Journal of Civil & Environmental Engineering

Latreche, J Civil Environ Eng 2014, 4:5
DOI: 10.4172/2165-784X.1000157

Research Article — Open Access

A Discrete-Time Quasi-Theoretical Solution of the Modified Riccati Matrix Algebraic Equation

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Abstract

In this paper, based on MacLaurin's series and the Riccati equation, an algebraic quadratic equation will be developed and hence, its two roots, which represent the minimizing and maximizing optimal control matrices, would be deducted easier. Otherwise, a step-by-step algorithm to compute the control matrix for every step of time according to the preceding responses and a new signal pick will be explained. The proposed method presents a new discrete-time solution for the problem of optimal control in the linear or nonlinear cases of systems subjected to arbitrary signals. As an example, a system (structure) of three degrees of freedom, subjected to a strong earthquake is analyzed. The versus time displacements and the stiffness forces versus displacements of the system, for the two uncontrolled and controlled cases are graphically shown and clarify the great reduction of the controlled system results, in comparison with the uncontrolled system ones. The curves of variations of the elements of the optimal control matrix versus discrete-time are also presented.

Keywords: Optimal control; Nonlinear systems; Modified riccati equation; Quasi-theoretical solution; Discrete-time algorithm

Introduction

It is well known that the QR is a widely used method for the optimal control of systems in engineering analysis and design practices. The method indeed, is based on the determination of the optimal control matrix, which is practically and optimally reduce the effect of the signals for which the systems are subjected. Moreover, the interest to resolve the Riccati matrix equation is appear clarify in the literature since decades, and several simple or complex algorithms are proposed [1-22], an iterative algorithm [17], a numerical algorithm using the iterative Newton-Raphson method [1], an algorithm and a software using the Eigen-solution [6], an iterative algorithm using the iterative Newton method [8] and a step-by-step algorithm for the resolution of the differential Riccati equation [12], present some from which is published in this domain.

Otherwise, based on the matrix algebraic equation of Riccati and the MacLaurin's series, a quadratic algebraic equation will be developed, such that its two roots which represent the closed-loop minimize and maximize optimal control matrices will be deducted easily. The deducted control matrices are computed for every so small step of time and for a given system proper matrices (deducted from the previously step of time). Therefore, the step-by-step algorithm proposed, means that the system behaves linearly during every step of time; but changing its properties from step to step in terms of the responses computed previously, so actually the proposed algorithm could be considered as a nonlinear algorithm and resolve nonlinear problems, such that the results of this method stretching well to the exact solution as-well-as the time step taken should be refined as possible.

As testing numerical example, to demonstrate the efficiency of the proposed method, a nonlinear structure of three degrees of freedom, subjected to El-Centro earthquake was analyzed, in the two uncontrolled case and controlled one using the proposed method. The curves of the responses and the acted stiffness forces on every degree of freedom show the great reductions of the results in the case of the controlled structure, compared with the others of the same uncontrolled one. Moreover, the variations of the optimal control matrix elements versus time are also shown.

Firstly, we start in the second section to transform the classical Riccati algebraic equation to a quadratic one, and therefore, we deduct in the third section, the two quasi-theoretical minimize and maximize optimal control matrices and force vector. The fourth section is appearing in the numerical example to present the results which could the proposed method offers. The following sections dealing with the explanation of results and the conclusion.

The Transformation of the Riccati Equation

The state space formulation of a controlled dynamic system is stated by the ordinary differential equation

$$Z(t) = A(t)\,Z(t) + Bf_e(t) + Bf_{c(t)} \tag{1}$$

Where, $f_{c(t)}$ is the controlled force vector deducted after computing the optimal control matrix, $f_e(t)$, Z(t), A(t), B, are respectively, the seismic force vector, the state space response vector, and the state space matrices, given by

$$f_e(\mathrm{t}) = -\mathrm{M}\,\Gamma\,\mathrm{a}_g \quad Z(\mathrm{t}) = \begin{Bmatrix} U(\mathrm{t}) \\ \dot{U}(\mathrm{t}) \end{Bmatrix} \quad \boldsymbol{A}(\boldsymbol{t}) = \begin{bmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \quad B = \begin{Bmatrix} 0 \\ M^{-1} \end{Bmatrix}$$

I represents the unity matrix, r a unity vector and a_g the ground acceleration.

Suppose that the optimal control matrix $P(t)_{2n\times 2n}$ (such that *n* represents the number of the structure's degrees of freedom), is related by

$$P(t) = \lambda(t)\,Z^{-1}(t) \tag{2}$$

By differentiating the Hamiltonian function we conclude the vector relations

$$\begin{cases} \dot{\ddot{e}}(t) = -\boldsymbol{A}^T(t)\lambda(t) - Q(t)Z(t) \\ \dot{\boldsymbol{Z}}(t) = -B\boldsymbol{R}^{-1}\boldsymbol{B}^T\lambda(t) + A(t)Z(t) \\ f_c(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\lambda(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^T P(t)Z(t) \end{cases} \tag{3}$$

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Received October 07, 2014; **Accepted** October 21, 2014; **Published** October 27, 2014

Citation: Latreche T (2014) A Discrete-Time Quasi-Theoretical Solution of the Modified Riccati Matrix Algebraic Equation. J Civil Environ Eng 4: 157. doi:10.4172/2165-784X.1000157

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Expressions, such that $R_{n\times n}$ and $Q_{2n\times 2n}$ represent the weighting matrices.

The first equation of the expressions (3) can be rewritten, after replacing λ(t) by its expression (2), and differentiating according to t

$$\dot{P}(t)Z(t)+P(t)\dot{Z}(t)=-A^T(t)P(t)Z(t)-Q(t)Z(t) \tag{4}$$

Replacing Z(t) by its relation (3) and simplifying by Z(t), we can getting then

$$\dot{P}(t)-P(t)BR^{-1}B^TP(t)+P(t)A(t)=-A^T(t)P(t)-Q(t) \tag{5}$$

Therefore, the differential matrix equation of Riccati, could be concluded as following

$$\dot{P}(t)=P(t)BR^{-1}B^TP(t)-P(t)A(t)-A^T(t)P(t)-Q(t) \tag{6}$$

Assuming that $\dot{P}(t) = F(P) = 0$, hence we could write

$$F(P) = PBR^{-1}B^TP - PA - A^TP - Q \tag{7}$$

According to MacLaurin's series, the equation (7) would be developed as follows

$$\mathbf{F(P) = F(0) + F_{,p}(0)\, P + F_{,pp}(0)\, P^2/2} \tag{8}$$

$$\begin{cases} F(0)=0BR^{-1}B^T0-0A-A^T0-Q=-Q \\ F_{,P}(0)=BR^{-1}B^T0+0BR^{-1}B^T-(A+A^T)=-(A+A^T) \\ F_{,PP}(0)=2BR^{-1}B^T \end{cases} \tag{9}$$

Replacing F(0), $\mathbf{F_{,p}(0)}$ and $\mathbf{F_{,pp}(0)}$ by their values (9), we get hence

$$\dot{P}(t) = BR^{-1}B^TP^2(t) - (A(t) + A^T(t))P(t) - Q(t) \tag{10}$$

The solution of the differential equation (10) couldn't indeed extends a minimize solution, because is appearing as a mixed solution, and to separate between the minimize and maximize solutions we suppose that $\dot{P}(t) = 0$. Therefore, the following algebraic equation will be deducted

$$BR^{-1}B^TP^2 - (A + A^T)P - Q = 0 \tag{11}$$

This equation represents a transformation of the matrix algebraic equation of Riccati to a quadratic one, such that its two roots could be deducted explicitly. Contrary to the linear systems, such that this equation would be resolved once, for the nonlinear systems, the equation (11) should be resolved for every step of time, according to the case of the system which is defined in term of preceding responses (in structural engineering, the displacements and velocities). Therefore, the two roots of this equation represent clearly and optimally the minimizing and maximizing solutions. These roots of this equation could then be deducted easier as it is will be demonstrated in the following section for every step of time and in terms of the proper matrices of the system, computed by the step which precedes.

The Optimal Control Matrices and Force Vector

The two matrix roots of the above equation (11), are given by the following expressions, which are represent the minimize and maximize optimal control matrices respectively

$$\begin{cases} P_1(t)=\left[2BR^{-1}B^T\right]^{-1}\left[\left(A(t)+A^T(t)\right)-\left[\left(A(t)+A^T(t)\right)^2+4BR^{-1}B^TQ(t)\right]^{1/2}\right] \\ P_2(t)=\left[2BR^{-1}B^T\right]^{-1}\left[\left(A(t)+A^T(t)\right)+\left[\left(A(t)+A^T(t)\right)^2+4BR^{-1}B^TQ(t)\right]^{1/2}\right] \end{cases} \tag{12}$$

Otherwise, the matrix $BR^{-1}B^T$ has no inverse such that it has the form

$$BR^{-1}B^T=\begin{bmatrix} 0 & 0 \\ 0 & M^{-1}R^{-1}M^{-1} \end{bmatrix} \tag{13}$$

Therefore, we should to proceed approximately. The proposed approximation procedure can be stated as follows

The square matrix $(BR^{-1}B^T + I)^2$ could be developed by the relation

$$(BR^{-1}B^T + I)^2 = (BR^{-1}B^T)^2 + 2BR^{-1}B^TI + I^2$$

Multiplying lefty the two sides of this equation by the term $(BR^{-1}B^T)^{-1}$, we get

$$(BR^{-1}B^T)^{-1}(BR^{-1}B^T + I)^2 = BR^{-1}B^T + 2I + (BR^{-1}B^T)^{-1}I^2$$

After simplification, this equation becomes

$$(BR^{-1}B^T)^{-1}[(BR^{-1}B^T + I)^2 - I^2] = BR^{-1}B^T + 2I \tag{14}$$

The matrix $[(BR^{-1}B^T + I)^2 - I^2]$ has the same form as $BR^{-1}B^T$, and couldn't also be inversing. Approximately, we propose to take the term μI instead of I^2 in the first side of equation (14), such that the coefficient μ is a, as possible, stretching to the unity but not equals to the strictly unity. According to the precision of the computer and/or the programming language used, we can take μ precise as possible. For example we can take μ – 0.9999999999999999. After simplifying equation (14), then we can get

$$(BR^{-1}B^T)^{-1} = [BR^{-1}B^T + 2I)]\,[(BR^{-1}B^T + I)^2 - \mu I]^{-1} \tag{15}$$

The second side of the expression (15), represent a good approximately inverse for the term $BR^{-1}B^T$**.** Replacing $(BR^{-1}B^T)^{-1}$ by its expression (15) in the two matrix roots (12), the minimize and maximize optimal control matrices which should to be computed for every step of time are then becomes

$$\begin{cases} P(t)=0.5\left[BR^{-1}B^T+2I\right]\left[\left(BR^{-1}B^T+I\right)^2-\mu I\right]^{-1}\left[\left(A(t)+A^T(t)\right)-\left[\left(A(t)+A^T(t)\right)^2+4BR^{-1}B^TQ(t)\right]^{1/2}\right] \\ P(t)=0.5\left[BR^{-1}B^T+2I\right]\left[\left(BR^{-1}B^T+I\right)^2-\mu I\right]^{-1}\left[\left(A(t)+A^T(t)\right)+\left[\left(A(t)+A^T(t)\right)^2+4BR^{-1}B^TQ(t)\right]^{1/2}\right] \end{cases} \tag{16}$$

The matrix $[(A(t) + A^T(t))^2 + 4BR^{-1}B^TQ(t)]^{1/2}$ can be computed using the iterative method [23], supposing that

$$X^2 - [(A(t) + A^T(t))^2 + 4BR^{-1}B^TQ(t)] = 0$$

Such that X is the matrix square root needed. Moreover, the method proposed in [23], represent an iteration algorithm of the fourth order Runge-Kutta method to resolve any scalar or matrix algebraic equation, according to a so small proposed ΔT (which is not the same step of time, for that the responses are computed) and a sufficient convergence tolerance. The following Fortran code algorithm explaining how to resolve the latest algebraic equation to get the square root X

```
A1=A+TRANSPOSE(A)
WHERE(X/=0.)X=0.
DT =0.004
TOL = 0.000000005
DO
K1=DT*(MATMUL(X,X)-(MATMUL(A1,A1)+4.*MATMUL(BR1BT,Q)))
K2=DT*(MATMUL(X+0.5*K1,X+0.5*K1)-(MATMUL(A1,A1)+4.*MATMUL(BR1BT,Q)))
K3=DT*(MATMUL(X+0.5*K2,X+0.5*K2)-(MATMUL(A1,A1)+4.*MATMUL(BR1BT,Q)))
K2=DT*(MATMUL(X+K3,X+K3)-(MATMUL(A1,A1)+4.*MATMUL(BR1BT,Q)))
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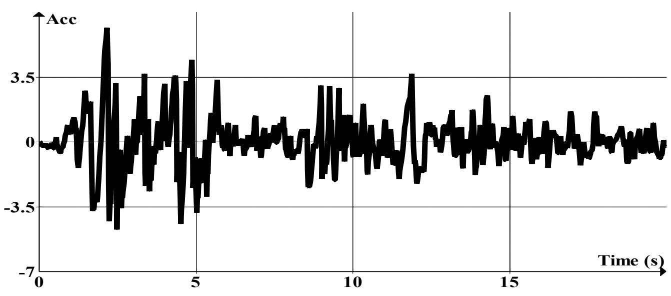

Figure 1: The acceleration of the ground (m/s²).

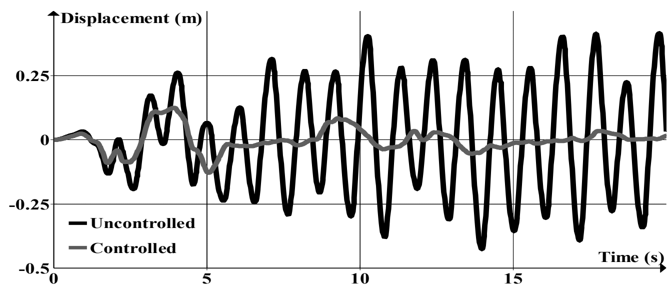

Figure 2: The first floor displacement vs. time (m).

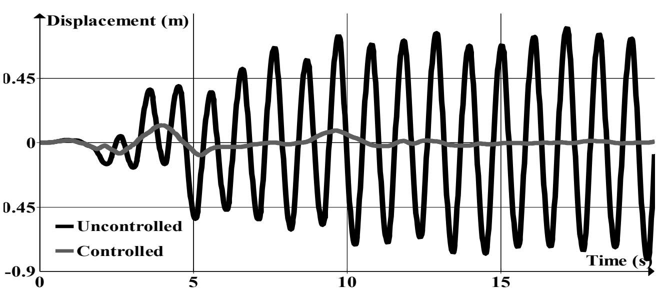

Figure 3: The second displacement vs. time.

X=X+ (K1+2.*K2+2.*K3+K4)/6.

EQA=MATMUL(X,X)-(MATMUL(A1,A1)+4.*MATMUL(BR1BT,Q))

! WRITE(*,*)MAXVAL(EQA), MINVAL(EQA)

IF(ABS(MAXVAL(EQA))<TOL.AND.ABS(MINVAL(EQA))<TOL)EXIT

ENDDO

The optimal control matrix is computed in terms the system proper matrices computed from the preceding step, such that they are in turns following the system nonlinearity laws in function of the responses computed. The optimal control forces acted to every degree of freedom of the system, for any step of time, are ranked in the vector of the third equation (3) in terms of the last step responses.

Therefore, the proposed method is summarized, for every step of time, in the following steps:

- Computing the proper matrices of the system, according to the previously results (responses)
- Computing, according the case, the minimize or the maximize optimal control matrix
- Deducting the optimal control force vector and adding it to the new exterior force vector
- Resolving the ordinary differential state space equation of the system to get the responses of this step of time
- According to the results found from the step (4.), starting a new loop with a new step of time

Numerical Example

The chosen numerical example, for the evaluation of the efficiency of the proposed method to reduce optimally the results of the systems subjected to arbitrary signals, is imitated in a structure of three degrees of freedom with concentrates masses at the level of every degree of freedom, and subjected to the Modified El-Centro Earthquake. The stiffness behavior of the material, which the structural elements has been fabricated, is assumed to be following a bilinear model, for which it consists of two branches, an elastic branch with the stiffness equals $\mathbf{K}_e$ and a plastic branch such that the stiffness equals $\mathbf{K}_p$. The mass, stiffness and damping matrices for the chosen structure are given by

$$M=\begin{bmatrix} m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & m\end{bmatrix}\quad K=\begin{bmatrix} k_1 & -k_1 & 0\\ -k_1 & k_1+k_2 & -k_2\\ 0 & -k_2 & k_2+k_3\end{bmatrix}\quad C=\alpha M+\beta K$$

m = 1kg and $k_{1\text{ or }2\text{ or }3}$ can take the values: k_e = 25 N/m (for the elastic linear branch) and $k_p = k_e/3$ (for the plastic branch), α and β represent the Rayleigh damping coefficients which are given in terms of the frequencies of the linear elastic structure ω_1 and ω_2 and the damping coefficient ξ. Assuming that the damping coefficient ξ = 0.05, then

$$\beta=0.1(\omega_2-\omega_1)/\left(\omega_2^2-\omega_1^2\right)=0.006559 \qquad \alpha=\omega_1\omega_2\beta=0.368488$$

The elastic limit displacement, with which the material stiffness transferring from the elastic to the plastic branch, is chosen to be equals 0.025 m, this limit indeed, that deciding if the proper structural stiffness and damping matrices changing or no from any step of to another. The ground acceleration variations are shown in Figure 1, such that the step of time separates two peaks is 0.02 s. The weighting matrices are chosen to be giving by

$$R=\begin{bmatrix} 0.1 & 0 & 0\\ 0 & 0.1 & 0\\ 0 & 0 & 0.1\end{bmatrix}\quad Q=\begin{bmatrix} 0 & 0\\ 0 & -MRM\left(CM^{-1}\right)^2\end{bmatrix}_{6\times 6}$$

The displacements curves of the three floors (degrees of liberty), for the two uncontrolled and controlled cases, versus time are shown by the Figures 2-4. The hystereses of the stiffness forces acted on the three degrees of freedom versus the displacements variations are clarified by the three Figures 5-7. To show the effect of the nonlinearity behavior of the system, some of the optimal control matrix elements variations versus time are be clearing in Figures 8-11.

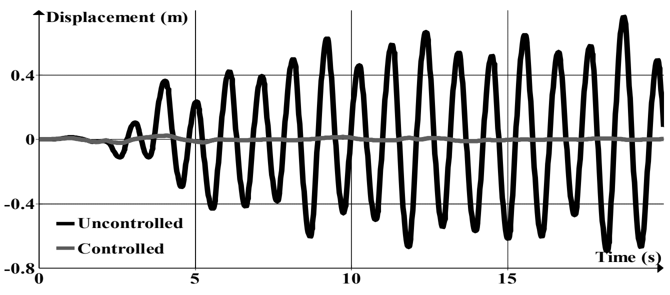

Figure 4: The third floor displacement vs. time.

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Results and Discussion

By perceiving of the Figures 2-4 and Table 1, we can remark the grand differences between the displacements of the three floors of the analyzed structure. Despite that the analyzed structure is subjected to a so strong earthquake, and the uncontrolled responses are so considerable (40 to 80 cm), we can see that the controlled responses are so moderate (3 to13 cm) despite the fact that the elastic displacement adopted is also so small (2.5 cm). By reference to Table 1 and the three Figures 2-4 respectively, one can remark the following: the percentage, of the difference between the uncontrolled and controlled maximal displacements (0.426 m and 0.124 m) of the first floor equals about 71%; for the second floor, for which the uncontrolled and controlled displacements are respectively, 0.816 m and 0.121 m and 0.121 m, the deducted percentage is 85%; and for the third floor, the maximal displacements for the two cases and the difference percentage are equal to 0.763 m, 0.024 m and 97% respectively. About the stiffness forces for three degrees of freedom, contrary to the uncontrolled forces which fluctuate between 4N and 7.2 N, these forces for the controlled case, alternate between 0.6 N and 1.5 N. Referencing to Table 2 and Figures 5-7 respectively, we can remark that, for the first floor, the uncontrolled and controlled stiffness forces and their difference percentage take the values 3.968 N, 1.493 N 62% respectively; for the second floor the

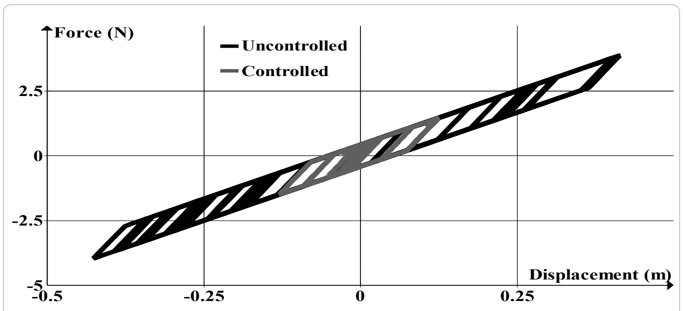

Figure 5: The first floor stiffness force vs. Displacement.

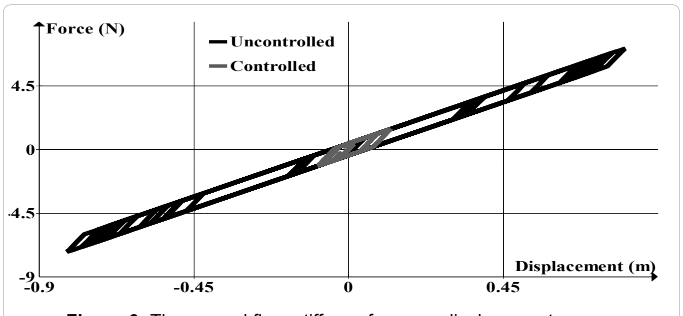

Figure 6: The second floor stiffness force vs. displacement.

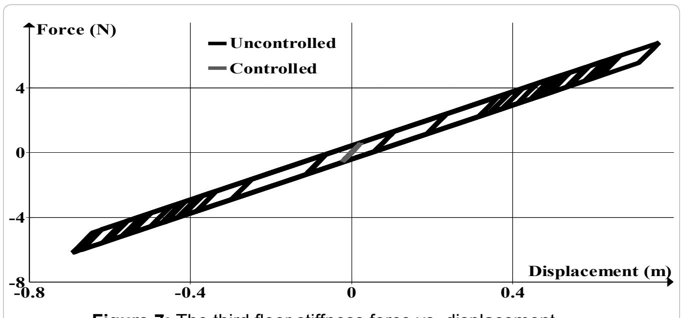

Figure 7: The third floor stiffness force vs. displacement.

Floors	Uncontrolled max. displacements	Controlled max. displacements	% Cont./Uncont.
1	0.426	0.124	29.11
2	0.816	0.121	14.83
3	0.763	0.024	3.15

Table 1: The maximal Uncontrolled, Controlled displacements and their fraction.

Floors	Uncontrolled max. forces	Controlled max. forces	% Cont./Uncont.
1	3.968	1.493	37.63
2	7.218	1.418	19.65
3	6.774	0.566	8.36

Table 2: The maximal Uncontrolled, Controlled forces and their fraction.

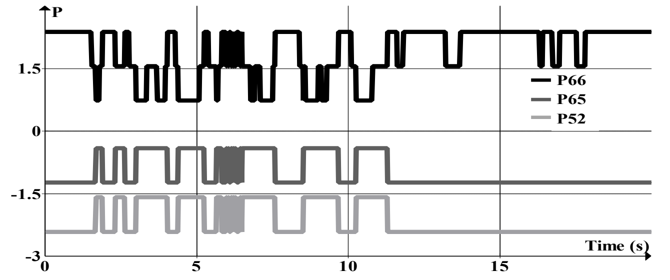

Figure 8: The P66, P65 and P52 elements of the optimal control matrix.

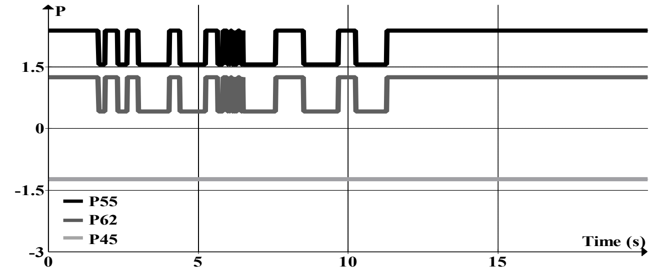

Figure 9: The P55, P62 and P45 elements of the optimal control matrix.

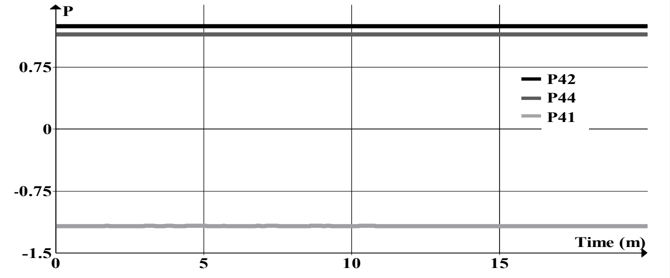

Figure 10: The P42, P44 and P41 elements of the optimal control matrix.

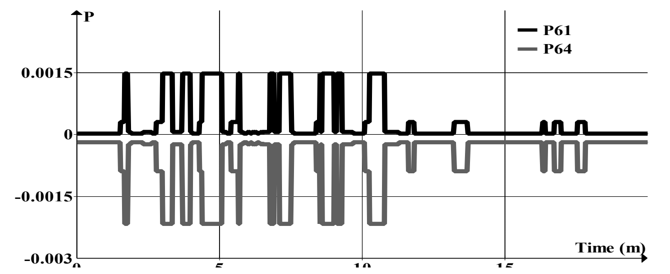

Figure 11: The P61 and P64 elements of the optimal control matrix.

stiffness forces and the percentage are given by 7.218 N, 1.418 N 80%; while for the last floor, are equal to 7.218 N, 1.418 N 80%.

Conclusion

The proposed method could be summarized in determining firstly, the minimize and maximize optimal control matrices (according the case) which are the matrices roots of the quadratic equation developed herein and secondly, to compute the roots by a discrete-time algorithm which aimed to evaluate the optimal control matrix for every step of time according to the nonlinear behavior of the analyzed system which in turn (the behavior of the system), is a function of the previously responses (the displacements and velocities in the case of structural engineering). This proposed method of the optimal control of systems subjected to arbitrary signals indeed, possesses the ability to offer a good control and a so sufficiently results. Furthermore, the method allows the analysis of nonlinear systems (i.e. real systems), because it is resolved for every step of time and according to the state of the system. The Figures 8-11, clearing show the effect of the nonlinearity of the structure adopted as an example, on the variations of the optimal control matrix.

The results (displacements and stiffness forces curves) of the proposed example show the efficiency of the getting solution. As it is seen by the figures and tables, the controlled results are very considerably reduced, which can surpass 90% of percentage of the differences between uncontrolled and controlled displacements and acted stiffness forces. In spite that the ground motion acceleration are very high, and this effect is shown by the uncontrolled structure responses; but the controlled results are too moderate and sufficiency, and for the third floor, the element was not plasticized even as shown by Figure 7, despite that the limit elastic displacement adopted is very small.

These excellent controlled results indeed, demonstrate the effect of the optimal control of structures using the Quadratic Regulator method, and the obtained solution of the optimal control matrix formulated and the effect of the nonlinear behavior of the system adopted.

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