A General System of Regularized Non-convex Variational Inequalities

Lee BS¹ and Salahuddin S²*
¹Department of Mathematics, Kyungsu University, Busan, 608-736, Korea
²Department of Mathematics, Jazan University, Jazan, Kingdom of Saudi Arabia

Abstract

In this communication, we introduced a general system of regularized non-convex variational inequalities (GSRNVI) and established an equivalence between this system and fixed point problems. By using this equivalence we define a projection iterative algorithm for solving GSRNVI, we also proved existence and uniqueness of GSRNVI. The convergence analysis of the proposed iterative algorithm is studied.

Keywords: General system of regularized non-convex variational inequalities; Uniformly t-prox regular sets; Iterative schemes; Convergence analysis

Introduction

The originally variational inequality problem introduced by Stampacchia [1] in the early sixties has a great impact and in influence in the development of almost all branches of pure and applied sciences and has witnessed an explosive growth in theoretical advances, algorithmic development. As a result of interaction between different branches of mathematical and engineering sciences, we now have a variety of tools to suggest and analyze various algorithms for solving variational inequalities and related optimizations [2-6]. Verma [7-10] studied some systems of variational inequality with single valued mappings and suggest some iterative algorithms to compute approximate solutions of these systems in Hilbert spaces. Agarwal et al. [11] studied sensitivity analysis for a system of generalized nonlinear mixed quasi variational inclusions with single valued mappings. Several authors studied different kinds of systems of variational inequalities and suggested iterative algorithms to find the approximate solutions of the systems [12-15]. We remark that the results regarding the existence of solutions and iterative schemes for solving the system of variational inequalities and related problems are being considered in the setting of convex sets and the technique defined on the characteristics of the projection operator over convex a set which does not hold in general when the sets are non-convex. It is well known that the uniform prox regular sets are convex and include the convex set as special cases. Wen [16] considered a system of non-convex variational inequalities with different nonlinear operator and asserted that this system is equivalent to the fixed point problem and suggested an iterative algorithm for the system of non-convex variational inequalities. The convergence analysis of the proposed iterative algorithm under some certain assumption is also studied. In [17] point out the equivalence formulation used by Wen [16] is not correct. Inspired and motivated by the works of [18-26], we introduced and studied a general systems of regularized non-convex variational inequalities. By using the equivalence, we defined a projection iteration algorithm for solving GSRNVI. Further, we proved the existence and uniqueness of solutions of general system of regularized non-convex variational inequalities. The convergence analysis of the proposed iterative algorithm is also studied.

Basic Foundation

Let H be a real Hilbert space endowed with norm \( \| \cdot \| \) and an inner product \( \langle \cdot , \cdot \rangle \) respectively. Let \( \Omega \) be nonempty closed subsets of H. We represent \( d_{\Omega}(.;\Omega) \) or \( d_{\Omega}(.;\Omega) \) the distance function from a point to a set \( \Omega \) that is

\[
d_{\Omega}(u) = \inf_{v \in \Omega} \| u - v \|
\]

\[\text{Definition 2.1:} \text{ Let } u \in H \text{ be a point not lying in } \Omega. \text{ A point } v \in \Omega \text{ is called a closed point or a projection of } u \text{ onto } \Omega \text{ if } d_{\Omega}(u) = \| u - v \|. \text{ The set of all such closed points is denoted by } P_{\Omega}(u), \text{ that is }
\]

\[
P_{\Omega}(u) = \{ v \in \Omega : d_{\Omega}(u) = \| u - v \| \}
\]

\[\text{Definition 2.2: The proximal normal cone of } \Omega \text{ at a point } u \in \Omega \text{ is given by }
\]

\[
Q_{\Omega}^{\alpha}(u) = \{ \zeta \in H : u \in Q_{\Omega}^{\alpha}(u + \alpha \zeta) \}
\]

where \( \alpha > 0 \) is a constant.

\[\text{Lemma 2.3: Let } \Omega \text{ be a nonempty closed subset of } H. \text{ Then } \zeta \in Q_{\Omega}^{\alpha}(u) \text{ if and only if there exists a constant } \alpha = \alpha(\zeta, u) > 0 \text{ such that } \langle \zeta, v - u \rangle \leq \alpha \| v - u \| \forall v \in \Omega.
\]

\[\text{Lemma 2.4: Let } \Omega \text{ be a nonempty closed and convex subset of } H. \text{ Then } \zeta \in Q_{\Omega}^{\alpha}(u) \text{ if and only if } \langle \zeta, v - u \rangle \leq 0 \forall v \in \Omega.
\]

\[\text{Definition 2.5: Let } f : H \to \mathbb{R} \text{ be a locally Lipschitz near a point } x. \text{ The Clark’s directional derivative of } f \text{ at } x \text{ in the direction } v, \text{ denoted by } f^{\circ}(x; v) \text{ is defined by }
\]

\[
f^{\circ}(x; v) = \limsup_{y \to x} \frac{f(y + tv) - f(y)}{t}
\]

where \( y \) is a vector in H and \( t \) is a positive scalar.

The tangent cone to \( \Omega \) at a point \( x \in \Omega \), denoted by \( T_{\Omega}(x) \) is defined by

\[
T_{\Omega}(x) = \{ v \in H : d_{\Omega}^{\circ}(x; v) = 0 \}
\]

The normal cone to \( \Omega \) at \( x \in \Omega \), denoted by \( Q_{\Omega}(x) \) is defined by

*Corresponding author: Salahuddin S, Department of Mathematics, Jazan University, Jazan, Kingdom of Saudi Arabia, Tel: 966 7 321 0061; E-mail: salahuddin12@mailcity.com

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\[ Q_\Omega(x) = \{ \zeta \in H : \langle \zeta, v \rangle \geq 0 \, \forall v \in T_\Omega(x) \} \]

The Clarke normal cone denoted by \( Q_\Omega^C(x) \) is defined by
\[
Q_\Omega^C(x) = \text{co}(Q_\Omega(x))
\]
where \( \text{co}(S) \) denotes the closure of the convex hull of \( S \) and \( Q_\Omega^C(x) \) is convex.

**Definition 2.6:** [4] For a given \( t \in (0, +\infty) \); a subset \( \Omega \) of \( H \) is called the normalized uniformly prox-regular (or uniformly t-prox-regular) if every nonzero proximal normal to \( \Omega \), can be realized by an \( t \)-ball.

That is for all \( \overline{x} \in \Omega \) and \( 0 \neq \zeta \in Q_{\Omega(t)}^C(\overline{x}) \)
\[
\zeta = \frac{1}{t} \left( x - \frac{\langle \zeta, x \rangle}{\| \zeta \|^2} \right), \quad x \in \Omega,
\]
Therefor for all \( \overline{x} \in \Omega \) and \( 0 \neq \zeta \in Q_{\Omega(t)}^C(\overline{x}) \) with \( \| \zeta \| = 1 \) we have
\[
\langle \zeta, x - \overline{x} \rangle \geq \frac{1}{t} \| \zeta \|^2, \quad x \in \Omega.
\]

**Lemma 2.7** [23] A closed set \( \Omega \subseteq H \) is convex if and only if it is uniformly \( t \)-prox-regular for every \( t > 0 \).

Let \( t \) be an uniformly \( \Omega_t \) prox-regular (nonconvex) set and \( g_i : \Omega_t \rightarrow \Omega_t \), \( i = 1, 2, 3 \) (the identity operator) and \( x^* = y^* = z^* = u \), then the system (3.1) reduces to the following classical variational inequalities defined on the nonconvex set \( \Omega \), find \( u \in \Omega \), such that
\[
0 \in T_u + Q_{\Omega_t}^C(u), \quad (3.3)
\]
where \( Q_{\Omega_t}^C(u) \) denotes the normal cone of \( \Omega_t \) at \( u \) over the non convex set.

**Lemma 3.1** \((x^*, y^*, z^*) \in \Omega_t \times \Omega_t \times \Omega_t \) is a solution of problem (3.1) if and only if
\[
g_1(x^*) = P_{\Omega_t}(g_1(x^*) - r_1T_x(y^*, z^*)),
g_2(y^*) = P_{\Omega_t}(g_2(z^*) - r_2T_y(z^*, y^*)),
g_3(z^*) = P_{\Omega_t}(g_3(x^*) - r_3T_z(x^*, z^*)),\quad (3.4)
\]

where \( P_{\Omega_t} \) is the projection of \( H \) on to the uniformly t-prox-regular set \( \Omega_t \). In the proof of Lemma 3.1, there occur three fatal errors. First in view of Proposition 2.8, for any \( t \in (0, 1) \) the projection of points in the tube \( U(t') = \{ u \in H : 0 < d_{\Omega_t}(u) < t' \} \) onto the set \( \Omega_t \) exists and unique, that is for any \( x \in U(t') \), the set \( P_{\Omega_t}(x) \) is nonempty and singleton. From the Lemma 3.1 and Proposition 2.6 the points \( g_1(y^*) - r_1T_x(y^*, z^*) \) and \( g_2(z^*) - r_2T_y(z^*, y^*) \) should be in \( U(t') \) for some \( t' \in (0, t) \) it is not necessary true, hence (3.4) are not necessarily well defined. If \( r_1 < 1 - \frac{1}{t'} \| (y', x') \| \), \( r_1 < 1 - \frac{1}{t'} \| (z', y') \| \), \( r_2 < 1 - \frac{1}{t'} \| (y', x') \| \), and \( t' \in (0, t) \). Then we have
\[
d_{\Omega_t}(g_1(y^*) - r_1T_x(y^*, z^*)) \leq d_{\Omega_t}(g_1(y^*)) + \frac{r_1 \| (y', x') \|}{1 + \frac{r_1 \| (y', x') \|}{t'}},
\]
Hence, \( (g_1(y^*) - r_1T_x(y^*, z^*)) \in U(t') \).

Similarly we have \( (g_2(z^*) - r_2T_y(z^*, y^*)) \in U(t') \) and \( (g_3(x^*) - r_3T_z(x^*, z^*)) \in U(t') \).

The problem (3.1) is called a general system of regularized non convex variational inclusions is equivalence to the system (3.1):
\[
0 \in r_1 T_x(y^*, x') + g_1(x) - g_1(y^*) + r_1 Q_{\Omega_t}(g_1(x)),
\]
\[
0 \in r_2 T_y(z^*, y') + g_2(z) - g_2(y^*) + r_2 Q_{\Omega_t}(g_2(z)),
\]
\[
0 \in r_3 T_z(x^*, z') + g_3(x) - g_3(x^*) + r_3 Q_{\Omega_t}(g_3(x)), \quad (3.5)
\]

Since \( Q_{\Omega_t}(g_1(x)) \), \( Q_{\Omega_t}(g_2(z)) \) and \( Q_{\Omega_t}(g_3(x)) \) are cone, the system (3.1) is equivalent to the following system:
\[
0 \in r_1 T_x(y^*, x') + g_1(x) - g_1(y^*) + Q_{\Omega_t}(g_1(x)),
\]
\[
0 \in r_2 T_y(z^*, y') + g_2(z) - g_2(y^*) + Q_{\Omega_t}(g_2(z)),
\]
\[
0 \in r_3 T_z(x^*, z') + g_3(x) - g_3(x^*) + Q_{\Omega_t}(g_3(x)). \quad (3.6)
\]

The system (3.1) is equivalent to the system (3.5) which is not true in general.

**Example 3.2** Let \( H = \mathbb{R} = [0, b] \times [c, d] \) be the union of two disjoint intervals \([0, b]\) and \([c, d]\) where \( b < c < d \). Then \( t \) is an uniformly \( t \)-prox-regular set with \( t = \text{cNd} 2 \). Define \( T : \Omega \times \Omega \rightarrow \Omega \) and \( g : \Omega \rightarrow \Omega \), by
\[
\langle Tu - v, u - v \rangle \geq 0, \forall v \in \Omega, \quad (3.2)
\]
and (3.2) is equivalent to find \( u \in \Omega \) such that
\[
0 \in Tu + Q_{\Omega_t}^C(u), \quad (3.3)
\]
where for \( i = 1, 2, 3 \), \( s_i, m \in \mathbb{R}, \theta_i < 0 \) and \( b^{-1} \leq k < \frac{c}{b^n} \) are arbitrary but fixed.

Assume \( x' = y' = z' = b \) and \( r > 0 \), \( i = 1, 2, 3 \) be the fixed arbitrary. Hence for all \( w \in \Omega \)

\[
\alpha > \max \left\{ -r_0 \theta_i e^{\psi_i}, -r_2 \theta_i e^{\psi_i}, -r_3 \theta_i e^{\psi_i} \right\}
\]

\[
c - kb^n < w - kb^n < d - kb^n
\]

\[
\alpha (c - kb^n) + r_0 \theta_i e^{\psi_i} \leq \alpha (w - kb^n) + r_2 \theta_i e^{\psi_i} \leq \alpha (d - kb^n) + r_3 \theta_i e^{\psi_i}
\]

\[
(w - kb^n) \geq 0 \quad \forall \; \psi \in \Omega
\]

(3.8)

From (3.7)-(3.8), we have

\[
\{r_1 T_i(y', x') + g_i(x') - g_i(y'), w - g_i(x') \} + \alpha \|w - g_i(x')\|^2
\]

Since \( r_0 \theta_i e^{\psi_i}(w - kb^n) < 0 \) for all \( w \in \{c, d\} \) i.e.,

\[
\{r_1 T_i(y', x') + g_i(x') - g_i(y'), w - g_i(x') \} < 0 \quad \forall \; \psi \in \Omega
\]

Hence

\[
\{r_1 T_i(y', x') + g_i(x') - g_i(y'), w - g_i(x') \} \geq 0, \; w \in \Omega
\]

cannot hold. Similarly we have

\[
\{r_2 T_i(y', x') + g_i(y') - g_i(x'), w - g_i(y') \} \geq 0, \; w \in \Omega
\]

while the inequality

\[
\{r_2 T_i(y', x') + g_i(y') - g_i(x'), w - g_i(y') \} \geq 0, \; \psi \in \Omega
\]

cannot hold. Again in similar way we have

\[
\{r_3 T_i(y', x') + g_i(z') - g_i(x'), w - g_i(z') \} \geq 0
\]

the inequality

\[
\{r_3 T_i(y', x') + g_i(z') - g_i(x'), w - g_i(z') \} \geq 0
\]

cannot hold. Therefore we can see that every solution of (3.3) is a solution of (3.3) but converse need not be true in general. On the basis of example we define as the general system of regularized non-convex variational inequality. For given nonlinear mappings \( T_i: H \times H \rightarrow H \) and \( g_i: H \times H \rightarrow H \) \( i = 1, 2, 3 \) we consider the general system of regularized non-convex variational inequality for finding \((x', y', z') \in H \times H \times H \)

such that \((g_i(x'), g_i(y'), g_i(z')) \in \Omega_i \times \Omega_i \times \Omega_i \) and

\[
\{r_1 T_i(y', x') + g_i(x') - g_i(y'), g_i(x') - g_i(x') \}
\]

\[
+ \frac{\alpha}{2} \|w - g_i(x')\|^2 \geq 0, \; \forall \; x \in \Omega
\]

\[
\{r_2 T_i(y', x') + g_i(y') - g_i(x'), g_i(y') - g_i(x') \}
\]

\[
+ \frac{\alpha}{2} \|w - g_i(x')\|^2 \geq 0, \; \forall \; x \in \Omega
\]

\[
\{r_3 T_i(y', x') + g_i(z') - g_i(x'), g_i(z') - g_i(x') \}
\]

\[
+ \frac{\alpha}{2} \|w - g_i(x')\|^2 \geq 0, \; \forall \; x \in \Omega
\]

Proposition 3.3: Let \( \Omega \) be an uniformly t-prox regular set. The system (3.9) is equivalent to the system (3.6).

Proof: Let \((x', y', z') \in H \times H \times H, \; g_i(x'), g_i(y'), g_i(z') \in \Omega_i \times \Omega_i \times \Omega_i \) \( i = 1, 2, 3 \) be a solution set of the system (3.9), then from Definition 2.6, \((x', y', z') \in H \times H \times H, \; g_i(x'), g_i(y'), g_i(z') \in \Omega_i \times \Omega_i \times \Omega_i \) \( i = 1, 2, 3 \) be a solution set of the system (3.9).

Lemma 3.4: For \( i=1, 2, 3 \) let \( T_i; T_i \) be the same as in the system (3.9), then \((x', y', z') \in H \times H \times H, \; g_i(x'), g_i(y'), g_i(z') \in \Omega_i \times \Omega_i \times \Omega_i \) \( i = 1, 2, 3 \) be a solution set of the system (3.9) if and only if (3.4) is satisfied by the system (3.4) with \( P_{\alpha} = (1 + q \Omega_i) \) and Proposition 3.3 we have

\[
0 \in \{r_1 T_i(y', x') + g_i(x') - g_i(y'), g_i(x') + Q_i g_i(0, x') \}
\]

\[
\Rightarrow g_i(y') - r_1 T_i(y', x') \in g_i(x') + Q_i g_i(0, x')
\]
\[ g_i(y') - r T_i(y', x') \in (I + Q^p_{\alpha_i})(g_i(x')) \]
\[ g_i(x') = P_{\Omega_i} [ g_i(y') - r T_i(y', x') ] \]
where \( I \) is an identity mapping. Similarly we have
\[ \exists (\Omega, y, z) (x, y, z) H \times H \times H \]
\[ g_i(x') - r T_i(x', z') \in g_i(z') + Q^p_{\alpha_i}(g_i(z')) \]
\[ g_i(z') - r T_i(z', y') \in (I + Q^p_{\alpha_i})(g_i(z')) \]
\[ g_i(y') = P_{\Omega_i} [ g_i(z') - r T_i(z', y') ] \]
and
\[ 0 \in r T_j(x', x') + g_i(x') - g_i(z') + Q^p_{\alpha_i}(g_i(x')) \]
\[ g_i(x') - r T_j(x', x') \in g_i(z') + Q^p_{\alpha_i}(g_i(z')) \]
\[ g_i(z') - r T_j(z', y') \in (I + Q^p_{\alpha_i})(g_i(z')) \]
\[ g_i(y') = P_{\Omega_i} [ g_i(z') - r T_j(z', y') ] \]
This completes the proof.

Existence and Convergence Analysis

Definition 4.1: A mapping \( T: H \times H \to H \) is said to be
(i) monotone in the first variable if for all \( x, y \in H \)
\[ \langle T(x, u) - T(y, v), x - y \rangle \geq 0 \quad \forall u, v \in H, \]
(ii) \( \mu \) -strongly monotone in the first variable if there exists a constant \( \mu > 0 \) such that
\[ \langle T(x, u) - T(y, v), x - y \rangle \geq \mu \| x - y \|^2 \quad \forall u, v \in H, \]
(iii) \( \mu \) -cocoercive if there exists a constant \( \mu > 0 \) such that
\[ \langle T(x, u) - T(y, v), x - y \rangle \geq \mu \| T(x, u) - T(y, v) \|^2 \quad \forall x, y \in H, \]
(iv) relaxed \( \mu \) -cocoercive if there exists a constant \( \mu > 0 \) such that
\[ \langle T(x, u) - T(y, v), x - y \rangle \geq \mu \| T(x, u) - T(y, v) \|^2 \quad \forall x, y, u, v \in H, \]
(v) \( \mu \) -Lipschitz continuous in the first variable if there exists a constant \( \mu > 0 \) such that for all \( x, y \in H \)
\[ \| T(x, u) - T(y, v) \| \leq \mu \| x - y \| \quad \forall x, y \in H \]
(i) \( \mu \) -monotone if there exists a constant \( \mu > 0 \) such that
\[ \langle g(x) - g(y), x - y \rangle \geq \mu \| g(x) - g(y) \|^2 \quad \forall u, v \in H \]
(ii) \( \xi \) -Lipschitz continuous if there exists a constant \( \xi > 0 \) such that for all \( x, y \in H \)
\[ \langle g(x) - g(y), x - y \rangle \geq \xi \| x - y \|^2 \quad \forall x, y \in H \]
Now we prove that existence and unique solution set of general system of regularized non-convex variational inequalities.

Theorem 4.3: Let the mappings \( T_i, g_i \) and \( r_i \) \( i = 1, 2, 3 \) be the same as in the system (3.1) such that \( g_i(H) \subseteq \Omega \). Let \( g_i \) be the \( \mu \) -cocoercive with constant \( \xi > 0 \) and Lipschitz continuous mapping with constant \( \mu > 0 \). Let \( T_i \) be the relaxed \( (\eta_i, v_i) \)-cocoercive with respect to the first variable with constants \( \eta_i, v_i > 0 \) and \( \lambda_i \)-Lipschitz continuous mapping with constant \( \lambda_i > 0 \). If the constant \( r_i \) \( i = 1, 2, 3 \) satisfy the following conditions:
\[ r_i < r_i \quad \text{for} \quad i = 1, 2, 3 \]
\[ r_i < r_i \quad \text{and} \quad r_i < r_i \quad \text{for} \quad y \in H, \text{for } t' \in (0, t) \]
And
\[ \xi_i - \sqrt{\lambda_i^2 - \delta^2} < \frac{\lambda_i^2}{\lambda_i^2 + \delta^2} \]
\[ \xi_i - \sqrt{\lambda_i^2 - \delta^2} < \frac{\lambda_i^2}{\lambda_i^2 + \delta^2} \]
\[ \delta = \frac{t}{t-i} \quad \text{then the system (3.9) admits a unique solutions.} \]

Proof: Define \( \phi, \psi, \Omega : H \times H \to H \) by
\[ \phi(x, y) = x - g_i(x) + \frac{\alpha_i}{r_i} [ g_i(y) - r T_i(y, x) ] \]
\[ \psi(y, z) = y - g_i(y) + \frac{\alpha_i}{r_i} [ g_i(x) - r T_i(x, z) ] \]
Since \( H \times H \to H \) is a Hilbert space, we define \( \cdot \) : \( H \times H \times H \to H \) by
\[ \zeta(x, y, z) = (\phi(x, y), \phi(y, z), \psi(x, z)), \quad \forall x, y, z \in H \]
we claim that \( \zeta \) is a contraction mapping. Indeed let \( (x, y, z) \in H \times H \times H \) and \( r' < \frac{t}{t-i} \) for \( t' \neq 0, t \), hence
\[ g_i(y) - r T_i(y, x) \in U(t') \]
and the t-prox-regularity of \( \Omega \) implies that \( P_{\Omega_i} [ g_i(y) - r T_i(y, x) ] \) exists and unique. Using

Proposition 2.8, we have
\[ \| g_i(y) - g_i(x') - r T_i(y, x') \| \leq \| g_i(y) - g_i(x') - r T_i(y, x') - (g_i(x) - g_i(x')) \| \]
\[ \leq \| x - x' - (g_i(x) - g_i(x')) \| + \| g_i(y) - g_i(x') - r T_i(y, x') - T_i(y, x') \| \]
\[ \leq \| x - x' - (g_i(x) - g_i(x')) \| + \| (g_i(y) - g_i(x') - r T_i(y, x')) \| \]
\[ \leq \| x - x' \|^2 \quad \text{and} \quad \| (g_i(y) - g_i(x') - r T_i(y, x')) \|^2 \]
\[ \leq \| x - x' \|^2 \quad \text{and} \quad \| (g_i(y) - g_i(x') - r T_i(y, x')) \|^2 \]
\[ \leq \| x - x' \|^2 \quad \text{and} \quad \| (g_i(y) - g_i(x') - r T_i(y, x')) \|^2 \]
\[ \leq \| x - x' \|^2 \quad \text{and} \quad \| (g_i(y) - g_i(x') - r T_i(y, x')) \|^2 \]
\[ \leq \| x - x' \|^2 \quad \text{and} \quad \| (g_i(y) - g_i(x') - r T_i(y, x')) \|^2 \]
\[ \leq \| x - x' \|^2 \quad \text{and} \quad \| (g_i(y) - g_i(x') - r T_i(y, x')) \|^2 \]
Similarly we obtain and 

For arbitrary initial points \( x_0, y_0, z_0 \in \mathbb{H} \), \( T_i \) with \( i=1; 2; 3 \) is a solution set of 

\[
(4.14)
\]

Again, since \( g_3 \) is \( \mu_3 \)-cocoercive and \( \lambda_3 \)-Lipschitz continuous and \( T_3 \) is relaxed \( (\eta_3, \mu_3) \)-cocoercive mapping and \( \lambda_3 \)-Lipschitz continuous with first variable, we obtain

\[
(4.12)
\]

Lemma 3.4 guarantees that \( \big( \hat{x}, \hat{y}, \hat{z} \big) \in \mathbb{H} \times \mathbb{H} \times \mathbb{H} \) with \( (g(x), g(y), g(z)) \in \Omega \times \Omega \times \Omega, \) \( i=1, 2, 3 \) is a solution set of \( \big( x_i \big) \), \( i=1, 2, 3 \) is an identity mappings, then we have the following algorithm.

Algorithm 4.5: Let mapping \( T_i \) and constant \( r > 0 \) for \( i=1, 2, 3 \) be the same as in the system (3.9). For arbitrary initial points \( x_0, y_0, z_0 \in \mathbb{H} \), compute the sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) in \( \mathbb{H} \) in the following way:

\[
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n \big[ x_n - g_n(x_n) + P_{\Omega} (g_n(y_n) - r_n T_n(x_n, y_n)) \big]
\]

\[
y_{n+1} = (1 - \alpha_n) y_n + \alpha_n \big[ y_n - g_n(y_n) + P_{\Omega} (g_n(z_n) - r_n T_n(z_n, y_n)) \big]
\]

\[
z_{n+1} = (1 - \alpha_n) z_n + \alpha_n \big[ z_n - g_n(z_n) + P_{\Omega} (g_n(x_n) - r_n T_n(x_n, z_n)) \big],
\]

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\). Again assume that \( g_i = 1, i=1, 2, 3 \) is an identity mappings, then we have the following algorithm.

Algorithm 4.6 Let mapping \( T_i \) and constant \( r > 0 \) for \( i=1, 2, 3 \) be the same as in the system (3.9). For arbitrary initial points \( x_0, y_0, z_0 \in \mathbb{H} \), compute the sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) in \( \Omega \) in the following way:

\[
x_{n+1} = P_{\Omega} \big( y_{n+1} - r_n T_n(x_n, y_n) \big)
\]

\[
y_{n+1} = P_{\Omega} \big( z_{n+1} - r_n T_n(z_n, y_n) \big)
\]

\[
z_{n+1} = P_{\Omega} \big( x_{n+1} - r_n T_n(x_n, z_n) \big),
\]

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\):

Now we prove the strong convergence of the sequences generated by Algorithm 4.4 to a unique solution set of the system.

Theorem 4.7 Let the mappings \( T_i \), \( g \) and \( r > 0 \) for \( i=1, 2, 3 \) be the same as in the system (3.9). For arbitrary initial points \( x_0, y_0, z_0 \in \mathbb{H} \), compute the sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) in \( \Omega \) in the following way:

\[
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n \big[ x_n - g_n(x_n) + P_{\Omega} (g_n(y_n) - r_n T_n(x_n, y_n)) \big]
\]

\[
y_{n+1} = (1 - \alpha_n) y_n + \alpha_n \big[ y_n - g_n(y_n) + P_{\Omega} (g_n(z_n) - r_n T_n(z_n, y_n)) \big]
\]

\[
z_{n+1} = (1 - \alpha_n) z_n + \alpha_n \big[ z_n - g_n(z_n) + P_{\Omega} (g_n(x_n) - r_n T_n(x_n, z_n)) \big],
\]

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\):

Now we prove the strong convergence of the sequences generated by Algorithm 4.4 to a unique solution set of the system.

Theorem 4.7 Let the mappings \( T_i \), \( g \) and \( r > 0 \) for \( i=1, 2, 3 \) be the same as in the system (3.9). For arbitrary initial points \( x_0, y_0, z_0 \in \mathbb{H} \), compute the sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) in \( \Omega \) in the following way:

\[
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n \big[ x_n - g_n(x_n) + P_{\Omega} (g_n(y_n) - r_n T_n(x_n, y_n)) \big]
\]

\[
y_{n+1} = (1 - \alpha_n) y_n + \alpha_n \big[ y_n - g_n(y_n) + P_{\Omega} (g_n(z_n) - r_n T_n(z_n, y_n)) \big]
\]

\[
z_{n+1} = (1 - \alpha_n) z_n + \alpha_n \big[ z_n - g_n(z_n) + P_{\Omega} (g_n(x_n) - r_n T_n(x_n, z_n)) \big],
\]

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\):

Now we prove the strong convergence of the sequences generated by Algorithm 4.4 to a unique solution set of the system.

Theorem 4.7 Let the mappings \( T_i \), \( g \) and \( r > 0 \) for \( i=1, 2, 3 \) be the same as in the system (3.9). For arbitrary initial points \( x_0, y_0, z_0 \in \mathbb{H} \), compute the sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) in \( \Omega \) in the following way:

\[
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n \big[ x_n - g_n(x_n) + P_{\Omega} (g_n(y_n) - r_n T_n(x_n, y_n)) \big]
\]

\[
y_{n+1} = (1 - \alpha_n) y_n + \alpha_n \big[ y_n - g_n(y_n) + P_{\Omega} (g_n(z_n) - r_n T_n(z_n, y_n)) \big]
\]

\[
z_{n+1} = (1 - \alpha_n) z_n + \alpha_n \big[ z_n - g_n(z_n) + P_{\Omega} (g_n(x_n) - r_n T_n(x_n, z_n)) \big],
\]

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\):

Now we prove the strong convergence of the sequences generated by Algorithm 4.4 to a unique solution set of the system.
\[ x' = (1 - \alpha_n) x + \alpha_n \left[ x' - g_i (x') + P_{T_i} (g_i (y') - \tau_i T_i (y', x')) \right] \]
\[ y' = (1 - \alpha_n) y + \alpha_n \left[ y' - g_2 (y') + P_{T_2} (g_2 (z') - \tau_i T_2 (y', z')) \right] \]
\[ z' = (1 - \alpha_n) z + \alpha_n \left[ z' - g_3 (z') + P_{T_3} (g_3 (x') - \tau_i T_3 (y', z')) \right], \]
(4.15)
where \( \alpha_n \) is a sequence in \([0, 1]\): Since \( n \in \mathbb{N} \), \((g(y^*_n), g(y_n)) \in \Omega_t\); \( g_1 \) is continuous mapping in the first variable and \( g_2 \) is \( \mu_1 \)-strong cocoercive mapping and \( \xi_1 \)-Lipschitz continuous mapping, as same way of (4.16) - (4.20), we get
\[ \| z_n - z' \| \leq (1 - \alpha_n) \| z_{n-1} - z' \| + \alpha_n (p_1 \| z_{n-1} - z' \| + \theta_1 \| y_n - y' \|) \]
(4.22)
where \( p_1 \) and \( \theta_1 \) are same as in (4.12). Now
\[ \| (x_{n+1}, y_{n+1}, z_{n+1}) - (x', y', z') \| \leq (1 - \alpha_n) \| (x_n, y_n, z_n) - (x', y', z') \| \]
\[ + \alpha_n ((p_1 + \theta_1) \| y_n - y' \| + (p_1 + \theta_1) \| y_{n-1} - y' \| + \| y_n - y_{n-1} \|) \| z_{n-1} - z' \| \]
\[ \leq (1 - \alpha_n) \| (x_n, y_n, z_n) - (x', y', z') \| + \alpha_n \| (x_n, y_n, z_n) - (x', y', z') \|, \]
(4.23)
where \( z' \) is same as in (4.13). From condition (4.2) we get \( \sum_{n=0}^{\infty} \alpha_n = \infty \) we get
\[ \lim_{n \to \infty} \prod_{i=0}^{n} (1 - (1 - \alpha_n)) = 0 \]
(4.24)
Therefore from (4.23)-(4.24)
\[ \| (x^n, y^n, z^n) - (x^*, y^*, z^*) \| \to 0 \text{ as } n \to \infty \]
and the sequences \( \{(x_n, y_n, z_n)\} \) suggested by algorithm 4.4, converges strongly to a unique solution set \( (x^*, y^*, z^*) \) of the general system of regularized nonconvex variational inequalities. In a similar way, we can prove the convergence of iterative sequences generated by Algorithm 4.5 and Algorithm 4.6.

References


15. Inchan I, Petrot N (2011) System of general variational inequalities involving different nonlinear operators related to fixed point problems and its applications. Fixed point Theory.


