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A Convergent C9 Continuous Binary Non-Stationary Subdivision Technique

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Abstract

This paper comprises a smooth limiting curve having C9 continuity using a non-stationary binary six-point approximating subdivision technique. The proposed technique is more efficient and produces more smooth results having a very large domain as compared with its stationary counterpart.

Keywords: Refinement • Convergence • Stationary • Limit function • Convexity

Introduction

In the area of computer aided geometric design (CAGD), subdivision technique has attracted the attention because the results produce by subdivision techniques are highly efficient and effective. In subdivision method, calculations can be done with very less cost and the problems occur in calculations like complications are negligible. Subdivision is based on initial discrete set of control points which connects one another by a relation. A refinement rule is defined and is recursively applied on the chosen points to get the required form of curves or surfaces.

Subdivision (SD) techniques are efficient technique for producing arcs and shapes from distinct set of control points. The essential techniques for applications are the techniques for surfaces, yet these may produce curves involving a fundamental technique for the structure, study and comprehension of technique creating surfaces. Subdivision techniques are broadly utilized in numerous zones including CAGD, Graphics and related fields. The SD gained a huge popularity in CAGD due to its greater effectiveness and small multifaceted nature of calculation. A univariate SD process characterizes a bend as the point of confinement of a grouping of refinements performed on an underlying polyline. The greater part of the current univariate SD technique is paired, stationary and rectilinear.

The idea of SD was firstly proposed by De Rahm. Due to numerous generalizations and analysis of SD, added more important points which increases its perfectness in CAGD. Most of the present stationary, linear and binary SD techniques are univariate. Currently many techniques are there which cannot be useful for the preservation of monotonicity.

Many papers have been published among last decade which gave power to the subdivision family. We can separate the SD techniques into various kinds in various behaviors. Now we are considering stationary and non-stationary subdivision techniques. The arrangement of mask decides the subdivision formula at k-th level and is known as the kth level cover. If the cover is autonomous of k-th level, the SD technique is known as the stationary SD technique otherwise known as non-stationary. The stationary SD technique only generate the algebraic polynomial but non-stationary part can reproduce cardioid, astride and conic section which is not possible in the stationary case.

The smoothness, preservation of shape and the supporting size of the tension parameter are the main issues occur at the time of implementation of SD technique. Also, the problems may occur in the analysis of the mathematical structure produced by the algorithm of SD. These issues in SD can be resolved easily using different ways. To verify the convergence of the technique, asymptotical equivalence relation is helpful. SD technique form smooth curves and surfaces using iterative refining procedure on initial points. The area of utilization of SD is very wide such as computer graphics, computer animations and processing of image are mainly concerned with SD. Computer aided geometric design (CAGD) plays an important rule to make the physical model for two and three-dimensional objects. It is used to make curves and figures in two-dimensional (2D) space and create surfaces in three-dimensional (3D) space. We can observe its influence in geological and medical sciences as it plays a vital rule in image processing and geological system of information.

Cai [1] discussed the four points interpolating SD technique that generates the curve of C2 continuity. To meet different needs Hernandez et al. [2] proposed several SD techniques which gives the continuity of different order and preserve the shape of designing curve. Daniel and shunmugaraj [3] introduced the non-stationary technique for curve designing and the given technique has large variety in domain as compared with the stationary one. For the creation of smooth bends, the variation in binary 4-point interpolating technique was given by Augsdorfer et al. [4]. Siddiqi and Rehan [5] proposed a new formula for corner cutting SD technique which give rise to curve of C1 continuity. Deng and Wang [6] proposed a method in which they used a geometric approach for the generation of C1 continuous and for the generation of SD arcs. After that they used initial points and their curvature vector to get the figure of curve. A new ternary six-point SD technique with the shape preserving operator was given by Mustafa and Ashraf [7] having C2 continuity. Floater [8] defined the order of the approximation for four-point interpolatory curve SD which based on the property of locally cubical polynomial. He interpolates by the initial spaced points and a limit curve is obtained with fourth derivative bounded. Hao at el. [9] proved the higher order continuity of the six-point approximating binary SD technique. In CAGD, for the generation of smooth curves passing through the polygonal meshes, an interpolating SD technique was given by Sharon and Dyn [10]. They also gave the idea of reduction of bivariate SD technique to a finite number of univariate, non-stationary and interpolatory SD technique. Siddiqi and Younis [11] construct m-point binary approximating SD technique for smooth curve. They also discussed the reproduction of polynomial and basis function for proposed technique. Shujin et al. [12] presented new interpolating SD technique which is C1 continuous and the technique produces surfaces at distinct as well as regular points which defined an interpolating subdivision

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scheme for the design of the smooth curve in CAGD. To generate C3 continuity and keeps the divided difference of second order. Tan et al. [13] gave a method which is based on new 4-point SD technique and the generating curve was monotonicity and convexity preserving. Tan et al. [14] discussed the preservation of convexity of five-point binary SD technique with a single constraint, uniform convergence and Ck continuity also investigate by them. With the help of hyperbolic function Siddiqi et al. [15] developed a binary three-point and four-point non-stationary SD technique. Convergence of technique was checked using asymptotically equivalent relation method. Conti et al. [16] showed the asymptotical equivalence between the non-stationary and stationary convergent technique. They introduced a new notation termed as asymptotical similarity which is coarser than asymptotical equivalence. Siddiqi et al. [17] introduce a new non-stationary binary six-point SD technique in hyperbolic form with the help of hyperbolic function and the proposed technique is consistent to generate the smoother curve. Using the characteristic of good smoothness and conic precision of SD plots Novara and Romani [18] give rise a new family of non-stationary SD techniques. Rehan and Siddiqi [19] introduced family of different point SD techniques and generates the smoother arcs of order C1and C2 respectively. The continuity properties of these schemes were analyzed by the generating function and eigenvalue analysis. Siddiqi and Noreen [20] analyzed the condition that when does the control points become convex strictly and give C2 smooth curve. Tan et al. [21] presented the initial control parameters which are differ from each other and generate a very vast variety of C3 continuous limiting curve. Conti et al. [22] developed the idea that how a non-stationary technique is asymptotically equivalent to the similar stationary one by using the elementary limit function of sequence. They discussed the exponential polynomial reproduction under the assumption of similar asymptote. Rehan and Sabri [23] introduced a technique which yield approximation arcs up to continuity of order C3 and interpolating continuity of order C1. For checking the derivative continuity of the technique polynomial Laurent method is used. Mustafa and Hameed [24] presented the family of parameter dependent univariate and bivariate SD techniques. Their technique provides a way to preserve the image of the limit curve locally. A generalized method was developed by Akram et al. [25] to preserve the shape of the object with the help of initial tension Parameter. The new technique also preserves both the monotonicity and convexity of the object. Li and Chang [26] constructed a generalized form of non-uniform four-point interpolatory SD surface for arbitrary mashes. It was first time in SD for the generalization of non-uniform SD surfaces with extraordinary points.

Stationary six-point subdivision technique

The refinement rule of stationary binary six-point subdivision technique is defined as

$$n_{2i}^{k+1} = g n_{i-2}^k + e n_{i-1}^k + d n_i^k + c n_{i+1}^k + b n_{i+2}^k + a n_{i+3}^k$$
$$n_{2i+1}^{k} = a n_{i-2}^k + b n_{i-1}^k + c n_i^k + d n_{i+1}^k + e n_{i+2}^k + g n_{i+3}^k,$$

Where $n^0 = \{n_i^0\}$ are at zero level be a part of discrete set of control points. The technique coefficients satisfy the relation $a+b+c+d+e+g=1$, where

$$a = \frac{11}{1024}, b = \frac{165}{124}, c = \frac{231}{512}, d = \frac{165}{512}, e = \frac{55}{1024}, g = \frac{1}{1024}.$$

The technique (2.1) has the smoothness

- C^3- continuous when $0.0062 < g < 0.0096$
- C^4-continous when $-0.0043 < g < -0.0029$
- C^5-continous when $-0.0078 < g < -0.0029$
- C^6-continous when $-0.0057 < g < -0.0053$
- C^7-continous when $0.0019 < g < 0.0059$
- C^8-continous when $0 < g < 0.0019$
- C^9-continous when $= \frac{1}{1024}$.

Non-stationary six-point subdivision technique

The refinement rule of non-stationary binary six-points subdivision technique are defined as

$$n_{2i}^{k+1} = z_0^k n_{i-2}^k + z_1^k n_{i-1}^k + z_2^k n_i^k + z_3^k n_{i+1}^k + z_4^k n_{i+2}^k + z_5^k n_{i+3}^k$$
$$n_{2i+1}^{k} = z_5^k n_{i-2}^k + z_4^k n_{i-1}^k + z_3^k n_i^k + z_2^k n_{i+1}^k + z_1^k n_{i+2}^k + z_0^k n_{i+3}^k \quad (3.1)$$

Where $n^0 = \{n_i^0\}$ are at zero level be a part of discrete set of control points. The technique coefficients satisfy the relation $z_0^k + z_1^k + z_2^k + z_3^k + z_4^k + z_5^k = 1$.

The mask of newly generated non-stationary binary subdivision technique are represented as

$$z_0^k = n(g^{k+1}),\ z_1^k = 19n(g^{k+1}) + \frac{9}{256}, z_2^k = -6n(g^{k+1}) + \frac{21}{64}$$

$$z_3^k = -42n(g^{k+1}) + \frac{63}{128}, z_4^k = 21n(g^{k+1}) + \frac{9}{64},\ z_5^k = 7n(g^{k+1}) + \frac{1}{256},$$

Where,

$$n(g^{k+1}) = \frac{1}{2[(g^{k+1})^2 - 64]}, \quad (3.2)$$

With

$$g^{k+1} = \sqrt{g^k + 552} \text{ And } g^0 \epsilon [-552, -488) \cup (-488, \infty), \quad (3.3)$$

Using (3.3), we can calculate the coefficients g_i^k separately at different vlaues k with initial parameters $g^0 \epsilon [-552, 488) \cup (-488, \infty)$.

The value of the initial parameter starting from $g^0 \geq -552$ and gives $g^k + 552 \geq 0, \forall k \epsilon z_+$, so that g^{k+1} is well organized. A wide range of definitions is helpful to get the sufficient variation in the form of continuous curve.

Remark 1. From equation (3.3) and Figure1shows that, the shape of the smooth curve changes a lot at first as we increase the initial tension parameter g^0 within it define domain and then curve moves towards the approximation when g^0⟶∞.

Remark 2. Equation (3.3) elaborates that we can get the stationary sequence {g^k} if the initial chosen value of g is 24 and at the end g^k is also 24. In this case the define non-stationary SD technique becomes the stationary one.

Analysis of Smoothness

In this section, we analysed the smoothness of the new approximating technique (3.2) which is non-stationary and gives the C9 continuous curves for any value of the initial parameter g^0 within the given range defined in equation (3.3).

Theorem 1

The non-stationary binary six-point SD technique defined in equation (3.1) is asymptotically equivalent to the stationary technique defined in equation (2.1). Hence, it gives the smooth curves having C9continuity for the large range of parameter for g^0 ϵ[-552,-488)∪(-488,∞).

Proof

To prove the theorem, for newly developed non-stationary SD technique which have the convergence to C9 limit curve, we required its ninth divided difference mask. The mask of the technique is

$$m^k = \Big[n(g^{k+1}), 7n(g^{k+1}) + \frac{1}{256}, 19n(g^{k+1}) + \frac{9}{256}, 21n(g^{k+1}) + \frac{9}{64}, -6n(g^{k+1}) + \frac{21}{64}, -42n(g^{k+1}) + \frac{63}{128}, -6n(g^{k+1}) + \frac{21}{64}, 21n(g^{k+1}) + \frac{9}{64}, 19n(g^{k+1}) + \frac{9}{256}, 7n(g^{k+1}) + \frac{1}{256}, n(g^{k+1})\Big]$$

Then its 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th and 9th divided difference mask becomes

$$l_1^k = 2\Big[n(g^{k+1}), 6n(g^{k+1}) + \frac{1}{256}, 13n(g^{k+1}) + \frac{1}{32}, 8n(g^{k+1}) + \frac{7}{64}, -14n(g^{k+1}) + \frac{7}{32}, -28n(g^{k+1}) + \frac{35}{128}, -14n(g^{k+1}) + \frac{7}{32}, 8n(g^{k+1}) + \frac{7}{64}, 13n(g^{k+1}) + \frac{1}{32}, 6n(g^{k+1}) + \frac{1}{256}, n(g^{k+1})\Big]$$

$$l_2^k = 4\Big[n(g^{k+1}), 5n(g^{k+1}) + \frac{1}{256}, 8n(g^{k+1}) + \frac{7}{256}, \frac{21}{256}, -14n(g^{k+1}) + \frac{35}{256}, -14n(g^{k+1}) + \frac{35}{256}, \frac{21}{256}, 8n(g^{k+1}) + \frac{7}{256}, 5n(g^{k+1}) + \frac{1}{256}, n(g^{k+1})\Big]$$

$$l_3^k = 8\left[n(g^{k+1}), 4n(g^{k+1}) + \frac{1}{256}, 4n(g^{k+1}) + \frac{3}{128}, -4n(g^{k+1}) + \frac{15}{256}, -10n(g^{k+1}) + \frac{5}{128}, -4n(g^{k+1}) + \frac{15}{256}, 4n(g^{k+1}) + \frac{3}{128}, 4n(g^{k+1}) + \frac{1}{256}, n(g^{k+1})\right]$$

$$l_4^k = 16\left[n(g^{k+1}), 3n(g^{k+1}) + \frac{1}{256}, n(g^{k+1}) + \frac{5}{256}, -5n(g^{k+1}) + \frac{5}{128}, -5n(g^{k+1}) + \frac{5}{128}, -n(g^{k+1}) + \frac{5}{256}, 3n(g^{k+1}) + \frac{1}{256}, n(g^{k+1})\right]$$

$$l_5^k = 32\left[n(g^{k+1}), 2n(g^{k+1}) + \frac{1}{256}, -n(g^{k+1}) + \frac{1}{64}, -4n(g^{k+1}) + \frac{3}{128}, -n(g^{k+1}) + \frac{1}{64}, 2n(g^{k+1}) + \frac{1}{256}, n(g^{k+1})\right]$$

$$l_6^k = 64\left[n(g^{k+1}), n(g^{k+1}) + \frac{1}{256}, -2n(g^{k+1}) + \frac{3}{256}, -2n(g^{k+1}) + \frac{3}{256}, n(g^{k+1}) + \frac{1}{256}, n(g^{k+1})\right]$$

$$l_8^k = 256\left[n(g^{k+1}), -n(g^{k+1}) + \frac{1}{256}, -n(g^{k+1}) + \frac{1}{256}, n(g^{k+1})\right]$$

$$l_9^k = 512\left[n(g^{k+1}), -2n(g^{k+1}) + \frac{1}{256}, n(g^{k+1})\right]$$

Remark 2 imply that,

$l_9^\infty = l_9^k = 512[\frac{1}{1024}, \frac{1}{512}, \frac{1}{1024}]$ (4.1)

Equation (4.1) consists of the coefficients of the ninth divided difference of the stationary SD technique defined in (2.1). The case shows that the stationary technique has C^9 continuity. The associated technique with l_9^∞ will be C^9 continuous. For checking the two techniques are equivalently asymptotic, we need to prove,

$$\sum_{k=0}^{\infty} \|l_9^k - l_9^\infty\|_\infty < +\infty. \ (4.2)$$

Since,

$$l_9^k - l_9^\infty = 512\left[n(g^{k+1}) - \frac{1}{1024}, 2\left(\frac{1}{1024} - n(g^{k+1})\right), n(g^{k+1}) - \frac{1}{1024}\right]$$

$$\|l_9^k - l_9^\infty\|_\infty = 512 max\left\{2\left|n(g^{k+1}) - \frac{1}{1024}\right|, 2\left|\frac{1}{1024} - n(g^{k+1})\right|\right\}$$

$$=1024\left|n(g^{k+1}) - \frac{1}{1024}\right|$$

Now for our desired result we need to prove smoothness of the series

$\sum_{k=0}^{\infty} \left|n(g^{k+1}) - \frac{1}{1024}\right|$. (4.3)

that depends on the function $n(g^{k+1})$. Now, as $n(g^{k+1})$ is expressed in terms of g^{k+1}, we will see the performance of g^{k+1} that changes within the interval [0, +∞).Since

$n(g^{k+1}) - \frac{1}{1024} = 0$ $g^{k+1} = 24$(i.e.,$g^k = 24$. (4.4)

$n(g^{k+1}) - \frac{1}{1024} > 0$ $g^{k+1} \epsilon (8,24)$. (4.5)

$n(g^{k+1}) - \frac{1}{1024} < 0$ $g^{k+1} \epsilon$ [0,8)∪(24,∞) . (4.6)

Now, we discuss the following cases to verify the smoothness of (4.3).

Case 1: g^0=24 (i.e.,g^{k+1}=24)

Thus, we have

$\|l_9^k - l_9^\infty\|_\infty = 0.$

Smoothness of (4.3) is verified.

Case 2: $g^0 \epsilon (-488, 24)$ $(i.e., g^k \epsilon (8,24))$

In this case,

$\|l_9^k - l_9^\infty\|_\infty$ =1024$\left|n(g^{k+1}) - \frac{1}{1024}\right|$

It is sufficient to prove,

$\sum_{k=0}^{\infty} \left|n(g^{k+1}) - \frac{1}{1024}\right| = \sum_{k=0}^{\infty} \left(\frac{1}{2(g^{k+1})^2 - 64} - \frac{1}{1024}\right) < +\infty.$

To verify the result ratio test is useful. Since $n(g^{k+1}) - \frac{1}{1024} > 0$ and the sequence $\{g^k\}_{k\epsilon N}$ for the case must be strictly increasing using remark (2), we have

$$\frac{\left(\frac{1}{2(g^{k+2})^2 - 64} - \frac{1}{1024}\right)}{\left(\frac{1}{2(g^{k+1})^2 - 64} - \frac{1}{1024}\right)} < 1.$$

And successively in this way ratio test has been applied and convergence of (4.3) is satisfied.

Case3: $g^0 \epsilon [-552, -488) \cup (24, \infty)$ **(i.e.,** $g^{k+1} \epsilon [0,8) \cup (24, \infty)$)

For this case,

$\|l_9^k - l_9^\infty\|_\infty = 1024\left|n(g^{k+1}) - \frac{1}{1024}\right|$

Next to prove,

$\sum_{k=0}^{+\infty} \left|n(g^{k+1}) - \frac{1}{1024}\right| = \sum_{k=0}^{+\infty} \left(\frac{1}{2(g^{k+1})^2 - 64} - \frac{1}{1024}\right) < +\infty.$

We discuss the following subcase.

Case 3.1: $g^0 \epsilon (24, \infty)$ **(i.e.,** $g^{k+1} \epsilon (24, \infty)$ **)**. By remark (2), the sequence $\{g^k\}_{k\epsilon N}$ is strictly decreasing for this case and using the ratio test, we have

$$\frac{\left(\frac{1}{2(g^{k+2})^2 - 64} - \frac{1}{1024}\right)}{\left(\frac{1}{2(g^{k+1})^2 - 64} - \frac{1}{1024}\right)} < 1.$$

Smoothness of (4.3) is proved.

Case 3.2: $g^0 \epsilon [-552, -488) (i.e., g^1 \in [0,8))$ and $g^{k+1} \in (3, \infty), k = 2,3,4, \ldots$ which return back into case 2 and smoothness of (4.3) is proved

When we combined these three cases the non-stationary SD technique defined by the coefficients in (3.1) is asymptotically equivalent to the stationary technique in (2.1) and generates the C^9 continuous limiting curve.

This completes the proof of theorem 1.

Property of the proposed technique

For initial data of the proposed non-stationary technique, the basis function is denoted by B and it is demonstrating as a limit function.

$n_j^0 = \{0, j \neq 01, j = 0$ (5.1)

Theorem 2

The symmetry of the basis function of the technique defined in (5.1) is about y-axis.

Proof

First, we define, $F_m = \left\{\frac{j}{2^m} j\epsilon z\right\}$, such that $B\left(\frac{j}{2^m}\right) = n_j^m$ for all $j\epsilon z$ and mathematical induction is applied to prove the theorem on when $m = 0$, we have $B(j) = n_j^0 = n_{-j}^0 = B(-j)$and this implies that $B\left(\frac{j}{2^m}\right) = B\left(\frac{-j}{2^m}\right)$ $\forall$ $j\epsilon z$. It can also be written as $B\left(\frac{j}{2^{m+1}}\right) = B\left(\frac{-j}{2^{m+1}}\right)$ $\forall$ $j\epsilon z$.

Now,

$$B\left(\frac{2^{j+1}}{2^{n+1}}\right) = g n_{j-2}^k + e n_{j-1}^k + d n_j^k + c n_{j+1}^k + b n_{j+2}^k + a n_{j+3}^k$$

$$= gB\left(\frac{j-2}{2^m}\right) + eB\left(\frac{j-1}{2^m}\right) + dB\left(\frac{j}{2^m}\right) + cB\left(\frac{j+1}{2^m}\right) + bB\left(\frac{j+2}{2^m}\right) + aB\left(\frac{j+3}{2^m}\right)$$

$$= gB\left(\frac{-j+2}{2^m}\right) + eB\left(\frac{-j+1}{2^m}\right) + dB\left(\frac{-j}{2^m}\right) + cB\left(\frac{-j-1}{2^m}\right) + bB\left(\frac{-j-2}{2^m}\right) + aB\left(\frac{-j-3}{2^m}\right)$$

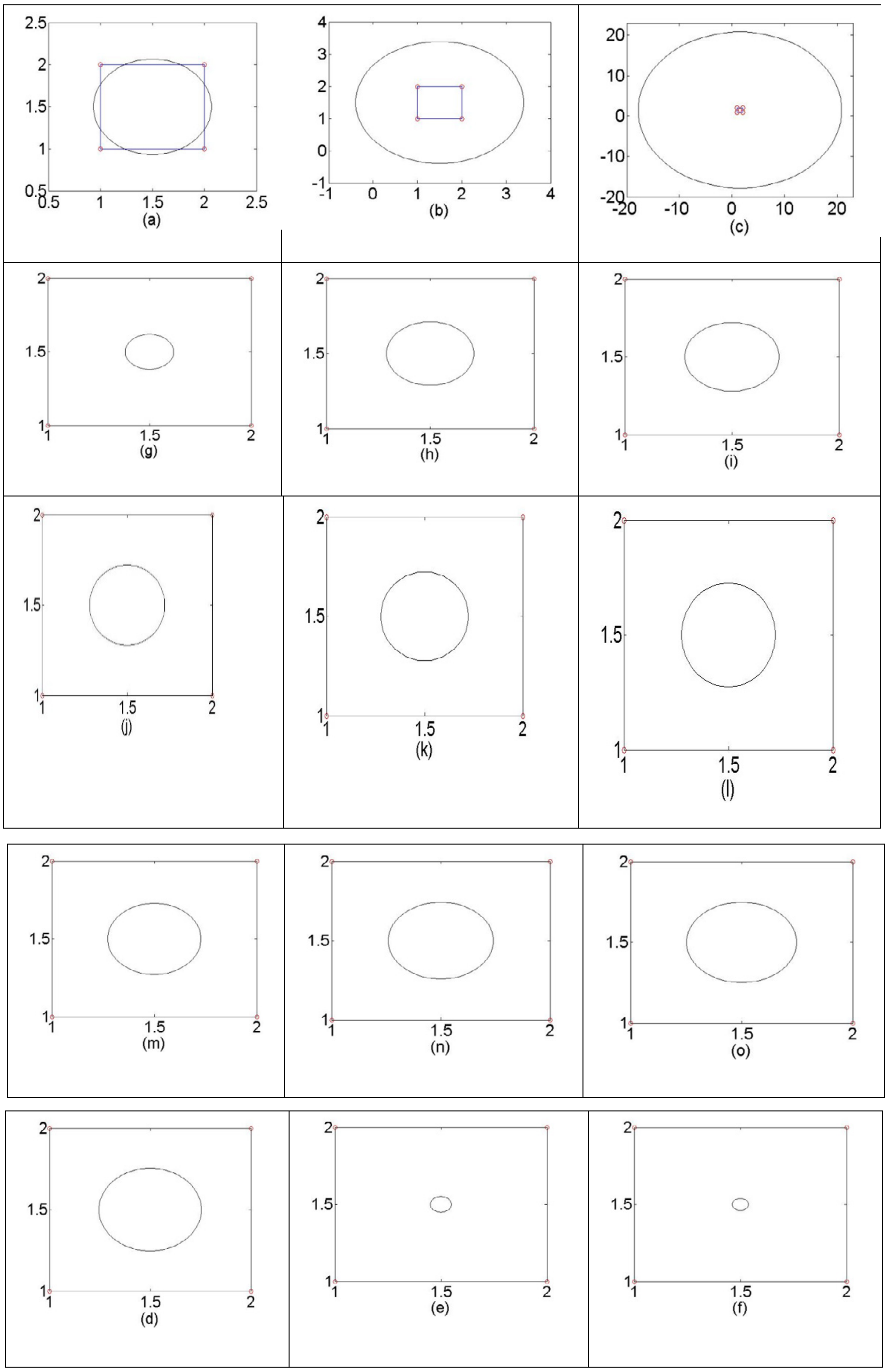

Figure 1. Generate the wide range of C^9 limiting curve by using the developed technique (3.2) for different values of parameter. (a) $g^0 = -552$, (b) $g^0 = -500$, (c) $g^0 = -487$, (d) $g^0 = -450$, (e) $g^0 = -425$, (f) $g^0 = -400$, (g) $g^0 = -350$, (h) $g^0 = -100$, (i) $g^0 = -10$, (j) $g^0 = 0$ (k) $g^0 = 24$, (l) $g^0 = 50$, (m) $g^0 = 100$, (n) $g^0 = 500$, (o) $g^0 = 1000$.

$$= B\left(\frac{-2j-1}{2^m}\right), \forall j \epsilon z$$

Thus, $B\left(\frac{j}{2^m}\right) = B\left(\frac{-j}{2^m}\right), \forall j \epsilon z$.which is the required result.

Graphical View

In this section, we would like to elaborate the benefits of the developed technique (3.2) with the help of an example. As discussed in section (3), the generated curves expand from inner side to outside of discrete polygonal when $g^0 \to \infty$.

In Figure 1, The geometric behavior of the non- stationary binary six-point approximating SD technique is illustrated in Figure 1 and the technique generates a smooth limiting curve of C^9 continuous for the shape control parameter,

(a) $g^0 = -552$, (b) $g^0 = -500$, (c) $g^0 = -487$, (d) $g^0 = -450$, (e) $g^0 = -425$, (f) $g^0 = -400$, (g) $g^0 = -350$, (h) $g^0 = -100$, (i) $g^0 = -10$, (j) $g^0 = 0$ 0,

(k) $g^0 = 24$, (l) $g^0 = 50$, (m) $g^0 = 100$, (n) $g^0 = 500$, (o) $g^0 = 1000$

Conclusion

This study strongly conclude that a new non-stationary binary six-point approximating subdivision technique is helpful in producing the smooth curve with C^9 continuity. The power of the technique is that it gives smooth curves for the larger domain which is useful in geometric design for making the character flexible.

Conflicts of Interest

Authors wish to declare that there is no conflict of interest for this study.

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