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A COUPLED ROBUST DESIGN USING GAME THEORY

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Abstract
Robust design is a proven process to achieve insensitivity. From the view point of numerical optimization, the robustness of the objective function makes the system performance insensitive to uncertainties. To better manage the uncertainties, the Taguchi method, reliability-based optimization and robust optimization can be used. This study suggests how to analyze a robust design problem using axiomatic design concept. The design axioms provide a general framework for design methodologies. Two axioms are (1) Independence Axiom and (2) Information Axiom. These axioms can be applied to all design processes in a general way. The first axiom illustrates the relationship between functional requirements (FRs) and design parameters (DPs). Then, the designs of products can be classified into three types: uncoupled, decoupled and coupled. Two goals of robust design can be defined as two functional requirements. One is to reduce the distribution of a response. The other is to set the mean of a response to its target. In general, it is easy to determine the robust solution for a uncoupled or decoupled design. However, the coupled design cannot currently give true robustness, leading to a trade-off between performance and robustness. In this paper, game theory is applied to optimize the trade-off between two functional requirements.

Keywords: Axiomatic design, Coupled design, Robust design, Game theory

1. INTRODUCTION

A design procedure performed by the axioms is called axiomatic design. A special feature of the first axiom is that the design parameters are determined independently for the corresponding functional requirements [1]. The axiomatic design provides the idea to analyze the design quality. This research classifies the robust design problem into three types by applying the first axiom.

Robust designs improve product quality and reliability in industrial engineering. The concept of robust design was introduced by Dr. G. Taguchi in the late 1940s, and his technique based on this concept has become known commonly as the Taguchi method or the robust design. Since 1980s, the Taguchi method has been applied to numerical optimization, complementing the deficiencies of deterministic optimization. This newly developed optimization method is often called robust optimization, and it overcomes the limitations of deterministic optimization, which neglects the effect of uncertainties in design variables and/or design parameters [1-3].

The uncertainties, which can be the tolerances of design variables and/or the variations in design parameters, often induce severe variations in the response function. Evidently, the robust design concept is essential to the design problem with variations. To consider robustness, statistics such as the mean and variance (or standard deviation) of a response should be calculated in the robust optimization process. Variations of a response are generated from uncertainties in design variables and/or design parameters. The purpose of a robust design is to find a design that will give target response with the smallest variation [3, 4].

Axiomatic design has been used to provide a framework for design problems. It states that a design should be defined by independent functional requirements (FRs) and corresponding design parameters (DPs), and designers should minimize the information content of the designs [5, 6, 7]. Designers have to choose a correct set of DPs to be able to satisfy the FRs. Essentially, the FR is what we want to achieve and the DP is how we achieve it. The basic postulate of axiomatic design is that there are two fundamental axioms that govern a design process. This study utilizes the first axiom, called the independence axiom, to analyze the system.

A mechanical system can be divided into three categories under the independence axiom. When the independence axiom is satisfied, the design matrix takes the form of a diagonal matrix; this is called an uncoupled design. An uncoupled design that satisfies the imposed design requirements and has the minimum information content is the optimum design. When the design matrix has a triangular form, the design is called a decoupled design, and when a design matrix cannot be reorganized to a triangular form, the design is called a coupled design [5, 6].

Minimizing the standard deviation and the difference between the mean and the target value of a response can be regarded as functional requirements that must be met to obtain a robust design. For an uncoupled or a decoupled system, a robust design can be easily achieved. Unfortunately, real design problems are often not included in uncoupled or decoupled designs. These designs cannot currently accomplish real robustness; thus, a trade-off between performance and robustness has to be made. In this research, game theory is applied to optimize the trade-off.

Game theory is divided into two categories of non-cooperative and cooperative games. A cooperative game is a game in which players make decisions in cooperation with any player; on the contrary, a non-cooperative game is a game in which players do not cooperate with each other to reach the same goal. Thus, a robust design problem pertaining to a coupled design can be treated as cooperative game [8, 9, 10]. The mean and the standard deviation of the multiobjective function in a robust design can be treated as the responses. In addition, design variable set can be decomposed into the strategy set of two players.

In this study, a two-bar design problem is solved to obtain its robust design through the suggested procedures. By the game theory approach using a bargaining function, the optimal solution is found [9].

2. COUPLED DESIGN AND SIGNAL TO NOISE RATIO

2.1. Coupled Design

Axiomatic design is the framework for a good design. It helps to create synthesized solutions that satisfy perceived needs by mapping between FRs and DPs. An FR is the goal to achieve and is defined in the functional requirement domain, and a DP is the means to achieve this goal and is determined in the physical domain. Relevant DP can be chosen in the physical domain by the mapping process to satisfy a given FR in the functional domain [5, 6, 7].

Axiomatic design presents two axioms. One is the independence axiom, and the other is the information axiom. By the independence axiom, the FRs should be independently defined from the relation between FRs and DPs. That is, the FRs should be maintained independently. On the contrary, the information axiom requires that a design should minimize the information content. This study utilizes the independence axiom to take the coupled design out of an arbitrary dsign problem.

In the axiomatic approach, FRs should be independently defined from the relation between FRs and DPs. The relationship between FRs and DPs are represented as

$$\mathbf{FR} = \mathbf{ADP} \qquad (1)$$

where **FR** is the functional requirement vector, **DP** is the design parameter vector, and **A** is the design matrix, respectively.

Based on the independence axiom, designs are classified into three types: uncoupled, decoupled and coupled. For a design problem with two functional requirements and two design parameters, the design equations for uncoupled, decoupled and coupled designs are represented as Eqs. (2)~(4), respectively.

$$\begin{Bmatrix} FR_1 \\ FR_2 \end{Bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{Bmatrix} DP_1 \\ DP_2 \end{Bmatrix} \qquad (2)$$

$$\begin{Bmatrix} FR_1 \\ FR_2 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{Bmatrix} DP_1 \\ DP_2 \end{Bmatrix} \qquad (3)$$

$$\begin{Bmatrix} FR_1 \\ FR_2 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} DP_1 \\ DP_2 \end{Bmatrix}. \qquad (4)$$

where A_{ij} is the element value that cannot be negligible.

To satisfy the independence axiom, the design matrix must either be diagonal or triangular. When the design matrix is diagonal, an FR can be satisfied independently by one DP; such a design is called an uncoupled design. When the matrix is triangular, the independence of FRs can be guaranteed if and only if the DPs are determined in a proper sequence; such a design is called a decoupled design. Any other form of a design matrix is called a coupled design [6, 7].

2.2. Taguchi Method and Signal-to-Noise (S/N) Ratio [4]

By the end of the 1940s, the Taguchi method had been developed by Dr. G. Taguchi for quality improvement of a product. His technique in quality engineering is referred as the Taguchi method or robust design. While the Taguchi method was successfully applied at the Electrical Communications Laboratories of the Nippon Telephone and Telegraph Company, AT & T Bell Laboratories became interested in it. In 1980, Phadke invited Taguchi to AT & T Bell Laboratories. After being impressed by the Taguchi method, Phadke published a textbook on the Taguchi method in 1989.

The Taguchi method has greatly contributed to the quality improvement of various designs. In early case studies, the Taguchi method was applied to process designs rather than product designs because it was regarded as a method for design of experiments rather than a design methodology.

Taguchi introduced a quadratic loss function to represent robustness as

$$L(f) = k(f - m_f)^2 \qquad (5)$$

where f is the response function, m_f is the target value, and k is the loss constant , respectively. The expected value of the loss function is defined as

$$Q = E[L(f)] = k[\sigma_f{}^2 + (\mu - m_f)^2] \qquad (6)$$

where μ and σ_f are the mean and the standard deviation of the response function, f, respectively.

Robust design is a design with minimum average loss. Dr. Taguchi suggested that a product or a process design be composed of three levels: system, parameter, and tolerance designs. In the system design step, new ideas are generated to provide products to customers. In the parameter design step, the designer determines the optimum setting for control factors using orthogonal arrays and S/N ratios. The manufacturing cost will not be affected by the parameter design step since tolerances are fixed. The ultimate goal of the parameter design step is to make products insensitive to noise factors without eliminating them. The tolerance design step is implemented to improve quality at a minimum cost. However, it should be used when the sensitivity of the responses in the parameter design step is not within a satisfactory range. The parameter design scheme of the Taguchi method, in particular, is adopted for robust design.

In the parameter design step, control factors are determined to reduce the effect of noise factors. Therefore, noise factors are not directly considered. On the other hand, noise factors are directly controlled in the tolerance design step. In the parameter design step, the quality of a product or process is improved without cost increase, whereas in the tolerance design step, quality is improved with cost increase.

When the target value of a response is given, the Taguchi method determines the optimum setting of the control factors so that the variation of the response is minimized, although any uncontrollable factor may exist. Eq. (6) can be regarded as an index to finding a robust design. Suppose that we have a scaling factor s to adjust the current mean to the target value. Scaling factor z is described as

$$z = \frac{m}{\mu}. \tag{7}$$

When the current mean is adjusted to the target value, the average loss function of Eq. (6) is changed to

$$Q_a = k\{(\mu\frac{m}{\mu} - m)^2 + (\frac{m}{\mu}\sigma_f)^2\} = km^2\frac{\sigma_f^{\ 2}}{\mu^2} \tag{8}$$

Then, to enhance the additivity effect of the control factors, Eq. (8) is transformed to

$$\eta = 10\log_{10}\frac{\mu^2}{\sigma_f^{\ 2}} \tag{9}$$

Eq. (9) is the ratio of the power of the signal factors, μ ,and the power of the noise factors, σ_f. Thus, it is called the S/N ratio. Maximizing Eq. (9) is equivalent to minimizing Eq. (8). That is, a robust design is obtained by maximizing Eq. (9).

2.3. Taguchi Method versus Design Matrix

In Eq. (9), if any factors increase the mean of $\mu_{f,}$ instead of decreasing σ_f to maximize the S/N ratio, an incoherent answer can be obtained. Therefore, Eq. (9) is a function of mean and variance. Considerable risk of conflicting variability is involved [3].

There are two goals in performing a robust design. One is to minimize the variability produced by the noises factors. The other is to approach the target value as close as possible. The design for one goal is not usually consistent with the one for the other goal. To meet both goals, Taguchi developed a two-step optimization strategy. The first step reduces the variation, and the second step adjusts the mean on the target. That is, if the factors affecting the variance and the mean are divided, a robust design can be obtained [4].

Suppose that we have one response function, f. Then, its robust design would have two requirements: one is to minimize the standard deviation or the variance, and the other is to minimize the difference between the mean and the target value of the response. The design equation for the robust design can be represented as

$$\begin{Bmatrix} Minimize\ (\mu - m_f)^2 \\ Minimize\quad \sigma_f^{\ 2} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} DP_1 \\ DP_2 \end{Bmatrix} \tag{10}$$

The target value, m_f, can be a positive infinite, a negative infinite or specified value for smaller-the-better, larger-the-better or nominal-the-best case, respectively. When the design matrix in Eq. (10) is either diagonal or triangular, a robust design can be easily achieved. However, in general, the design matrix for a robust design has the form of a coupled design.

3. ROBUST DESIGN USING GAME THEORY

A robust design, which is a function of standard deviation and mean of the responses, can be represented as [9]

$$Minimize\ [f_1(\mathbf{x})=|\mu(\mathbf{x})- m_f|, f_2(\mathbf{x})=s(\mathbf{x}), \ldots, f_k(\mathbf{x})\ ] \tag{11}$$

$$Subject\ to\ g_j(\mathbf{x}) \le 0,\ j=1,\ldots,m \tag{12}$$

where $\mathbf{x}$ is the design variable vector, $f_k(\mathbf{x})$ is the k-th objective function, s is the standard deviation, $g_j(\mathbf{x})$ is the j-th constraint function, and m is the number of constraints, respectively.

The weighting method, constraint method, global criterion method, goal programming, etc are approaches that can solve multiobjective problems. In this study, the bargaining function is investigated to solve Eqs. (11)~(12).

3.1. Game Theory

Game theory is a branch of mathematics dealing with decision making in conflict situations. In the conflict situation, two or more players exist. The players have their own objective. Due to these features, game theory is often utilized to solve numerical optimization problems such as multiobjective optimization problems.

Game theory is divided into two branches of non-cooperative and cooperative branches. A cooperative game consists of a set of players and a characteristic function. A cooperative game is a game in which players make decisions in cooperation with each other. Thus, a robust design problem pertaining to a coupled design is treated as a cooperative game in this study.

3.2. Bargaining Function [8, 9, 10]

Suppose that we have twp players A and B. Each player can select a strategy, vector $\mathbf{x}$ from an admissible strategies set, S. Let U be a payoff function. Then, the payoffs of players A and B are u=$U_A(\mathbf{x})$ and v=$U_B(\mathbf{x})$, respectively. In cooperative games, individual players will bargain in order to maximize their payoffs in cooperation with each other.

The bargaining function is defined as

$$B(\mathbf{x}) = (U_A(\mathbf{x}) - u')(U_B(\mathbf{x}) - v') \tag{13}$$

where U is the utility function, and u' and v' are arbitrary utility function values, respectively. Thus, each player will find his best strategy, $\mathbf{x}$, such that the payoff becomes maximum, maintaining the positive sign in each term of Eq. (13). Introducing the response function, f, the bargaining function is represented as

$$B(\mathbf{x}) = \prod_{i=1}^{k} (f_{iw}(\mathbf{x}) - f_i(\mathbf{x})) \quad (14)$$

where f_{iw} is the worst function value of the i-th objective function. Alternatively, the following bargaining function can be utilized. Let $\mathbf{x}_1$*, $\mathbf{x}_2$*, …, $\mathbf{x}_k$* be the optima determined from a single objective function. Then, F_{iu} is defined as

$$F_{iu}=\text{Max}\ [\ F_i(\mathbf{x}_j^*),\ j=1,2,\ldots,k] \quad (15)$$

where F_i=c$_j f_i(\mathbf{x})$ and it is treated as a constant. The bargaining function can be written as

$$B(\mathbf{x}) = \prod_{i=1}^{k} (F_{iu}(\mathbf{x}) - F_i(\mathbf{x}_C^*)) \quad (16)$$

where $\mathbf{x}_c$* is the Pareto-optimum determined from the multiobjective function. When the weighting method is introduced, the following formulation should be solved to evaluate Eq. (16).

$$\textit{Find} \qquad \mathbf{w} = [w_1\ w_2\ \ldots\ w_k], \mathbf{x} \quad (17)$$

$$\textit{Minimize} \qquad \sum_{i=1}^{k} w_i f_i(\mathbf{x}) / f_i^*(\mathbf{x}) \quad (18)$$

$$\textit{Subject to} \qquad g_j(\mathbf{x}) \le 0, \quad j = 1,\ldots,m \quad (19)$$

where $f_i^*(\mathbf{x})$ is the function value at the optimum only considering the i-th objective function.

4. TWO-BAR DESIGN PROBLEM

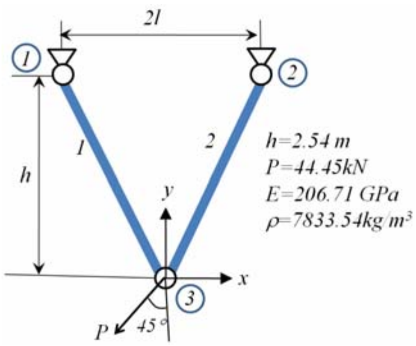

Fig. 1 Two-bar truss

The responses of the two-bar shown in Fig. 1 are calculated as [2, 8]

$$Q = 2\rho h x_2 \sqrt{1+x_1^2} A_{ref} \quad (20)$$

$$\delta = \frac{Ph(1+x_1^2)^{1.5}\sqrt{1+x_1^4}}{2\sqrt{2}Ex_1^2 x_2 A_{ref}} \quad (21)$$

$$\sigma_1 = \frac{P(1+x_1)\sqrt{1+x_1^2}}{2\sqrt{2}x_1 x_2 A_{ref}} \quad (22)$$

$$\sigma_2 = \frac{P(-1+x_1)\sqrt{1+x_1^2}}{2\sqrt{2}x_1 x_2 A_{ref}} \quad (23)$$

where x_1=l/h, x_2=A/A_{min}, A_{min}=1(×645.16×10^{-6} m^2), A_{ref}=1, 10 ≤ x_1≤200 (×0.0254m), and 0.1 ≤ x_2≤2.5 (×645.16×10^{-6}m^2).

The allowable stress of this structure, σ_a ,is 137MPa. It is assumed that the tolerance of x_i, Δx_i (i=1,2) is 10% of the current design variable, and its standard deviation σ_{xi} is Δx_i/3, and each variable is statistically independent, random and normally distributed.

Suppose for the robust design problem, we find that the design is insensitive to the displacement at node 3. The design equation is represented as

$$\begin{Bmatrix} \textit{Minimize} & \bar{\delta} \\ \textit{Minimize} & s \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad (24)$$

where $\bar{\delta}$ is the mean of the displacement at node 3.

First, the orthogonal array is utilized to find the design variable sensitive to the FRs and insensitive to the FRs, respectively. To determine the elements of the design matrix, the ANOVA (analysis of variance) is performed. The three levels are set to the lower bound, the average between two bounds, and the upper bound of each design variable. Table 1 shows the results of the 3 level-orthogonal array experiments. The mean and the standard deviation are calculated by

$$\bar{\delta} \approx \delta(x_1, x_2) \quad (25)$$

$$s \approx \sqrt{\sum_{i=1}^{2} \left(\frac{\partial \delta}{\partial x_i}\right)^2 \cdot s_{xi}^2}. \quad (26)$$

Table 1 Full combination experiment for 2 D.V. and 3 levels

No.	x_1 (*in*)	x_2 (*in*)	δ (*in*)	s (*in*)
1	10	0.1	11.936	3.988
2	10	1.3	0.920	0.024
3	10	2.5	0.479	0.007
4	105	0.1	0.485	1.700
5	105	1.3	0.037	0.010
6	105	2.5	0.019	0.003
7	200	0.1	1.358	9.054
8	200	1.3	0.104	0.054
9	200	2.5	0.054	0.014

The sums of squares with respect to the mean are S_{x1}=33.96, S_{x2}=37.59, respectively. On the contrary, the sums of squares with respect to the standard deviation are S_{x1}=9.56, S_{x2}=47.93, respectively. Thus, the non-diagonal elements in Eq. (24), A_{12} and A_{21}, cannot be neglected. It is concluded that this design problem is a coupled design problem. The multiobjective function to obtain the robust design for this problem is represented as

$$\textit{Minimize}\ [f_1(\mathbf{X})=\delta, f_2(\mathbf{X})=s, f_3(\mathbf{X})=\textit{weight}] \quad (27)$$

$$\textit{Subject to}\ \ \sigma_i - \sigma_a \le 0\ ,\ i=1,2 \quad (28)$$

The above formulation is solved by 1) weighting method, 2) global criterion method, 3) bargaining function of Eq. (16) and global criterion method, 4) bargaining function of Eq. (16) and weighting method and 5) bargaining function of Eq. (14). The initial values of the design variables are x_1=200 and x_2=2.5. Each case is solved by the modified feasible direction method. The value, p , is set to 2 in the global criterion method. In the global criterion method, the objective is defined as

$$Minimize \quad \left[\sum_{i=1}^{k} \left| \frac{f_i * - f_i(X)}{f_i *} \right|^p \right]^{1/p} . \qquad (29)$$

The results are summarized in Table 2. The optimum of Case 2 is the same as Case 3, and the optimum of Case 1 is close to that of Case 4. The comparison of the bargaining function value showed that Case 3 and Case 4 are better than Case 1 and Case 2. However, Case 5 decreases the weight only, although each function is scaled.

Table 2 Solutions for multi-objective optimization

Case	x_1, x_2 (*in*)	δ (*in*)	*s* (*in*)	weight (lb_f)
1	62.18, 2.38	0.02242	0.00196	158.6
2	59.25, 1.82	0.03071	0.00334	119.7
3	59.53, 1.83	0.03043	0.00331	120.4
4	64.38, 2.47	0.02096	0.00182	166.3
5	71.57, 0.65	0.07370	0.02698	45.4

5. CONCLUSIONS

The following statements are summarized. First, the first axiom is introduced to evaluate the design quality. That is, the design equation is composed to investigate the relationship between FRs and DPs. The design matrix is defined by calculating the sum of squares of mean and standard deviation. Second, for an uncoupled or a decoupled design problem, the design variable sensitive to mean and the design variable sensitive to standard deviation are divided. Then, each optimum is obtained by solving each optimization formulation. That kind of design problem is ideal case, considering the axiomatic design concept. Third, for a coupled design problem, its robust design is determined by solving multiobjective function. In this stage, the bargaining function is successfully applied. As a future work, the practical structural design will be applied.

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