_r + (t_1 - t_2)\beta_{r-1} - \beta_{r-2} \geq 0 \text{ for } r=1,2,3,\ldots..,m+1,$$

$$t_1t_2\beta_r + (t_1 - t_2)\beta_{r-1} - \beta_{r-2} \leq 0 \text{ for } r=m+2,\ldots..,n+1,$$

$0 \leq k \leq n-1, 0 \leq m-1 \leq n, \alpha_{-1} = \beta_{-1} = \alpha_{n+1} = \beta_{n+1} = 0, \alpha_n > 0$ then all the zeros of P(z) lie in

$$\left|z + \frac{\alpha_{n-1} - (t_1 - t_2)\alpha_n}{a_n}\right| \leq 2(\alpha_{k+1}t_2 + \alpha_k)\frac{t_1^{k+1}}{|a_n|t_1^n} + 2(\beta_{m+1}t_2 + \beta_m)\frac{t_1^{m+1}}{|a_n|t_1^n} - \frac{t_2\alpha_n + t_1\beta_n + \alpha_{n-1}}{|a_n|} \quad (1.4)$$

For $\beta_j = 0, j=0,1,2,\ldots,n$ in the Theorem (1.4), we obtain the following result.

Corollary 1.5: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \geq 3$ with real and positive coefficients. If $t_1 \geq t_2 \geq 0, t_1 \neq 0$ can be found such that

$$t_1t_2\alpha_r + (t_1 - t_2)\alpha_{r-1} - \alpha_{r-2} \geq 0 \text{ for } r=1,2,3,\ldots..,k+1,$$

$$t_1t_2\alpha_r + (t_1 - t_2)\alpha_{r-1} - \alpha_{r-2} \leq 0 \text{ for } r=k+2,\ldots..,n+1,$$

$0 \leq k \leq n-1$ $a_{-1} = a_{n+1} = 0$, then all the zeros of P (z) lie in

$$\left|z + \frac{a_{n-1}}{a_n} - (t_1 - t_2)\right| \leq t_2 + \frac{a_{n-1}}{a_n} + \frac{2t_2\alpha_{k+1} + 2\alpha_k}{a_n t_1^{n-k+1}} \quad (1.5)$$

Remark

In general Theorem 1.4 also gives much better result than Theorem 1.3 for $0 \leq k \leq n-1$. For this we show that the circle defined by (1.5) is contained in the circle defined by (1.3). Let z=w be any point belonging to the circle defined by (1.4) then

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$$\left|w+\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right|\le 2(\alpha_{k+1}t_2+\alpha_k)\frac{t_1^{k+1}}{|a_n|t_1^n}+2(\beta_{m+1}t_2+\beta_m)\frac{t_1^{m+1}}{|a_n|t_1^n}-\frac{t_2\alpha_n+t_1\beta_n+\alpha_{n-1}}{|a_n|}$$ This gives

$$|w|=\left|w+\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}-\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right|$$

$$\le\left|w+\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right|+\left|\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right|$$

$$\le 2(\alpha_{k+1}t_2+\alpha_k)\frac{t_1^{k+1}}{|a_n|t_1^n}+2(\beta_{m+1}t_2+\beta_m)\frac{t_1^{m+1}}{|a_n|t_1^n}-\frac{t_2\alpha_n+t_1\beta_n+\alpha_{n-1}}{|a_n|}+\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{|a_n|}$$

$$=\frac{t_1}{|a_n|}\left\{2t_1^{k-n}(\alpha_k+t_2\alpha_{k+1})+2t_1^{m-n}(\beta_m+t_2\beta_{m+1})-(\alpha_n+\beta_n)\right\}$$

Hence the point z = w belongs to the circle defined by (1.3) and therefore, the circle defined by (1.4) is contained in the circle (1.3).

Proof of the Theorem

Proof of Theorem 1.4: Consider the polynomial

$$G(z)=(t_1-z)(t_2+z)P(z)$$

$$=-a_nz^{n+2}+\{a_n(t_1-t_2)-a_{n-1}\}z^{n+1}$$

$$+\sum_{v=2}^{n}\{a_vt_1t_2+a_{v-1}(t_1-t_2)-a_{v-2}\}z^v$$

$$+\{a_1t_1t_2+a_0(t_1-t_2)\}z+a_0t_1t_2$$

$$=-a_nz^{n+2}+\{a_n(t_1-t_2)-a_{n-1}\}z^{n+1}$$

$$+\sum_{v=2}^{n}\{a_vt_1t_2+a_{v-1}(t_1-t_2)-a_{v-2}\}z^v \qquad (a_{-2}=a_{-1}=0)$$

Let $|z|>t_1$, then

$$|G(z)|\ge|z|^{n+1}\left\{|a_nz+a_{n-1}-(t_1-t_2)a_n|-\sum_{v=0}^{n}|a_vt_1t_2+a_{v-1}(t_1-t_2)-a_{v-2}|\frac{1}{|z|^{n-v+1}}\right\}$$

$$>|z|^{n+1}\left\{|a_nz+\alpha_{n-1}-(t_1-t_2)\alpha_n|-|\beta_{n-1}|-(t_1-t_2)|\beta_n|-\sum_{v=0}^{n}|a_vt_1t_2+a_{v-1}(t_1-t_2)-a_{v-2}|\frac{1}{t_1^{n-v+1}}\right\}$$

(1.6)

Now by hypothesis

$$\sum_{v=0}^{n}|a_vt_1t_2+a_{v-1}(t_1-t_2)-a_{v-2}|t_1^v\le\sum_{v=0}^{n}|\alpha_vt_1t_2+\alpha_{v-1}(t_1-t_2)-\alpha_{v-2}|t_1^v+\sum_{v=0}^{n}|\beta_vt_1t_2+\beta_{v-1}(t_1-t_2)-\beta_{v-2}|t_1^v$$

$$\le\sum_{v=0}^{k+1}|\alpha_vt_1t_2+\alpha_{v-1}(t_1-t_2)-\alpha_{v-2}|t_1^v$$

$$\sum_{v=k+2}^{n}|\alpha_vt_1t_2+\alpha_{v-1}(t_1-t_2)-\alpha_{v-2}|t_1^v$$

$$+\sum_{v=0}^{m+1}|\beta_vt_1t_2+\beta_{v-1}(t_1-t_2)-\beta_{v-2}|t_1^v$$

$$+\sum_{v=m+1}^{n}|\beta_vt_1t_2+\beta_{v-1}(t_1-t_2)-\beta_{v-2}|t_1^v$$

$$=2(\alpha_{k+1}t_2+\alpha_k)t_1^{k+2}-(\alpha_nt_2+\alpha_{n-1})t_1^{n+1}$$

$$+2(\beta_{m+1}t_2+\beta_m)t_1^{m+2}-(\beta_nt_2+\beta_{n-1})t_1^{n+1}$$

Using this in (1.6), we obtain

$$|G(z)|\ge|z|^{n+1}$$

$$\left\{\begin{array}{l}|a_nz+\alpha_{n-1}-(t_1-t_2)\alpha_n|\\ -(\beta_{n-1}-(t_1-t_2)\beta_n)-2(\alpha_{k+1}t_2+\alpha_k)\frac{1}{t_1^{n-k-1}}+(\alpha_nt_2+\alpha_{n-1})-2(\beta_{m+1}t_2+\beta_m)\\ \frac{1}{t_1^{n-m-1}}+(\beta_nt_2+\beta_{n-1})\end{array}\right\}$$

$$=|z|^{n+1}\left\{|a_nz+\alpha_{n-1}-(t_1-t_2)\alpha_n|+t_1\beta_n-2(\alpha_{k+1}t_2+\alpha_k)\frac{1}{t_1^{n-k-1}}+(\alpha_nt_2+\alpha_{n-1})-2(\beta_{m+1}t_2+\beta_m)\frac{1}{t_1^{n-m-1}}\right\}>0$$

If

$$|a_nz+\alpha_{n-1}-(t_1-t_2)\alpha_n|>2(\alpha_{k+1}t_2+\alpha_k)\frac{t_1^{k+1}}{|a_n|t_1^n}+2(\beta_{m+1}t_2+\beta_m)\frac{t_1^{m+1}}{|a_n|t_1^n}$$

$$-(t_2\alpha_n+t_1\beta_n+\alpha_{n-1})$$

Hence all the zeros of $G(z)$ whose modulus is greater than t_1 lie in the circle

$$\left|z+\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right|\le 2(\alpha_{k+1}t_2+\alpha_k)\frac{t_1^{k+1}}{|a_n|t_1^n}+2(\beta_{m+1}t_2+\beta_m)\frac{t_1^{m+1}}{|a_n|t_1^n}$$

$$-\frac{t_2\alpha_n+t_1\beta_n+\alpha_{n-1}}{|a_n|} \qquad (1.7)$$

Now we show that all the zeros of $G(z)$ whose modulus is less than equal to t_1 also lie in the circle defined by (1.4). Let $|z|\le t_1$, then we have

$$|a_nz+\alpha_{n-1}-(t_1-t_2)\alpha_n|\le|a_n|t_1+|\alpha_{n-1}-(t_1-t_2)\alpha_n|$$

$$\le t_1\alpha_n+t_1\beta_n+\alpha_{n-1}-(t_1-t_2)\alpha_n$$

$$=2t_1\beta_n+2(t_2\alpha_n+\alpha_{n-1})$$

By hypothesis

$$\sum_{v=k+2}^{n}\frac{a_vt_1t_2+a_{v-1}(t_1-t_2)-a_{v-2}}{t_1^{n-v+1}}\le 0,\quad 0\le k\le n-1$$

This gives

$$2(t_2\alpha_n+\alpha_{n-1})\le\frac{2(\alpha_{k+1}t_2+\alpha_k)}{t_1^{n-k-1}} \qquad (1.8)$$

Similarly for $0\le m\le n-1$

$$2(t_2\beta_n+\beta_{n-1})\le\frac{2(\beta_{m+1}t_2+\beta_m)}{t_1^{n-m-1}} \qquad (1.9)$$

Also we have

$$t_1\beta_n\le t_2\beta_n+\beta_{n-1} \qquad (1.10)$$

Combining (1.9) and (1.10) , we obtain

$$2t_1\beta_n\le\frac{2(\beta_{m+1}t_2+\beta_m)}{t_1^{n-m-1}} \qquad (1.11)$$

Using (1.8) and (1.11) in (1.7), we have

$$|a_nz+\alpha_{n-1}-(t_1-t_2)\alpha_n|\le 2(\alpha_{k+1}t_2+\alpha_k)\frac{t_1^{k+1}}{|a_n|t_1^n}+2(\beta_{m+1}t_2+\beta_m)\frac{t_1^{m+1}}{|a_n|t_1^n}$$

$$-(t_2\alpha_n+t_1\beta_n+\alpha_{n-1})\cdot$$

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Since all the zeros of $P(z)$ are also the zeros of $G(z)$, we conclude that all the zeros of $P(z)$ lie in

$$\left|z+\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right| \le 2(\alpha_{k+1}t_2+\alpha_k)\frac{t_1^{k+1}}{|a_n|t_1^n}+2(\beta_{m+1}t_2+\beta_m)\frac{t_1^{m+1}}{|a_n|t_1^n} -\frac{t_2\alpha_n+t_1\beta_n+\alpha_{n-1}}{|a_n|}$$

This completes the proof of the theorem.

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