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2-Dimensional Algebras Application to Jordan, G-Associative and Hom-Associative Algebras

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Abstract

We classify, up to isomorphism, the 2-dimensional algebras over a field $\mathbb{K}$. We focuse also on the case of characteristic 2, identifying the matrices of $GL(2, \mathbb{F}_2)$ with the elements of the symmetric group Σ_3. The classification is then given by the study of the orbits of this group on a 3-dimensional plane, viewed as a Fano plane. As applications, we establish classifications of Jordan algebras, algebras of Lie type or Hom-Associative algebras.

Keywords: 2-Dimensional algebras; Classification; Hom-associative algebras

Introduction

An algebra $\mathcal{A}$ over a field $\mathbb{K}$ is $\mathbb{K}$-vector space equipped with a product which corresponds to a bilinear map on $\mathcal{A}$ with values in $\mathcal{A}$. For a given dimension, one of the basic problems is the determination up to linear isomorphism of all these algebras. Sub classes of algebras where widely studied. These subclasses where often obtained setting a quadratic relation on μ. Among other examples of such classes are Lie algebras (in this case μ is skewsymmetric and satisfies Jacodi identity), associative algebras, Lie-admissibles algebras, Pre-Lie algebras in particular. In all these examples, classifications where established in a general frame work, that is, with no other hypothesis on these classes and only in very small dimensions. For example for Lie algebras, we know the general classifications up to the dimension 6. In bigger dimension we impose additional algebraic properties if we hope to continue this classification. For example simple Lie algebras are fully classified since the work of Killing and Cartan, in any dimension. Unfortunately it is more and less the only solved case. If we consider complexe nilpotent Lie algebras, the classification is known only up to the dimension 7. It is the same for the associatives algebras. If we are only interested in general algebras, the only known cases are the dimension 2 and 3. It is true that the problem is equivalent to the classification of tensors of type (2,1) on a finite dimensional vector space. We are then facing to a basic multilinear algebra problem which is subject to a lack of informations on the tensors.

Here we reconsider this problem from the beginning, that is in dimension 2. This work is certainly not the first one of the subject. There is for example the work of Petersson. Our approach is not similar. We are not fully interested by the classification up to isomorphism but by the determination of subclasses, minimal in a certain sense, which are invariant up to isomorphism. The motivation comes from the constatation of what happen in greater dimensions for nilpotent Lie algebras for example In this case, the classification is established in dimension 7 but quasi unusable in its present forme. This means that if we have a precise example of nilpotent Lie algebra of this dimension, it is long and fastidious to recognize it in the given list because most of the time it is not adapted to the invariants used to established the classification. Moreover the length of the list can be puzzling. In greater dimensions, the number of isomorphy classes, the need to write invariant parametrized families seems to be an unrealistic goal. Hence the idea to reduce the classification problem to a determination of invariant classes. This is the aim of this work. However we will established the link with Petersson's work. Our approach is quite basic. In characteristic different from 2, we decompose a tensor μ as a skewsymmetric and symmetric one. Since the skewsymmetric case is elementary, we classify those which are symmetric modulo the automorphism group of the associated skeysymmetric law. In characteristic 2, the problem is equivalent to the determination of the orbits of the Fano plane modulo the symmetric group. Finally, we use these results to describe or find again certain classes of algebras whose a direct approach is rather difficult. In particular, we determine the 2-dimensional Jordan algebras and we find again the results of ref. [1], the G-associative algebras and the Hom-associative algebras.

We have begun the study of the determination of general algebras in ref. [2] which was specially an introduction to a more precise work developed in this paper but with the same idea to describe "minimal" families invariant by isomorphism rather than a precise list for which the use is difficult. Recently, we were acquainted with the work of Pertersson, based on an Kaplansky result which permits to describe all the algebras from some unital algebras and to give isomorphism criteria. We try in this paper to look our description in a Petersson point of view. We note also a recent work, on the same subject of H. Ahmed, U. Bekbaev and I. Rakhimov [3].

Generalities

Let $\mathbb{K}$ be a field whose characteristic will be precise later. An algebra over a field $\mathbb{K}$ is a $\mathbb{K}$ -vector space V with a multiplication given by a bilinear map

$$\mu: V \times V \rightarrow V.$$

We denote by $A=(V,\mu)$ a $\mathbb{K}$-algebra structure on V with multiplication μ. Throughout this paper we fix the vector space V. Since we are interested by the 2-dimensional case we could assume that $V=\mathbb{K}^2$. Two $\mathbb{K}$-algebras $A=(V,\mu)$ and $A'=(V,\mu')$ are isomorphic if there is a linear isomorphism,

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f:$V\to V$

such as;

$f(\mu(X,Y))=\mu'(f(X), f(Y))$,

for all $X,Y\in V$. The classification of 2-dimensional $\mathbb{K}$-algebras is then equivalent to the classification of bilinear maps on $V=\mathbb{K}^2$ with values in V. Let $\{e_1,e_2\}$ be a fixed basis of V. A general bilinear map μ has the following expression:

$$\begin{cases}\mu(e_1,e_1)=\alpha_1e_1+\beta_1e_2,\\ \mu(e_1,e_2)=\alpha_2e_1+\beta_2e_2,\\ \mu(e_2,e_1)=\alpha_3e_1+\beta_3e_2,\\ \mu(e_2,e_2)=\alpha_4e_1+\beta_4e_2,\end{cases}$$

and it is defined by 8 parameters. Let f be a linear isomorphism of V. In the given basis, its matrix M is non degenerate. If we put:

$$M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

Then,

$$M^{-1}=\frac{1}{\Delta}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

with $\Delta=ad-bc\neq0$. The isomorphic multiplication.

$$\mu'=f^{-1}\circ\mu\circ(f\times f)$$

Satisfies,

$$\begin{cases}\mu'(e_1,e_1)=\alpha'_1e_1+\beta'_1e_2,\\ \mu'(e_1,e_2)=\alpha'_2e_1+\beta'_2e_2,\\ \mu'(e_2,e_1)=\alpha'_3e_1+\beta'_3e_2,\\ \mu'(e_2,e_2)=\alpha'_4e_1+\beta'_4e_2,\end{cases}$$

With,

$$\begin{cases}\alpha'_1=(a^2\alpha_1+ac\alpha_2+ac\alpha_3+c^2\alpha_4)\frac{d}{\Delta}-(a^2\beta_1+ac\beta_2+ac\beta_3+c^2\beta_4)\frac{b}{\Delta}\\ \beta'_1=-(a^2\alpha_1+ac\alpha_2+ac\alpha_3+c^2\alpha_4)\frac{c}{\Delta}+(a^2\beta_1+ac\beta_2+ac\beta_3+c^2\beta_4)\frac{a}{\Delta}\\ \alpha'_2=(ab\alpha_1+ad\alpha_2+bc\alpha_3+cd\alpha_4)\frac{d}{\Delta}-(ab\beta_1+ad\beta_2+bc\beta_3+cd\beta_4)\frac{b}{\Delta}\\ \beta'_2=-(ab\alpha_1+ad\alpha_2+bc\alpha_3+cd\alpha_4)\frac{c}{\Delta}+(ab\beta_1+ad\beta_2+bc\beta_3+cd\beta_4)\frac{a}{\Delta}\\ \alpha'_3=(ab\alpha_1+bc\alpha_2+ad\alpha_3+cd\alpha_4)\frac{d}{\Delta}-(ab\beta_1+bc\beta_2+ad\beta_3+cd\beta_4)\frac{b}{\Delta}\\ \beta'_3=-(ab\alpha_1+bc\alpha_2+ad\alpha_3+cd\alpha_4)\frac{c}{\Delta}+(ab\beta_1+bc\beta_2+ad\beta_3+cd\beta_4)\frac{a}{\Delta}\\ \alpha'_4=(b^2\alpha_1+bd\alpha_2+bd\alpha_3+d^2\alpha_4)\frac{d}{\Delta}-(b^2\beta_1+bd\beta_2+bd\beta_3+d^2\beta_4)\frac{b}{\Delta}\\ \beta'_4=-(b^2\alpha_1+bd\alpha_2+bd\alpha_3+d^2\alpha_4)\frac{c}{\Delta}+(b^2\beta_1+bd\beta_2+bd\beta_3+d^2\beta_4)\frac{a}{\Delta}\end{cases} \quad (1)$$

These formulae describe an action of the linear group $GL(2,\mathbb{K})$ on $\mathbb{K}^8$ parameterized by the structure constants (α_i, β_i), i=1,2,3,4 and the problem of classification consists in describing an element of each orbit.

Algebras Over a Field of Characteristic Different from 2

We assume in this section that char($\mathbb{K}$)≠2. We consider the bilinear map μ_a and μ_s given by:

$$\mu_a(X,Y)=\frac{\mu(X,Y)-\mu(Y,X)}{2},\quad \mu_s(X,Y)=\frac{\mu(X,Y)+\mu(Y,X)}{2}$$

for all $X,Y\in V$. The multiplication μ_a is skew-symmetric and it is a Lie multiplication (any skew-symmetric bilinear application in $\mathbb{K}^2$ is a Lie bracket). It is isomorphic to one of the following:

1. $\mu_a^1(e_1,e_2)=e_1$,
2. $\mu_a^2=0$.

In fact, if μ_a is not trivial, thus $\mu_a(e_1,e_2)=\alpha e_1+\beta e_2$. If $\alpha\neq0$, we consider the change of basis:

$$e'_1=\alpha e_1+\beta e_2,\quad e'_2=\alpha^{-1}e_2$$

We have $\mu_a(e'_1,e'_2)=\mu_a(\alpha e_1+\beta e_2,\alpha^{-1}e_2)=\mu_a(e_1,e_2)=\alpha e_1+\beta e_2=e'_1$.

If $\alpha=0$, thus $\beta\neq0$ and we take:

$$e'_1=e_2,\quad e'_2=-\beta^{-1}e_1.$$

This gives $\mu_a(e'_1,e'_2)=\mu_a(e_2,-\beta^{-1}e_1)=\beta^{-1}\beta e_2=e_2=e'_1$. In any case, if $\mu_a\neq0$, then it is isomorphic to μ_a^1.

Case $\mu_a^1(e_1,e_2)=e_1$

An automorphism of the Lie algebra (A,μ_a^1) is a linear isomorphism $f\in GL(2,\mathbb{K})$ such that:

$$f(\mu_a^1(X,Y))=\mu_a^1(f(X),f(Y))$$

for every $X,Y\in A$. The set of automorphisms of this Lie algebra is denoted by $Aut(\mu_a^1)$.

Lemma 1: We have:

$$Aut(\mu_a^1)=\left\{M=\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix},\ a,b\in\mathbb{K},\ a\neq0\right\}.$$

Proof. In fact, assume that $M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is the matrix of the automorphism f in the given basis $\{e_1,e_2\}$. Then,

$$f(\mu_a^1(e_1,e_2))=f(e_1)=ae_1+ce_2,$$

and,

$$\mu_a^1(f(e_1),f(e_2))=\mu_a^1(ae_1+ce_2,be_1+de_2)=(ad-bc)e_1.$$

Then,

$c=0$, $a=ad$.

But $detM=ad\neq0$ so $a=ad$ implies that $d=1$ This gives the lemma.

Let μ be a general multiplication of 2-dimensional $\mathbb{K}$-algebra such that μ_a is isomorphic to μ_a^1. It is isomorphic to a the bilinear map (always denoted by μ) whose structural constants are given by:

$$\begin{cases}\mu(e_1,e_1)=\alpha_1e_1+\beta_1e_2,\\ \mu(e_1,e_2)=(\alpha_2+1)e_1+\beta_2e_2,\\ \mu(e_2,e_1)=(\alpha_2-1)e_1+\beta_2e_2,\\ \mu(e_2,e_2)=\alpha_4e_1+\beta_4e_2.\end{cases}$$

The classification, up to isomorphism, of the Lie algebras (V,μ) such that μ_a is isomorphic to μ_a^1 is equivalent to the classification up an isomorphism belonging to $Aut(\mu_a^1)$ of the abelian algebras isomorphic to:

$$\begin{cases}\mu_s(e_1,e_1)=\alpha_1e_1+\beta_1e_2,\\ \mu_s(e_1,e_2)=\mu_s(e_2,e_1)=\alpha_2e_1+\beta_2e_2,\\ \mu_s(e_2,e_2)=\alpha_4e_1+\beta_4e_2,\end{cases}$$

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In this case (1) is reduced to:

$$\begin{cases} \alpha'_1 = a\alpha_1 - ab\beta_1, \\ \beta'_1 = a^2\beta_1, \\ \alpha'_2 = \alpha'_3 = b\alpha_1 + \alpha_2 - b^2\beta_1 - b\beta_2, \\ \beta'_2 = \beta'_3 = ab\beta_1 + a\beta_2, \\ \alpha'_4 = \left(b^2\alpha_1 + 2b\alpha_2 + \alpha_4 - b^3\beta_1 - 2b^2\beta_2 - b\beta_4\right)\frac{1}{a}, \\ \beta'_4 = b^2\beta_1 + 2b\beta_2 + \beta_4. \end{cases} \quad (2)$$

1. Assume that $\beta_1 \neq 0$.

- Suppose that $\mathbb{K}$ is algebraically closed and consider the isomorphism $\begin{pmatrix} \sqrt{\beta_1} & \frac{\alpha_1}{\beta_1} \\ 0 & 1 \end{pmatrix}$. The isomorphic algebra is such that $\alpha'_1 = 0$ and $\beta'_1 = 1$. We deduce that in this case μ_s is isomorphic to:

$$\begin{cases} \mu_s(e_1,e_1) = e_2, \\ \mu_s(e_1,e_2) = \mu_s(e_2,e_1) = \alpha_2 e_1 + \beta_2 e_2, \\ \mu_s(e_2,e_2) = \alpha_4 e_1 + \beta_4 e_2. \end{cases}$$

Then μ is isomorphic to:

$$\begin{cases} \mu^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_1,e_1) = e_2, \\ \mu^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_1,e_2) = (\alpha_2+1)e_1 + \beta_2 e_2, \\ \mu^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_2,e_1) = (\alpha_2-1)e_1 + \beta_2 e_2, \\ \mu^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_2,e_2) = \alpha_4 e_1 + \beta_4 e_2, \end{cases}$$

with $\alpha_2, \beta_2, \alpha_4, \beta_4 \in \mathbb{K}$.

- If $\mathbb{K}$ is not algebraically closed (for example if $\mathbb{K}$ is a finite field), let $\mathbb{K}^{*2}$ be the multiplicative subgroup of elements a^2 with $a \in \mathbb{K}$. In this case μ is isomorphic to a Lie bracket belonging to the 4 parameters family:

with $\alpha_2, \beta_2, \alpha_4, \beta_4 \in \mathbb{K}$ and $\lambda \in \mathbb{K}/\mathbb{K}^{*2}$. For example, if $\mathbb{K}=\mathbb{R}$, then $\lambda \in \{-1,1\}$.

2. Assume $\beta_1 = 0$, $\beta_2 \neq 0$. In this case (1) is reduced to:

$$\begin{cases} \alpha'_1 = a\alpha_1, \\ \beta'_1 = 0, \\ \alpha'_2 = b\alpha_1 + \alpha_2 - b\beta_2, \\ \beta'_2 = a\beta_2, \\ \alpha'_4 = \left(b^2\alpha_1 + 2b\alpha_2 + \alpha_4 - 2b^2\beta_2 - b\beta_4\right)\frac{1}{a}, \\ \beta'_4 = 2b\beta_2 + \beta_4. \end{cases} \quad (3)$$

and taking $b = -\beta_4/2\beta_2$ and $a = \beta_2^{-1}$, we see that μ_s is isomorphic to:

$$\begin{cases} \mu_s(e_1,e_1) = \alpha_1 e_1, \\ \mu_s(e_1,e_2) = \mu_s(e_2,e_1) = \alpha_2 e_1 + e_2, \\ \mu_s(e_2,e_2) = \alpha_4 e_1. \end{cases}$$

We obtain the following multiplication, $\mathbb{K}$ being algebraically closed or not:

$$\begin{cases} \mu^2_{\alpha_1,\alpha_2,\alpha_4}(e_1,e_1) = \alpha_1 e_1, \\ \mu^2_{\alpha_1,\alpha_2,\alpha_4}(e_1,e_2) = (\alpha_2+1)e_1 + e_2, \\ \mu^2_{\alpha_1,\alpha_2,\alpha_4}(e_2,e_1) = (\alpha_2-1)e_1 + e_2, \\ \mu^2_{\alpha_1,\alpha_2,\alpha_4}(e_2,e_2) = \alpha_4 e_1. \end{cases}$$

3. Assume now that $\beta_1 = \beta_2 = 0, \alpha_1 \neq 0$. In this case (1) is reduced to:

$$\begin{cases} \alpha'_1 = \acute{a} \quad {}_1 \\ \beta'_1 = \beta'_2 = 0 \\ \alpha'_2 = \acute{b}\alpha_1 + \quad {}_2 \\ \alpha'_4 = \left(b^2\alpha_1 + 2b\alpha_2 + \alpha_4 - b\beta_4\right)\frac{1}{a} \\ \beta'_4 = \beta_4. \end{cases} \quad (4)$$

and taking $b = -\alpha_2/\alpha_1$ and $a = \alpha_1^{-1}$, we obtain $\alpha'_2 = 0$ and $\alpha'_1 = 1$. In this case, μ is isomorphic to:

$$\begin{cases} \mu^3_{\alpha_4,\beta_4}(e_1,e_1) = e_1, \\ \mu^3_{\alpha_4,\beta_4}(e_1,e_2) = e_1, \\ \mu^3_{\alpha_4,\beta_4}(e_2,e_1) = -e_1, \\ \mu^3_{\alpha_4,\beta_4}(e_2,e_2) = \alpha_4 e_1 + \beta_4 e_2. \end{cases}$$

4. Assume now that $\beta_1 = \beta_2 = 0, \alpha_1 = 0, 2\alpha_2 - \beta_4 \neq 0$. In this case, considering $b = -\alpha_4/(2\alpha_2 - \beta_4)$, the Lie bracket μ is isomorphic to:

$$\begin{cases} \mu^4_{\alpha_2,\beta_4}(e_1,e_1) = 0, \\ \mu^4_{\alpha_2,\beta_4}(e_1,e_2) = (\alpha_2+1)e_1, \\ \mu^4_{\alpha_2,\beta_4}(e_2,e_1) = (\alpha_2-1)e_1, \\ \mu^4_{\alpha_2,\beta_4}(e_2,e_2) = \beta_4 e_2, \end{cases}$$

5. Assume now that $\beta_1 = \beta_2 = 0, \alpha_1 = 0, 2\alpha_2 - \beta_4 = 0, \alpha_4 \neq 0$. The Lie bracket μ is isomorphic to:

$$\begin{cases} \mu^5_{\alpha_2}(e_1,e_1) = 0, \\ \mu^5_{\alpha_2}(e_1,e_2) = (\alpha_2+1)e_1, \\ \mu^5_{\alpha_2}(e_2,e_1) = (\alpha_2-1)e_1, \\ \mu^5_{\alpha_2}(e_2,e_2) = e_1 + 2\alpha_2 e_2, \end{cases}$$

6. If $\beta_1 = \beta_2 = 0, \alpha_1 = 0, 2\alpha_2 - \beta_4 = 0, \alpha_4 = 0$, then μ is isomorphic to $\mu^4_{\alpha_2,\beta_4}$ with $\beta_4 = 2\alpha_2$

Theorem 2: Any 2-dimensional non commutative algebras isomorphic to one of the following algebras:

- If $\mathbb{K}$ is algebraically closed:

$$\begin{cases} \mu^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_1,e_1) = e_2, \\ \mu^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_1,e_2) = (\alpha_2+1)e_1 + \beta_2 e_2, \\ \mu^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_2,e_1) = (\alpha_2-1)e_1 + \beta_2 e_2, \\ \mu^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_2,e_2) = \alpha_4 e_1 + \beta_4 e_2. \end{cases} \quad \begin{cases} \mu^2_{\alpha_1,\alpha_2,\alpha_4}(e_1,e_1) = \alpha_1 e_1, \\ \mu^2_{\alpha_1,\alpha_2,\alpha_4}(e_1,e_2) = (\alpha_2+1)e_1 + e_2, \\ \mu^2_{\alpha_1,\alpha_2,\alpha_4}(e_2,e_1) = (\alpha_2-1)e_1 + e_2, \\ \mu^2_{\alpha_1,\alpha_2,\alpha_4}(e_2,e_2) = \alpha_4 e_1. \end{cases}$$

$$\begin{cases} \mu^3_{\alpha_4,\beta_4}(e_1,e_1) = e_1, \\ \mu^3_{\alpha_4,\beta_4}(e_1,e_2) = e_1, \\ \mu^3_{\alpha_4,\beta_4}(e_2,e_1) = -e_1, \\ \mu^3_{\alpha_4,\beta_4}(e_2,e_2) = \alpha_4 e_1 + \beta_4 e_2. \end{cases} \quad \begin{cases} \mu^4_{\alpha_2,\beta_4}(e_1,e_1) = 0, \\ \mu^4_{\alpha_2,\beta_4}(e_1,e_2) = (\alpha_2+1)e_1, \\ \mu^4_{\alpha_2,\beta_4}(e_2,e_1) = (\alpha_2-1)e_1, \\ \mu^4_{\alpha_2,\beta_4}(e_2,e_2) = \beta_4 e_2, \end{cases} \quad \begin{cases} \mu^5_{\alpha_2}(e_1,e_1) = 0, \\ \mu^5_{\alpha_2}(e_1,e_2) = (\alpha_2+1)e_1, \\ \mu^5_{\alpha_2}(e_2,e_1) = (\alpha_2-1)e_1, \\ \mu^5_{\alpha_2}(e_2,e_2) = e_1 + 2\alpha_2 e_2. \end{cases}$$

with $\alpha_i, \beta_i \in \mathbb{K}$.

- If $\mathbb{K}$ is not algebraically closed:

$$\begin{cases}\varphi^{1,\lambda}_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_1,e_1)=\lambda e_2,\\ \varphi^{1,\lambda}_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_1,e_2)=(\alpha_2+1)e_1+\beta_2 e_2,\\ \varphi^{1,\lambda}_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_2,e_1)=(\alpha_2-1)e_1+\beta_2 e_2,\\ \varphi^{1,\lambda}_{\alpha_2,\beta_2,\alpha_4,\beta_4}(e_2,e_2)=\alpha_4 e_1+\beta_4 e_2,\end{cases} \mu^2_{\alpha_1,\alpha_2,\alpha_4},\ \mu^3_{\alpha_4,\beta_4},\ \mu^4_{\alpha_2,\beta_4},\ \mu^5_{\alpha_2},$$

$\alpha_i, \beta_i \in \mathbb{K}, \lambda \in \mathbb{K}/(\mathbb{K}^*)^2$.

Let us make the link with the results of Petersson [4]. The main idea of this work is to construct algebras from unital algebra. Recall that an algebra $A=(V,\mu)$ is called unital if there exists $1\in V$ such that $\mu(1,X)=\mu(X,1)=X$ for any $X\in V$ for any $X\in V$.

Lemma 3: If μ_a is not trivial, then A is not unital.

Proof. Assume that there exists 1 satisfying $\mu(1,X)=\mu(X,1)=X$, then:

$$0=\mu(1,X)-\mu(X,1)=\mu_a(1,X)-\mu_a(X,1)=2\mu_a(1,X)$$

for any $X\in V$. Then $\mu_a(1,X)=0$ for any X and 1 is in the center of $A_a=(V,\mu_a)$. But if μ_a is not trivial, the center of A_a is reduce to $\{0\}$. The algebra A cannot be unital.

The algebra $A=(V,\mu)$ is called regular if there exists $U,T\in V$ such that the linear applications:

$$L_U: X\to\mu(U,X),\quad R_T: X\to\mu(X,T)$$

are linear isomorphisms. From ref. [5], for any regular algebra $A=(V,\mu)$ there exist a unique, up an isomorphism, unital algebra $B=(V,\mu_u)$ and two linear isomorphisms f, g of V such that:

$$\mu(X,Y)=\mu_u(f(X),g(Y))$$

for any X, $Y\in V$. The algebra B is called the unital heart of A. To compare Theorem 2 with the Petersson results, we have to determine the regular algebras. Let us consider the first family. The application L_U is not regular for any U if and only if its determinant is identically null that is:

$$\alpha_2=-1,\quad \alpha_4=-2\beta_2,\quad \beta_4=\beta_2^2.$$

Likewise R_T is not regular for any T if and only if its determinant is identically null that is:

$$\alpha_2=1,\quad \alpha_4=2\beta_2,\quad \beta_4=\beta_2^2.$$

We deduce that any algebra $A^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}=\left(V,\mu^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}\right)$ is regular except the algebras given by:

$$\begin{cases}\mu^1_{-1,\beta_2,-2\beta_2,\beta_2^2}(e_1,e_1)=e_2,\\ \mu^1_{-1,\beta_2,-2\beta_2,\beta_2^2}(e_1,e_2)=\beta_2 e_2,\\ \mu^1_{-1,\beta_2,-2\beta_2,\beta_2^2}(e_2,e_1)=-2e_1+\beta_2 e_2,\\ \mu^1_{-1,\beta_2,-2\beta_2,\beta_2^2}(e_2,e_2)=-2\beta_2 e_1+\beta_2^2 e_2.\end{cases}\quad \begin{cases}\mu^1_{1,\beta_2,2\beta_2,\beta_2^2}(e_1,e_1)=e_2,\\ \mu^1_{1,\beta_2,2\beta_2,\beta_2^2}(e_1,e_2)=2e_1+\beta_2 e_2,\\ \mu^1_{1,\beta_2,2\beta_2,\beta_2^2}(e_2,e_1)=\beta_2 e_2,\\ \mu^1_{1,\beta_2,2\beta_2,\beta_2^2}(e_2,e_2)=2\beta_2 e_1+\beta_2^2 e_2.\end{cases}$$

Let us note that $A^1_{-1,\beta_2,-2\beta_2,\beta_2^2}$ is left-singular but right-regular and $A^1_{1,\beta_2,2\beta_2,\beta_2^2}$ is right-singular and left-regular. An algebra which is left and right singular is called bi-singular. We can summarize the results in the following array:

1. $A^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}$ regular except $A^1_{-1,\beta_2,-2\beta_2,\beta_2^2}$ and $A^1_{1,\beta_2,2\beta_2,\beta_2^2}$.
2. $A^1_{1,\beta_2,2\beta_2,\beta_2^2}$ is left-singular and right-regular,
3. $A^1_{1,\beta_2,2\beta_2,\beta_2^2}$ is right-singular and left-regular,
4. $A^2_{\alpha_1,\alpha_2,\alpha_4}$ is regular,
5. $A^3_{\alpha_4,\beta_4}$ is regular except $A^3_{\alpha_4,0}$,
6. $A^3_{\alpha_4,0}$ is bisingular.
7. $A^4_{\alpha_2,\beta_4}$ is regular except $A^4_{\alpha_2,0}, A^4_{1,\beta_4}, A^4_{-1,\beta_4}$
8. $A^4_{\alpha_2,0}$ is bisingular,
9. A^4_{1,β_4} is left-singular and right-regular as soon as $\beta_4\neq 0$,
10. A^4_{-1,β_4} is left-regular and right-singular as soon as $\beta_4\neq 0$,
11. $A^5_{\alpha_2}$ is regular except for $\alpha_2=0$, 1 or -1,
12. A^5_0 is bisingular,
13. A^5_1 is left-singular and right-regular as soon as $\beta_4\neq 0$,
14. A^5_{-1} is left-regular and right-singular as soon as $\beta_4\neq 0$,

We deduce.

Proposition 4: We consider the following algebras,

1. $A^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}$ with $(\alpha_2,\beta_2,\alpha_4,\beta_4)\neq(-1,\beta_2,-2\beta_2,\beta_2^2)$ or $(1,\hat{a}_2,2\hat{a}_2,\hat{a}_2^2)$,
2. $A^2_{\alpha_1,\alpha_2,\alpha_4}$,
3. $A^4_{\alpha_2,\beta_4}$ with $(\alpha_2,\beta_4)\neq(\alpha_2,0)$ or $(1,\beta_4)$ or $(-1,\beta_4)$,
4. $A^5_{\alpha_2}$ with $\alpha_2\neq 0,1,-1$.

For anyone of these algebras A, there exists an unital $\mathbb{K}$ algebra $B_A=(V,\mu_{u,A})$ and linear endomorphisms f_A, g_A such that the multiplication of A is given by:

$$\mu_A(X,Y)=\mu_{u,A}(f(X),g(Y)).$$

This unital algebra B_A is called the unital heart of A. Since B_A is unital, then [5] it is an etale algebra, that is $B_A\otimes\tilde{\mathbb{K}}=\tilde{\mathbb{K}}^2$ where $\tilde{\mathbb{K}}$ is the algebraic closure of A, or B_A is isomorphic to the dual algebra defined by $\mu_B(e_1,e_i)=\mu_B(e_i,e_1)=e_i, i=1,2$ and $\mu_B(e_2,e_2)=0$. To find this heart algebra we use the Kaplansky's Trick. If A is regular, we consider U and V such that L_U and R_V are non singular and $f=L_U^{-1}, g=R_T^{-1}$. The multiplication μ_u of the heart B is $\mu_u(X,Y)=\mu(g(X),f(Y)$ and the identity of B is $1_B=\mu(U,T)$.

1. Let be $A^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}$. If $\alpha_2\neq 1$ or -1 then L_{e_1} and R_{e_1} are not singular. In fact,

$$L_{e_1}=\begin{pmatrix}0 & \alpha_2+1\\ 1 & \beta_2\end{pmatrix},\quad R_{e_1}=\begin{pmatrix}0 & \alpha_2-1\\ 1 & \beta_2\end{pmatrix}$$

Thus,

$$f=\frac{-1}{\alpha_2+1}\begin{pmatrix}\beta_2 & -\alpha_2-1\\ -1 & 0\end{pmatrix},\quad g=\frac{-1}{\alpha_2-1}\begin{pmatrix}\beta_2 & -\alpha_2+1\\ -1 & 0\end{pmatrix}$$

Then the identity element of B_A is e_2 and,

$$\mu_B(e_1,e_1)=\mu_A(g(e_1)g(e_1))=\frac{1}{\alpha_2^2-1}(\beta_2 e_1-e_2)^2$$

and B_A is etale. If $\alpha_2=-1$, then we can take $U=e_2$ and $T=e_1$ as soon as $\alpha_4\beta_2\neq 2\beta_4$. If not we take $U=e_1+e_2$ and $T=e_1$. We have the same calcul for $\alpha_2=1$.

2. Let be $A^2_{\alpha_1,\alpha_2,\alpha_4}$. This algebra is regular. If $\alpha_1\neq 0$, then L_{e_1} and R_{e_1} are not singular and B_A is etale.

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Case $\mu_a(e_1,e_2)=0$

The multiplicatio μ n is symmetric. The group of automorphisms of μ_a is $GL(2,\mathbb{K})$. Moreover the multiplication writes:

$$\begin{cases}\mu(e_1,e_1)=\alpha_1e_1+\beta_1e_2,\\ \mu(e_1,e_2)=\alpha_2e_1+\beta_2e_2,\\ \mu(e_2,e_1)=\alpha_2e_1+\beta_2e_2,\\ \mu(e_2,e_2)=\alpha_4e_1+\beta_4e_2,\end{cases}$$

We assume that there exists two independent idempotent vectors. If e_1 and e_2 are these vectors, then:

$$\mu(e_1,e_1)=e_1,\ \ \mu(e_2,e_2)=e_2.$$

We obtain the following algebras:

$$\begin{cases}\mu^6_{\alpha_2,\beta_2}(e_1,e_1)=e_1,\\ \mu^6_{\alpha_2,\beta_2}(e_1,e_2)=\alpha_2e_1+\beta_2e_2,\\ \mu^6_{\alpha_2,\beta_2}(e_2,e_2)=e_2.\end{cases}$$

Remark that if any element is idempotent, thus $\mu(e_1,e_2)=\mu(e_2,e_1)=0$. In fact:

$$\mu(e_1+e_2,e_1+e_2)=e_1+e_2=\mu(e_1,e_1)+\mu(e_2,e_2)+2\mu(e_1,e_2)$$

In the general case, if ae_1+be_2 is an idempotent with $ab\neq0$, then a and b satisfy the system:

$$\begin{cases}a^2+2ab\alpha_2=a\\ b^2+2ab\beta_2=b.\end{cases}$$

If $4\alpha_2\beta_2=1$, then the system has solutions as soon as $\alpha_2=\beta_2=\frac{1}{2}$. In this case we obtain the multiplication $\mu^6_{\frac12,\frac12}$ and for any a, the vectors $ae_1+(1-a)e_2$ are idempotent. If $4\alpha_2\beta_2\neq1$, the vector:

$$v=\frac{1-2\alpha_2}{1-4\alpha_2\beta_2}e_1+\frac{1-2\beta_2}{1-4\alpha_2\beta_2}e_2$$

is an idempotent and the only idempotents are e_1, e_2 and v. The changes of basis $\{e_1,v\}$ or $\{e_2,v\}$ do not simplify the number of independent parameters.

We assume that there exists only one idempotent vector. If e_1 is this vector, thus $\mu(e_1,e_1)=e_1$. If we consider a vector $v=xe_1+ye_2$ such that $\mu(v,v)=v$, then x and y have to satisfy:

$$\begin{cases}x^2+2xy\alpha_2+y^2\alpha_4=x,\\ 2xy\beta_2+y^2\beta_4=y.\end{cases}\tag{5}$$

If we assume that $y\neq0$, the second equation gives as soon as $\beta_2\neq0$, $x=\frac{1-y\beta_4}{2\beta_2}$ and thus:

$$y^2\left(\beta_4^2-4\alpha_2\beta_2\beta_4+4\beta_2^2\alpha_4\right)+y\left(4\alpha_2\beta_2+2\beta_2\beta_4-2\beta_4\right)+1-2\beta_2=0.\tag{6}$$

Let us consider a change of basis which preserves e_1 that is,

$$\begin{cases}e'_1=e_1,\\ e'_2=be_1+de_2,\end{cases}\tag{7}$$

with $d\neq0$. Since in this new basis we have $\beta'_4=2b\beta_2+d\beta_4$, we can find b such that $\beta'_4=0$. Then we can assume that $\beta_4=0$.

If moreover $\alpha_2\neq0$, taking $d=\alpha_2^{-1}$, we obtain $\alpha'_2=1$ and we have the algebra:

$$\begin{cases}\mu(e_1,e_1)=e_1,\\ \mu(e_1,e_2)=e_1+\beta_2e_2,\\ \mu(e_2,e_2)=\alpha_4e_1.\end{cases}$$

Equation (6) simplifies as:

$$y^2\left(4\beta_2^2\alpha_4\right)+4\beta_2y+1-2\beta_2=0.\tag{8}$$

If we assume that $\mathbb{K}$ is algebraically closed, then this equation has in general two roots. It has no root if $\beta_2=0$ which is excluded. Then to have only one idempotent, 0 must be the only root which is equivalent to $\alpha_4=0$ and $\beta_2=1/2$. We obtain the following algebra:

$$\begin{cases}\mu^7(e_1,e_1)=e_1,\\ \mu^7(e_1,e_2)=e_1+\frac{1}{2}e_2,\\ \mu^7(e_2,e_2)=0.\end{cases}$$

If $\mathbb{K}$ is not algebraically closed, then we have no idempotent other than 0 if $\alpha_4=0$ and $\beta_2=1/2$ and we obtain the previous algebra μ^7 or if $y^2\left(4\beta_2^2\alpha_4\right)+4\beta_2y+1-2\beta_2$ is irreducible in $\mathbb{K}$. We obtain:

$$\begin{cases}\mu^7_R(e_1,e_1)=e_1,\\ \mu^7_R(e_1,e_2)=e_1+\beta_2e_2,\\ \mu^7_R(e_2,e_2)=\alpha_4e_1,\end{cases}$$

with $y^2\left(4\beta_2^2\alpha_4\right)+4\beta_2y+1-2\beta_2$ irreducible in $\mathbb{K}$ (so $\alpha_4\neq0$).

If $\alpha_2=0$ and if $\mathbb{K}$ is algebraically closed, we consider in the change of basis (7) defined above, $b=0$ and $d=\sqrt{\alpha_4}$ if $\alpha_4\neq0$:

$$\begin{cases}\mu(e_1,e_1)=e_1,\\ \mu(e_1,e_2)=\beta_2e_2,\\ \mu(e_2,e_2)=e_1.\end{cases}$$

There exits only one idempotent if and only if $\beta_2=1/2$. We obtain the following algebra:

$$\begin{cases}\mu^8(e_1,e_1)=e_1,\\ \mu^8(e_1,e_2)=\frac{1}{2}e_2,\\ \mu^8(e_2,e_2)=e_1.\end{cases}$$

If $\alpha_2=\alpha_4=0$, we have only one idempotent if and only if $2\beta_2\neq1$. We obtain:

$$\begin{cases}\mu^9(e_1,e_1)=e_1,\\ \mu^9(e_1,e_2)=\beta_2e_2,\quad(\beta_2\neq1/2)\\ \mu^9(e_2,e_2)=0.\end{cases}$$

Assume $\mathbb{K}$ not algebraically closed and $\alpha_2=0$. If the equation $d^2\alpha_4$ has a root in $\mathbb{K}$, we find μ^8. If not, let $\lambda_2\in\mathbb{K}/(\mathbb{K}^*)^2$ such that $d^2\alpha_4=\lambda_2$. In this case we have only one idempotent if and only if $(2\beta_2=1)$ or $\left(1-2\beta_2\notin\left(\mathbb{K}^*\right)^2\right)$. We obtain:

$$\begin{cases}\mu^{8,1}_R(e_1,e_1)=e_1,\\ \mu^{8,1}_R(e_1,e_2)=\frac{1}{2}e_2,\\ \mu^{8,1}_R(e_2,e_2)=\lambda_2e_1,\end{cases}$$

and,

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$$\begin{cases} \mu_R^{8,2}(e_1,e_1) = e_1, \\ \mu_R^{8,2}(e_1,e_2) = \beta_2 e_2, \quad 1-2\beta_2 \notin (\mathbb{K}^*)^2, \\ \mu_R^{8,2}(e_2,e_2) = \lambda_2 e_1. \end{cases}$$

Assume now that $\beta_2=0$. Then (5) implies $y^2\beta_4=y$. If $\beta_4=0$, then $y=0$ and we have:

$$\begin{cases} \mu(e_1,e_1) = e_1, \\ \mu(e_1,e_2) = \alpha_2 e_1, \\ \mu(e_2,e_2) = \alpha_4 e_1. \end{cases}$$

The change of basis $e'_1 = e_1, e'_2 = be_1 + de_2$ gives $\alpha'_2 = d\alpha_2, \alpha_{4'} = d^{\check{Z}}\alpha_4$. We obtain:

$$\begin{cases} \mu^{10}(e_1,e_1) = e_1, \\ \mu^{10}(e_1,e_2) = e_1, \\ \mu^{10}(e_2,e_2) = \alpha_4 e_1. \end{cases}$$

if $\alpha_2\neq0$. Assume now that $\alpha_2=0$ and $\alpha_4\neq0$. If $\mathbb{K}$ is algebraically close, we obtain:

$$\begin{cases} \mu^{11}(e_1,e_1) = e_1, \\ \mu^{11}(e_1,e_2) = 0, \\ \mu^{11}(e_2,e_2) = e_1, \end{cases}$$

$$\begin{cases} \mu_R^{11}(e_1,e_1) = e_1, \\ \mu_R^{11}(e_1,e_2) = 0, \\ \mu_R^{11}(e_2,e_2) = \lambda_2 e_1 \end{cases}$$

with $\lambda_2\in\mathbb{K}(\mathbb{K}^*)^2$. If $\alpha_4=0$,

$$\begin{cases} \mu^{12}(e_1,e_1) = e_1, \\ \mu^{12}(e_1,e_2) = 0, \\ \mu^{12}(e_2,e_2) = 0 \end{cases}$$

No vector is idempotent. If there exists v with $\mu(v,v)\neq0$, thus we can consider that $\mu(e_1,e_1)=e_2$ that is,

$$\begin{cases} \mu(e_1,e_1) = e_2, \\ \mu(e_1,e_2) = \mu(e_2,e_1) = \alpha_2 e_1 + \beta_2 e_2, \\ \mu(e_2,e_2) = \alpha_4 e_1 + \beta_4 e_2. \end{cases}$$

1. If $\alpha_4=0$, that is $\mu(e_2,e_2)=\beta_4 e_2$, then the vector $e'_2 = \beta_4^{-1} e_2$ is idempotent as soon as $\beta_4\neq0$. Then the hypothesis implies $\beta_4=0$. Let be $v=xe_1+ye_2$. The equation $\mu(v,v)=v$ is equivalent to:

$$x^2 e_2 + 2xy(\alpha_2 e_1 + \beta_2 e_2) = 2xy\alpha_2 e_1 + (x^2 + 2xy\beta_2)e_2 = xe_1 + ye_2.$$

that is,

$$2xy\alpha_2 = x, \ \ x^2 + 2xy\beta_2 = y.$$

If $\alpha_2=0$, then $x=y=0$, and no elements are idempotent. We obtain the algebras, corresponding to $\beta_2\neq0$ or $\beta_2=0$

$$\begin{cases} \mu^{13}(e_1,e_1) = e_2, \\ \mu^{13}(e_1,e_2) = e_2, \\ \mu^{13}(e_2,e_2) = 0. \end{cases}$$

$$\begin{cases} \mu^{14}(e_1,e_1) = e_2, \\ \mu^{14}(e_1,e_2) = 0, \\ \mu^{14}(e_2,e_2) = 0. \end{cases}$$

If $\alpha_2\neq0$ and $y = (2\alpha_2)^{-1}$ then x satisfies the equation:

$$x^2 + \left(\frac{\beta_2}{\alpha_2}\right)x - \frac{1}{2\alpha_2} = 0 \tag{9}$$

If $\mathbb{K}$ is algebraically closed, such equation admits a non trivial solution. This is not compatible with our hypothesis. Assume that $\mathbb{K}$ is not algebraically closed. If $\beta_2\neq0$, the change of basis $e'_1 = \beta_2^{-1}e_1$ and $e'_2 = \beta_2^{-2}e_2$ permits to consider $\beta_2=1$ and the (9) becomes,

$$x^2 + \frac{1}{\alpha_2}x - \frac{1}{2\alpha_2} = (x + \frac{1}{2\alpha_2})^2 - \frac{1+2\alpha_2}{4\alpha_2^2}$$

This equation has a non solution if $1+2\acute{a}_2 \notin (\mathbb{K})^2$ where $(\mathbb{K})^2=\{\lambda^2,\lambda\mathbb{K}\}$. We obtain the algebras:

$$\begin{cases} \mu_R^{14,1}(e_1,e_1) = e_2, \\ \mu_R^{14,1}(e_1,e_2) = \alpha_2 e_1 + e_2, \quad 2\alpha_2 + 1 \notin (\mathbb{K})^2, \\ \mu_R^{14,1}(e_2,e_2) = 0, \end{cases}$$

and,

$$\begin{cases} \mu_R^{14,2}(e_1,e_1) = e_2, \\ \mu_R^{14,2}(e_1,e_2) = \alpha_2 e_1 \quad 2\alpha_2 \notin (\mathbb{K})^2, \\ \mu_R^{14,2}(e_2,e_2) = 0. \end{cases}$$

2. If $\alpha_4\neq0$ the vector $v=xe_1+ye_2$ is idempotent if and only if:

$$\begin{cases} 2xy\alpha_2 + y^2\alpha_4 = x, \\ x^2 + 2xy\beta_2 + y^2\beta_4 = y. \end{cases}$$

Then $x = \dfrac{y^2\alpha_4}{1-2y\alpha_2}$. Let us note that $1-2y\alpha_2\neq0$ because $1-2y\alpha_2=0$ implies $y^2\alpha_4=0$ that is $y=0$ and in this case $x=0$ and $v=0$. We deduce that y is a root of the equation:

$$\left(\frac{y^2\alpha_4}{1-2y\alpha_2}\right)^2 + 2\frac{y^2\alpha_4}{1-2y\alpha_2}y\beta_2 + y^2\beta_4 - y = 0$$

that is:

$$-1 + y(4\alpha_2 + \beta_4) + y^2(2\alpha_4\beta_2 - 4\alpha_2^2 - 4\alpha_2\beta_4) + y^3(\alpha_4^2 - 4\alpha_2\alpha_4\beta_2 + 4\alpha_2^2\beta_4) = 0.$$

If $\mathbb{K}$ is algebraically closed, this equation admits always a solution except if:

$$\begin{cases} 4\alpha_2 + \beta_4 = 0, \\ 2\alpha_4\beta_2 - 4\alpha_2^2 - 4\alpha_2\beta_4 = 0, \\ \alpha_4^2 - 4\alpha_2\alpha_4\beta_2 + 4\alpha_2^2\beta_4 = 0. \end{cases}$$

Then $\beta_4 = -4\alpha_2, \alpha_4\beta_2 = -6\alpha_2^2, \alpha_4^2 = -8\alpha_2^3$. We note that $\beta_2=0$ implies, if the characteristic of $\mathbb{K}$ is not 3, $\alpha_2=\alpha_4=0$. From hypothesis, we can assume that $\beta_2\neq0$ and the change of basis $e'_1 = ke_1, e'_2 = k^2e_2$ which preserves the condition $e_1e_1=e_2$ changes β_2 in $k\beta_2$ and we can take $\beta_2=3$. Then $\alpha_4 = -2\alpha_2^2, \alpha_4^2 = 4\alpha_2^4 = -8\alpha_2^3$, then $\alpha_2=-2$ and $\alpha_4=4$, $\beta_4=8$ and we obtain the algebra:

$$\begin{cases} \mu^{15}(e_1,e_1) = e_2, \\ \mu^{15}(e_1,e_2) = -2e_1 + 3e_2, \\ \mu^{15}(e_2,e_2) = -8e_1 + 8e_2. \end{cases}$$

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Let us note that if the characteristic of $\mathbb{K}$ is 3, then $\alpha_4\beta_2=0$ and $\beta_2=0$. This gives $\alpha_2(\alpha_2+\beta_4)=0$ and $\alpha_4^2+4\alpha_2^2\beta_4=0$. Since $\alpha_2=0$ implies $\alpha_4=0$ and $4_2+\beta_4=\alpha_2+\beta_4=0$ we obtain $\beta_4=2\alpha_2$ and $\alpha_4^2=2\alpha_2^2\beta_4=\alpha_2^3$. By a change of basis we can take $\alpha_2=1$ and we obtain the algebra:

$$\begin{cases}\mu_{(3)}^{15}(e_1,e_1)=e_2,\\ \mu_{(3)}^{15}(e_1,e_2)=e_1,\\ \mu_{(3)}^{15}(e_2,e_2)=e_1+2e_2.\end{cases}$$

which correspond to μ_{15} in characteristic 3.

If $\mathbb{K}$ is not algebraically closed, we have to consider all the algebras for which the polynomial:

$$P_A(y)=-1+y(4\alpha_2+\beta_4)+y^2(2\alpha_4\beta_2-4\alpha_2^2-4\alpha_2\beta_4)+y^3(\alpha_4^2-4\alpha_2\alpha_4\beta_2+4\alpha_2^2\beta_4)\ (10)$$

has no root this is equivalent to say that P_A is irreducible. If we consider the coefficient of y^3, that is $q_3(A)=\alpha_4^2-4\alpha_2\alpha_4\beta_2+4\alpha_2^2\beta_4$, it is equal to the discriminant of the determinant of the endomorphism L_V, that is $q_3(A)=\mathrm{Disc}(\det(L_V))$. We deduce:

Proposition 5: The algebra A is regular if and only if $P_A(y)$ is strictly of degree 3.

It remains to examine the case $\mu(v,v)=0$ for any v. That is:

$$\begin{cases}\mu(e_1,e_1)=0,\\ \mu(e_1,e_2)=\alpha_2e_1+\beta_2e_2,\\ \mu(e_2,e_2)=0.\end{cases}$$

If $\alpha_2\beta_2\neq0$ we can find some idempotents. In all the others cases, we have no idempotent. We obtain:

$$\begin{cases}\mu^{16}(e_1,e_1)=0,\\ \mu^{16}(e_1,e_2)=e_1,\\ \mu^{16}(e_2,e_2)=0,\end{cases}$$

and

$$\begin{cases}\mu^{17}(e_1,e_1)=0,\\ \mu^{17}(e_1,e_2)=0,\\ \mu^{17}(e_2,e_2)=0.\end{cases}$$

Theorem 6: Any commutative 2-dimensional algebra over an algebraically closed field is isomorphic to one of the following:

$$\begin{cases}\mu^{6}(e_1,e_1)=e_1,\\ \mu^{6}(e_1,e_2)=\alpha_2e_1+\beta_2e_2,\\ \mu^{6}(e_2,e_2)=e_2.\end{cases}\quad \begin{cases}\mu^{7}(e_1,e_1)=e_1,\\ \mu^{7}(e_1,e_2)=e_1+\frac{1}{2}e_2,\\ \mu^{7}(e_2,e_2)=0.\end{cases}\quad \begin{cases}\mu^{8}(e_1,e_1)=e_1,\\ \mu^{8}(e_1,e_2)=\frac{1}{2}e_2,\\ \mu^{8}(e_2,e_2)=e_1.\end{cases}$$

$$\begin{cases}\mu^{9}(e_1,e_1)=e_1,\\ \mu^{9}(e_1,e_2)=\beta_2e_2,\quad(\beta_2\neq1/2),\\ \mu^{9}(e_2,e_2)=0.\end{cases}\quad \begin{cases}\mu^{10}(e_1,e_1)=e_1,\\ \mu^{10}(e_1,e_2)=e_1,\\ \mu^{10}(e_2,e_2)=\alpha_4e_1.\end{cases}\quad \begin{cases}\mu^{11}(e_1,e_1)=e_1,\\ \mu^{11}(e_1,e_2)=0,\\ \mu^{11}(e_2,e_2)=e_1.\end{cases}$$

$$\begin{cases}\mu^{12}(e_1,e_1)=e_1,\\ \mu^{12}(e_1,e_2)=0,\\ \mu^{12}(e_2,e_2)=0.\end{cases}\ \hat{E}\begin{cases}\mu^{13}(e_1,e_1)=e_2,\\ \mu^{13}(e_1,e_2)=e_2,\\ \mu^{13}(e_2,e_2)=0.\end{cases}\quad \begin{cases}\mu^{14}(e_1,e_1)=e_2,\\ \mu^{14}(e_1,e_2)=0,\\ \mu^{14}(e_2,e_2)=0.\end{cases}$$

$$\begin{cases}\mu^{15}(e_1,e_1)=e_2,\\ \mu^{15}(e_1,e_2)=-2e_1+3e_2,\\ \mu^{15}(e_2,e_2)=-8e_1+8e_2.\end{cases}\quad \begin{cases}\mu^{16}(e_1,e_1)=0,\\ \mu^{16}(e_1,e_2)=e_1,\\ \mu^{16}(e_2,e_2)=0.\end{cases}\quad \begin{cases}\mu^{17}(e_1,e_1)=0,\\ \mu^{17}(e_1,e_2)=0,\\ \mu^{17}(e_2,e_2)=0.\end{cases}$$

If $\mathbb{K}$ is not algebraically closed, we have also the following algebras where $\lambda_2\in\mathbb{K}/(\mathbb{K}^*)^2$:

$$\begin{cases}\mu_R^{8,1}(e_1,e_1)=e_1,\\ \mu_R^{8,1}(e_1,e_2)=\frac{1}{2}e_2,\\ \mu_R^{8,1}(e_2,e_2)=\lambda_2e_1.\end{cases}\quad \begin{cases}\mu_R^{8,2}(e_1,e_1)=e_1,\\ \mu_R^{8,2}(e_1,e_2)=\beta_2e_2,\ 1-2\beta_2\notin(\mathbb{K}^*)^2,\\ \mu_R^{8,2}(e_2,e_2)=\lambda_2e_1.\end{cases}\quad \begin{cases}\mu_R^{11}(e_1,e_1)=e_1,\\ \mu_R^{11}(e_1,e_2)=0,\\ \mu_R^{11}(e_2,e_2)=\lambda_2e_1.\end{cases}$$

$$\begin{cases}\mu_R^{14,1}(e_1,e_1)=e_2,\\ \mu_R^{14,1}(e_1,e_2)=\alpha_2e_1+e_2,\quad 2\alpha_2+1\notin\mathbb{K}^2,\\ \mu_R^{14,1}(e_2,e_2)=0.\end{cases}\quad \begin{cases}\mu_R^{14,2}(e_1,e_1)=e_2,\\ \mu_R^{14,2}(e_1,e_2)=\alpha_2e_1\quad 2\alpha_2+1\notin\mathbb{K}^2,\\ \mu_R^{14,2}(e_2,e_2)=0.\end{cases}$$

$$\begin{cases}\mu_R^{15,1}(e_1,e_1)=e_2,\\ \mu_R^{15,1}(e_1,e_2)=\alpha_2e_1+\beta_2e_2, P_A(y)\ \ without\ \ roots\\ \mu_R^{15,1}(e_2,e_2)=\alpha_4e_1+\beta_4e_2.\end{cases}$$

Let us examine the property of regularity for these algebras. Since they are commutative, the left and right regularity are equivalent notions. Computing directly the determinant of the operator $L_{xe_1+ye_2}$ we deduce in the case $\mathbb{K}$ algebraically closed:

1. The algebras $A^6=(V,\mu_6),\ A^7=(V,\mu_7),\ A^8=(V,\mu_8),\ A^{10}=(V,\mu_{10}),\ A^{15}=(V,\mu_{15})$ are regular,
2. $A^9=(V,\mu_9)$ is regular if $\beta_2\neq0$,
3. The algebras $A^{11}=(V,\mu_{11}), A^{12}=(V,\mu_{12}),\ A^{13}=(V,\mu_{13}),\ A^{14}=(V,\mu_{14}),\ A^{16}=(V,\mu_{16})$ and $A^{17}=(V,\mu_{17})$ are bisingular.

Algebras Over a Field of Characteristic 2

Let $\mathbb{F}$ be a field of characteristic 2. Assume that $\mathbb{F}=\mathbb{F}_2$. If A is a 2-dimensional $\mathbb{F}$-algebra and if $\{e_1, e_2\}$ is a basis of A, then the values of the different products belong to $\{e_1, e_2, e_1+e_2\}$. If f is an isomorphism of A, it is represented in the basis $\{e_1, e_2\}$ by one of the following matrices:

$$M_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ M_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ M_3=\begin{pmatrix}1&0\\1&1\end{pmatrix}$$

$$M_4=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ M_5=\begin{pmatrix}0&1\\1&1\end{pmatrix},\ M_6=\begin{pmatrix}1&1\\1&0\end{pmatrix}$$

Each of these matrices corresponds to a permutation of the finite set $\{e_1, e_2, e_3=e_1+e_2\}$. If fact we have the correspondance:

$GL(A)$	Ó$_3$
M_1	Id
M_2	τ_{12}
M_3	τ_{13}
M_4	τ_{23}
M_5	c
M_6	c^2

where ${}_{ij}$ is the transposition between i and j and c the cycle $\{231\}$. In fact, the matrix M_2 corresponds to the linear transformation $f_2(e_1)=e_2$, $f(e_2)=e_1$ and in the set (e_1, e_2, e_3) we have the transformation whose image is (e_1, e_2, e_3) that is the transposition τ_{12}. The matrix M_3 corresponds to the linear transformation $f_2(e_1)=e_1+e_2$, $f(e_2)=e_2$ which corresponds to the permutation (e_3, e_2, e_1) that is τ_{13}. For all other matrices we have similar results. We deduce:

Theorem 7: There is a one-to-one correspondance between the change of $\mathbb{F}$-basis in $A$and the group Σ_3.

If we want to classify all these products of A, we have to consider all the possible results of these products and to determine the orbits of the action of Σ_3. More precisely the product $\mu(e_i,e_j)$ is in values in the set (e_1, e_2, e_3=e_1+e_2). If we write $\mu(e_i, e_j)=ae_1+be_2+ce_3$, thus the matrix ($a$, b, c) is one of the following:

$$R_0=(0,0,0)=0, R_1=(1,0,0),\ \ R_2=(0,1,0),\ \ R_3=(0,0,1)$$

Let us consider the following sequence:

$$\mu(e_1,e_1),\mu(e_1,e_2),\mu(e_2,e_1),\mu(e_2,e_2),\mu(e_1,e_3),\mu(e_2,e_3),\mu(e_3,e_1),\mu(e_3,e_2),\mu(e_3,e_3))$$

As $\mu(e_1,e_3)=\mu(e_1,e_1+e_2)$, if $\mu(e_1,e_1)=R_i$ and $\mu(e_1,e_2)=R_j$ then $\mu(e_1,e_3)=R_i+R_j$ with the relations:

$$R_i+R_i=0,\ \ R_i+R_j=R_k,$$

for i, j, k all different and non zero. Thus the four first terms of this sequence determine all the other terms. More precisely, such a sequence writes:

$$\left(R_i,R_j,R_k,R_l,R_i+R_j,R_k+R_l,R_i+R_k,R_j+R_l,R_i+R_j+R_k+R_l\right)$$

Consequence: We have 4^4=256 sequences, each of these sequences corresponds to a 2-dimensional $\mathbb{F}$-algebra.

Let us denote by S the set of these sequences. We have an action of $\sum_3$ on S: if $\sigma\in\sum_3$ and $s\in S$, thus $s'=\sigma s$ is the sequence:

$$\begin{pmatrix}\mu\left(e_{\sigma(1)},e_{\sigma(1)}\right),\mu\left(e_{\sigma(1)},e_{\sigma(2)}\right),\mu\left(e_{\sigma(2)},e_{\sigma(1)}\right),\mu\left(e_{\sigma(2)},e_{\sigma(2)}\right),\mu\left(e_{\sigma(1)},e_{\sigma(3)}\right),\\ \mu\left(e_{\sigma(2)},e_{\sigma(3)}\right),\mu\left(e_{\sigma(3)},e_{\sigma(1)}\right),\mu\left(e_{\sigma(3)},e_{\sigma(2)}\right),\mu\left(e_{\sigma(3)},e_{\sigma(3)}\right)\end{pmatrix}$$

with $\mu\left(e_{\sigma(i)},e_{\sigma(j)}\right)=R_{\sigma^{-1}(k)}$ when $\mu(e_i,e_j)=R_k$ and $R_k\neq0$. If $R_k=0$, then $\mu\left(e_{\sigma(i)},e_{\sigma(j)}\right)=0$. The classification of the 2-dimensional $\mathbb{F}$-algebras corresponds to the determination of the orbits of this action. Recall that the subgroups of $\sum_3$ are $G_1=\{Id\},G_2=\{Id,\tau_{12}\},G_3=\{Id,\tau_{13}\},G_4=\{Id,\tau_{23}\},G_5=\{Id,c,c^2\},G_6=\Sigma_3$.

1. The isotropy subgroup is $\sum_3$. In this case we have the following sequence (we write only the 4 first terms which determine the algebras:

$$s_1=(0,0,0,0)$$
$$s_2=(R_1,R_3,R_3,R_2)$$

Recall that $\mu(e_1,e_1)=R_1$ means $\mu(e_1,e_1)=e_1$,$\mu(e_1,e_2)=R_3$ means $\mu(e_1,e_2)=e_3$ and so on.

2. The isotropy subgroup is $G_5=\{Id, c, c^2\}$ We have only one orbit:

s	$\mathcal{O}(s)$
$s_3=(R_3,R_2,R_2,R_1)$	$s_3,(R_2,R_1,R_1,R_3)$

3. The isotropy subgroup is of order 2.

s	$\mathcal{O}(s)$
$s_4=(0,R_1,R_2,0)$	$s_4,(R_1,R_3,R_2,0),(0,R_1,R_3,R_2)$
$s_5=(0,R_2,R_1,0)$	$s_5,(R_1,R_2,R_3,0),(0,R_3,R_1,R_2)$
$s_6=(0,R_3,R_3,0)$	$s_6,(0,R_1,R_1,0),(0,R_2,R_2),0)$
$s_7=(R_1,0,0,R_2)$	$s_7,(R_1,R_2,R_2,R_2),(R_1,R_1,R_1,R_2)$
$s_8=(R_1,R_1,R_2,R_2)$	$s_8,(0,R_1,0,R_2),(R_1,0,R_2,0)$
$s_9=(R_1,R_2,R_1,R_2)$	$s_9,(0,0,R_1,R_2),(R_1,R_2,0,0)$
$s_{10}=(R_2,0,0,R_1)$	$s_{10},(R_1,R_3,R_3,R_3),(R_3,R_3,R_3,R_1)$
$s_{11}=(R_2,R_1,R_2,R_1)$	$s_{11},(0,0,R_1,R_3),(R_3,R_2,0,0)$
$s_{12}=(R_2,R_2,R_1,R_1)$	$s_{12},(0,R_1,0,R_3),(R_3,0,R_2,0)$
$s_{13}=(R_2,R_3,R_3,R_1)$	$s_{13},(R_1,R_2,R_2,R_3),(R_3,R_1,R_1,R_2)$
$s_{14}=(R_3,0,0,R_3)$	$s_{14},(0,R_1,R_1,R_1),(R_2,R_2,R_2,0)$
$s_{15}=(R_3,R_1,R_2,R_3)$	$s_{15},(R_1,R_2,R_3,R_1),(R_2,R_3,R_1,R_3)$
$s_{16}=(R_3,R_2,R_1,R_3)$	$s_{16},(R_1,R_3,R_2,R_1),(R_2,R_1,R_3,R_2)$
$s_{17}=(R_3,R_3,R_3,R_3)$	$s_{17},(0,0,0,R_1),(R_2,0,0,0)$

4. The isotropy subgroup is trivial. In this case any orbit contains 6 elements. As there are 256–46=46=210 elements having $\sum_3$ as isotropy group, we deduce that we have 35 distinguished non isomorphic classes.

Conclusion

We have 52 classes of non isomorphic algebras of dimension 2 on the field F_2.

Applications : 2-dimensional *G*-associative and Jordan algebras

G-associative commutative algebras

The notion of G-associativity has been defined in ref. [4]. Let G be a subgroup of the symmetric group $\sum_3$. An algebra whose multiplication is denoted by μ is G-associative if we have:

$$\sum_{\sigma\in G}\varepsilon,(\sigma)\mu\left(\mu\left(x_{\sigma(i)},x_{\sigma(j)}\right),x_{\sigma(k)}\right)-\mu(x_{\sigma(i)},\mu\left(x_{\sigma(j)},x_{\sigma(k)}\right)=0$$

where $\varepsilon(\sigma)$ is the signum of the permutation . Since we assume that μ is commutative, all these notions are trivial or coincide with the simple associativity. Now, if the algebra is of dimension 2, then the associativity is completely determined by the identities:

$$\mu(\mu(e_1,e_1),e_2)-\mu(e_1,\mu(e_1,e_2))=0,\quad \mu(\mu(e_1,e_2),e_2)-\mu(e_1,\mu(e_2,e_2))=0$$

We deduce that the only associative commutative 2-dimensional algebras are:

- μ^6 for $(\alpha_2, \beta_2)\in\{(0,1),(1,0),(0,0)\}$,
- μ^9 for β_2=0 or 1,
- μ^{12}, μ^{16}, μ^{17} .
- if $\mathbb{K}=\mathbb{R}$: μ_R^8 for β_2=1 and λ=–1.

We find again the classical list [6].

G-associative noncommutative algebras

Let us consider now the noncommutative case. From Theorem 2, the multiplication μ is isomorphic to some μ^i,i=1,⋯,5 (we consider here that $\mathbb{K}$ is algabraically closed). Let $A\mu$ be the associator of μ, that is $A_\mu=\mu\circ(\mu\otimes Id)-\mu\circ(Id\otimes\mu)$ and μ is associative if and only if $A\mu$=0. The examination of this list allows to find the classification of the 2-dimensional noncommutative associative algebras: these algebras are isomorphic to one of the following:

1. $\mu^4_{-1,-2}$ that is $\begin{cases}e_1e_1=0,\\ e_1e_2=0,\\ e_2e_1=-2e_1,\\ e_2e_2=-2e_2.\end{cases}$
2. $\mu^4_{1,2}$ that is $\begin{cases}e_1e_1=0,\\ e_1e_2=2e_1,\\ e_2e_1=0,\\ e_2e_2=2e_2.\end{cases}$

Now, for any nonassociative algebra, we examine the G_i-associativity. Note that all these algebras are Lie-admissible, that is $\sum_3$-associative. We focuse essentially on the G_2-associativity, G_2={Id, τ_{12}}, because we deduce immediately the affine structures on the associated Lie algebra μ_a. Then we compute for any algebra $A_\mu(e_1,e_2,e_1)-A_\mu(e_2,e_1,e_1)$ and $A_\mu(e_1,e_2,e_2)-A_\mu(e_2,e_1,e_2)$. We deduce that $\mu^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}$ is G_2-associative if and only if β_2=α_4=0 and α_2=–1, β_4=–4. The algebras μ^2

and μ^3 are never G_2-associative, $\mu^4_{\alpha_2,\alpha_4}$ is G_2-associative for α_2=−1 or (β_4=α_2−1). Likewise, $\mu^5_{\alpha_2}$ is G_2-associative for α_2=−1 or α_2=1.

Proposition 8: Any 2-dimensional noncommutative G_2-associative algebra is isomorphic to one of the following:

1. $\mu^4_{-1,-2}$ or $\mu^4_{1,2}$, that is μ is associative,
2. $\mu^1_{-1,0,0,-4}$ that is $\begin{cases} e_1e_1 = e_2, \\ e_1e_2 = 0, \\ e_2e_1 = -2e_1, \\ e_2e_2 = -4e_2. \end{cases}$
3. μ^4_{-1,β_4} that is $\begin{cases} e_1e_1 = 0, \\ e_1e_2 = 0, \\ e_2e_1 = -2e_1, \\ e_2e_2 = \beta_4 e_2. \end{cases}$
4. $\mu^4_{\alpha_2,\alpha_2+1}$ that is $\begin{cases} e_1e_1 = 0, \\ e_1e_2 = (\alpha_2+1)e_1, \\ e_2e_1 = (\alpha_2-1)e_1, \\ e_2e_2 = (\alpha_2+1)e_2. \end{cases}$
5. μ^5_1 that is $\begin{cases} e_1e_1 = 0, \\ e_1e_2 = 2e_1, \\ e_2e_1 = 0, \\ e_2e_2 = e_1 + 2e_2. \end{cases}$
6. μ^5_{-1} that is $\begin{cases} e_1e_1 = 0, \\ e_1e_2 = 0, \\ e_2e_1 = -2e_1, \\ e_2e_2 = e_1 - 2e_2. \end{cases}$

Jordan algebras

In a Jordan algebra, the multiplication μ satisfies:

$$\begin{cases} \mu(v,w) = \mu(w,v) \\ \mu(\mu(v,w),\mu(v,v)) = \mu(v,\mu(w,\mu(v,v))) \end{cases}$$

for all v,w. We assume in this section that $\mathbb{K}$ is algebraically closed and that the Jordan algebra are of dimension 2. Thus the multiplication μ is isomorphic to μ_i for i=11,⋯,16. To simplify the notation, we will write vw in place of $\mu(v,w)$. If v is an idempotent, thus v^2=v and the Jordan identity gives:

$v(vw)$=$v(vw)$

for any w, that is, this identity is always satisfied.

Lemma 9: If v_1 and v_2 are idempotent vectors, thus:

$$(v_1v_2)((v_1+v_2)w) = (v_1+v_2)((v_1v_2)w)$$

for any w.

Proof. In the Jordan identity, we replace v by v_1+v_2. We obtain:

$$v_1^2(v_2w) + 2(v_1v_2)((v_1+v_2)w) + v_2^2(v_1w) = v_1(v_2^2w) + v_2(v_1^2w) + 2(v_1+v_2)((v_1v_2)w)$$

Since v_1 and (v_2) are idempotent, this equation reduces:

$$(v_1v_2)((v_1+v_2)w) = (v_1+v_2)((v_1v_2)w).$$

Proposition 10: If v_1 and v_2 are idempotent vectors such that v_1v_2 and v_1+v_2 are independent, thus the Jordan algebra is associative.

Proof. Let x and y be two vectors of the algebra. Thus, by hypothesis, $x = x_1v_1v_2 + x_2(v_1+v_2)$ and $y = y_1v_1v_2 + y_2(v_1+v_2)$. Thus:

$$x(yw) = x_1y_1(v_1v_2)((v_1v_2)w) + (x_1y_2 + x_2y_1)(v_1v_2)((v_1+v_2)w) + x_2y_2(v_1+v_2)((v_1+v_2)w)$$

and,

$$x(yw) = y(xw)$$

By commutativity we obtain:

$$x(yw) = x(wy) = y(xw) = (xw)y$$

this proves that the algebra is associative.

If μ is given by,

$$\begin{cases} \mu(e_1,e_1) = e_1, \\ \mu(e_1,e_2) = \alpha_2 e_1 + \beta_2 e_2, \\ \mu(e_2,e_2) = e_2 \end{cases}$$

the Jordan algebra admits two idempotents e_1 and e_2. Since $e_1e_2 = \alpha_2e_1 + \beta_2e_2$, the vectors e_1e_2 and e_1+e_2 are independent if and only if $\alpha_2 \neq \beta_2$. In this case the algebra can be associative and we obtain the following associative Jordan algebra corresponding to:

1. α_2=1, β_2=0
2. α_2=0, β_2=1

These Jordan algebras are isomorphic. This gives the following Jordan algebra:

$$J_1 = \begin{cases} e_1e_1 = e_1, \\ e_1e_2 = e_2e_1 = e_2 \\ e_2e_2 = e_2. \end{cases}$$

If e_1e_2 and e_1+e_2 are dependent, that is e_1e_2=$\lambda(e_1+e_2)$, then λ=−1 or $\frac{1}{2}$ or 0. If e_1e_2=0, the product is not a Jordan product. If λ=−1 the product is never a Jordan product. If $\ddot{e} = \frac{1}{2}$, we obtain the following Jordan algebra,

$$J_2 = \begin{cases} e_1e_1 = e_1, \\ e_1e_2 = e_2e_1 = \frac{1}{2}(e_1+e_2) \\ e_2e_2 = e_2. \end{cases}$$

μ is given by:

$$\begin{cases} \mu(e_1,e_1) = e_1, \\ \mu(e_1,e_2) = \beta_2 e_2, \\ \mu(e_2,e_2) = 0. \end{cases}$$

This product is a Jordan product if β_2=1 or 0. We obtain:

$$J_3 = \begin{cases} e_1e_1 = e_1, \\ e_1e_2 = e_2e_1 = e_2 \\ e_2e_2 = 0. \end{cases}, \quad J_4 = \begin{cases} e_1e_1 = e_1, \\ e_1e_2 = e_2e_1 = 0 \\ e_2e_2 = 0. \end{cases}$$

If μ=μ_{11} we have also a Jordan structure,

$$J_5=\begin{cases} e_1e_1=e_2 \\ e_1e_2=e_2e_1=0 \\ e_2e_2=0 \end{cases}$$

μ=0, we have the trivial Jordan algebra.

• If $\mathbb{K}$ is not algebraically closed, we consider,

$$\begin{cases} \mu_R^{8,2}(e_1,e_1)=e_1, \\ \mu_R^{8,2}(e_1,e_2)=\beta_2 e_2, \quad 1-2\beta_2\notin(\mathbb{K}^*)^2, \\ \mu_R^{8,2}(e_2,e_2)=\lambda e_1, \end{cases}$$

We obtain a Jordan structure:

$$J_6=\begin{cases} e_1e_1=e_1 \\ e_1e_2=e_2e_1=e_2 \\ e_2e_2=\lambda e_1. \end{cases}$$

We find the list established in ref. [1].

2-dimensional Hom-algebra

The notion of Hom-algebra was introduced to generalized form of Hom-Lie algebra which appeared naturally when we are interested by the notion of q-derivation on the Witt algebra. In dimension 2, this notion is equivalent to the classical notion of Lie algebra. In dimension 3, we have shown that any skew-symmetric algebra is a Hom-Lie algebra. Then our interest concerns Hom-associative algebra [7,8], that is algebra $A=(V,\mu)$ such that there exists $f\in \mathrm{End}(V)$ satisfying the Hom-Ass identity:

$$\mu(\mu(X,Y),f(Z))=\mu(f(X),\mu(Y,Z))$$

for any X, Y, $Z\in V$. Using previous notations, we consider the algebras $A^{(Id,f)}$ and its opposite $A^{(f,Id)}$. Their multiplication law are respectively defined by:

$$\mu_{R,f}(X,Y)=\mu(X,f(Y)), \quad \mu_{L,f}(X,Y)=\mu(f(X),Y)$$

and the Hom-Ass identity can be written:

$$\mu_{R,f}\circ(\mu\otimes Id)-\mu_{L,f}\circ(Id\circ\mu)=0.$$

Assume now that the algebra A is regular. In this case, assuming that the field is algebraically closed, there exists an unital algebra whose product is denoted $X\cdot Y$ and two endomorphisms u and v of V such that:

$$\mu(X,Y)=u(X)\cdot v(Y)$$

Then,

$$\mu_{R,f}(X,Y)=u(X)\cdot v\circ f(Y), \quad \mu_{L,f}(X,Y)=u\circ f(X)\cdot v(Y).$$

Then the Hom-Ass identity becomes:

$$u(u(X)\cdot v(Y))\cdot v\circ f(Z)-u\circ f(X)\cdot(v(u(Y)\cdot v(Z))=0.$$

Maybe, it is better to look the Hom-Ass identity from the previous list. Assume that A is non commutative.

1. $A=A^1_{\alpha_2,\beta_2,\alpha_4,\beta_4}=(V,\mu^1)$, let f be an endomorphism of V satisfying the Hom-Ass identity. To simplify notations we write XY for $\mu(X,Y)$ and $[X,Y]$ for $\mu_a(X,Y)$. We have in particular:

$$(e_1e_1)f(e_1)-f(e_1)(e_1e_1))=[e_2,f(e_1)]=0\,.$$

We deduce $f(e_1)=ae_2$. Likewise we have $[e_2e_2,f(e_2)]=0$ and $f(e_2)=k(\alpha_4e_1+\beta_4e_1)$. Other identities give :

(a) $(e_1e_2)f(e_1)-f(e_1)(e_2e_1)=0$ implies $a=0$ or $e_2e_2=0$.

(b) If $a=0$, then $(e_1,e_2)f(e_2)(e_1e_1)=0$ implies $f(e_2)e_2=0$ and $(e_1,e_1)f(e_2)-f(e_1)(e_1e_2)=0$ implies $e_2f(e_2)=0$. Then $[e_2, f(e_2)]=0$ and $f(e_2)=ke_2$. This gives $0=f(e_2)e_2=be_2e_2$ that is $f=0$ or $e_2e_2=0$. But we have seen that $f(e_2)=k(\alpha_4e_1+\beta_4e_2)$, then in all the cases, $f=0$.

(c) If $a\neq 0$, then $e_2e_2=0$ and $f(e_2)=0$. We deduce that $(e_1e_2)f(e_1)-f(e_1)(e_2e_1)=0$ implies $\alpha_2=\beta_2=0$. Thus $(e_2e_1)f(e_1)-f(e_2)(e_1e_1)=-a(e_1e_2)=-ae_1=0$ and $a=0$.

We deduce that the algebra $A_{\alpha_2,\beta_2,\alpha_4,\beta_4}$ is not a Hom-associative algebra.

2. $A=A^2_{\alpha_1,\alpha_2,\alpha_4}$. With similar simple computation we can look that also this algebra is not a Hom-Ass algebra.

3. $A=A^3_{\alpha_4,\beta_4}$. In this case also, if we compute $(e_1e_1)f(e_1)-f(e_1)(e_1e_1)=[e_1,f(e_1)]=0$, we obtain $f(e_1)=k_1e_1$. Also we have $(e_1e_2)f(e_1)-f(e_1)(e_2e_1)=2k_1e_1=0$ and $f(e_1)=0$. We deduce $e_1f(e_2)=0$ and $f(e_2)e_1=0$ and $f(e_2)=0$. Thus $f=0$ and A^3 is not a Hom-associative algebra.

4. $A=A^4_{\acute{a}_2,\hat{a}_4}$. If $\beta_4\neq 0$, then the Hom-Ass condition implies $\alpha_2=1$ or -1. We obtain the following Hom-Ass algebras:

$$\begin{cases} \mu^4_{1,\hat{a}_4}(e_1,e_1)=0, \\ \mu^4_{1,\hat{a}_4}(e_1,e_2)=2e_1, \\ \mu^4_{1,\hat{a}_4}(e_2,e_1)=0, \\ \mu^4_{1,\hat{a}_4}(e_2,e_2)=\beta_4e_2, \end{cases}, \begin{cases} \mu^4_{-1,\hat{a}_4}(e_1,e_1)=0, \\ \mu^4_{-1,\hat{a}_4}(e_1,e_2)=0, \\ \mu^4_{-1,\hat{a}_4}(e_2,e_1)=-2e_1, \\ \mu^4_{-1,\hat{a}_4}(e_2,e_2)=\beta_4e_2. \end{cases}$$

In each of these two cases, f is a diagonal endomorphism. These algebras are for $\beta_4\neq 2$ or -2, not associative.

5. $A=A^5_{\acute{a}_2}$. If $\alpha_2=0$, any linear endomorphism with values in $\mathbb{K}\{e_1\}$ satisfies the Hom-Ass identity. Then the following algebra is Hom-associative:

$$\begin{cases} \mu^5_0(e_1,e_1)=0, \\ \mu^5_0(e_1,e_2)=e_1, \\ \mu^5_0(e_2,e_1)=-e_1, \\ \mu^5_0(e_2,e_2)=e_1. \end{cases}$$

Assume now that $\alpha_2\neq 0$. If $\acute{a}_2\neq\pm 1$, then any endomorphism satisfying the Hom-Ass identity is trivial. If $\alpha_2=1$ or -1, we have non trivial solution and the following algebras are Hom-associative algebras:

$$\begin{cases} \grave{\imath}^5_{-1}(e_1,e_1)=0, \\ \grave{\imath}^5_{-1}(e_1,e_2)=0, \\ \grave{\imath}^5_{-1}(e_2,e_1)=-2e_1, \\ \grave{\imath}^5_{-1}(e_2,e_2)=e_1-2e_2. \end{cases}, \begin{cases} \grave{\imath}^5_{1}(e_1,e_1)=0, \\ \grave{\imath}^5_{1}(e_1,e_2)=2e_1, \\ \grave{\imath}^5_{1}(e_2,e_1)=0, \\ \grave{\imath}^5_{1}(e_2,e_2)=e_1+2e_2. \end{cases}$$

with $f=\begin{pmatrix} -4x & x \\ 0 & -2x \end{pmatrix}$ in the first case and $f=\begin{pmatrix} 4x & x \\ 0 & 2x \end{pmatrix}$ in the second case.

Then we have the list of noncommutative Hom-associative algebras. The commutative case can be established in the same way. In this case the Hom-Ass identity is reduced to:

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$(e_1e_1)f(e_2)-(e_1e_2)f(e_1)=0,\ (e_1e_2)f(e_2)-(e_2e_2)f(e_1)=0.$

Then f is in the kernel of the linear system whose matrix is:

$$HA_A=\begin{pmatrix} -\alpha_2\alpha_1-\beta_2\alpha_2 & -\alpha_2^2-\beta_2\alpha_4 & \alpha_1^2+\beta_1\alpha_2 & \alpha_1\alpha_2+\beta_1\alpha_4 \\ -\alpha_2\beta_1-\beta_2^2 & \alpha_2\beta_2-\beta_2\beta_4 & \alpha_1\beta_1+\beta_1\beta_2 & \alpha_1\hat{a}_2+\beta_1\beta_4 \\ -\alpha_4\alpha_1-\beta_4\alpha_2 & -\alpha_4\alpha_2-\beta_4\alpha_4 & \alpha_2\alpha_1+v_2\alpha_2 & \alpha_2^2+\beta_2\alpha_4 \\ -\alpha_4\beta_1-\beta_4\beta_2 & -\alpha_4\beta_2-\beta_4^2 & \alpha_2\beta_1+\beta_2^2 & \alpha_2\beta_2+\beta_2\beta_4 \end{pmatrix}$$

Then A is a Hom-associative algebra if and only if $H(A)=\det(HA_A)=0$. We deduce that the set of 2-dimensional commutative Hom-associative algebra can be provided with an algebraic hypersurface embedded in the affine variety $\mathbb{K}^6$. From Theorem 6, when $\mathbb{K}$ is algebraically closed, we obtain:

1. $H(A^6)=\alpha_2\beta_2(1-\alpha_2-\beta_2-3\alpha_2\beta_2+2\alpha_2^2\beta_2+\alpha_2^3\beta_2+2\alpha_2\beta_2^2+2\alpha_2^2\beta^2+\alpha_2\beta_2^3)$. It is equal to 0 for $\alpha_2=0$ or $\beta_2=0$ or $\alpha_2=1-\beta_2$ or $\alpha_2=\dfrac{-3\beta_2-\beta_2^2-(1+\beta_2)\sqrt{\beta_2}\sqrt{4+\beta_2}}{2\beta_2}$ or $\alpha_2=\dfrac{-3\beta_2-\beta_2^2+(1+\beta_2)\sqrt{\beta_2}\sqrt{4+\beta_2}}{2\beta_2}$.
2. $H(A^7)=-\dfrac{1}{4}$ and A^7 is not a Hom-associative algebra.
3. $H(A^8)=-\dfrac{9}{64}$ and A^8 is not a Hom-associative algebra.
4. $H(A^i)=0$ for $i=9,10,11,12,13,14,15,16,17$ and $A^9,A^{10},A^{11},A^{12},A^{13},A^{14},A^{15},A^{16},A^{17}$ are a Hom-associative algebras.

References

1. Bermuœdez AJM, Fresan J, Margalef Bentabol J (2011) Contractions of low-dimensional nilpotent Jordan algebras. Comm Algebra 39 (3): 1139-1151.
2. Goze M, Remm E (2011) 2-dimension algebras. African Journal of Mathematical Physics 110: 81-91.
3. Ahmed H, Bekbaev U, Rakhimov I (2017) Complete classification of two-dimensional algebras. AIP Conference Proceedings 1830 (1): 10.1063/1.4980965.
4. Goze M, Remm E (2007) A class of nonassociative algebras. Algebra Colloq 14(2): 313-326.
5. Petersson HP (2000) The classification of two-dimensional nonassociative algebras. Result Math 3: 120-154.
6. Goze M, Remm E (2003) Affine structures on abelian Lie groups. Linear Algebra Appl 360: 215-230.
7. Makhlouf A (2010) Paradigm of nonassociative Hom-algebras and Hom-superalgebras.Proceedings of Jordan Structures in Algebra and Analysis Meeting, Editorial Círculo Rojo, Almería, pp: 143-177.
8. Shanghua Z, Li G (2016) Free involutive Hom-semigroups and Hom-associative algebras. Front Math China 11 (2): 497-508.